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Linear Models with Nuisance Parameters and Deformation Measurement

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Abstract

The aim of the paper is an investigation of a possibility to study the deformation measurement in the framework of the linear model with nuisance parameters. It is proved that when the deformations are investigated only then it is possible to neglect the nuisance parameters.

Key words: Multiepoch linear regression model, nuisance parameters, BLUE.

1991 Mathematics Subject Classification: 62J05

1 Introduction, notations

The task to verify the stability of many engineer's construction works (dams, bridges) or to study the course of its deformation in time can be solved by suitably ordered measurements replicated at suitably chosen moments-epochs. These replicated measurements are modelled by multiepoch models. (Cf. [1, Chapter 9]).

Two fundamental types of multiepoch models may occur ([1, p. 366]).

a) Models with stable and variable parameters:
repeated measurements studying existence of deformation of some object and its

course are realized in separate networks especially constructed for this purpose. It consists of a group of supporting points whose position is assumed to be stable (this assumption is verified during the measurement) and a group of points whose movements related to the position of the stable points are investigated (the coordinates of the group of the stable points are a priori unknown). After finishing each epoch both the coordinates of the supporting points and the coordinates of investigated points are to be determined. The former serve to verify the hypothesis on the stableness of the group of supporting points.

b) Models with variable parameters only:
the network for studying the dynamism of a locality is joint to the stable points of a geodetic network (these represent the stable supporting points of the proceding type of the network, in contradiction to it, their coordinates are a priory known). The coordinates of the group of the points studied from the viewpoint of the dynamism are being determined.

The task of the following is to investigate the model of the first type. The main problem is whether the nuisance parameters (i.e. coordinates of the stable points) can or cannot be neglected.

The following notation will be used throughout the paper:

R^n	the space of all n-dimensional real vectors;
$\boldsymbol{u}_p$	the real column p-dimensional vector,
$\boldsymbol{A}_{m,n}$	the real $m \times n$ matrix;
$\boldsymbol{A}'$, $\mathscr{R}(\boldsymbol{A})$ $\mathscr{N}(\boldsymbol{A})$, $r(\boldsymbol{A})$	the transpose, the range, the null space and the rank of the matrix $\boldsymbol{A}$;
$vec(\boldsymbol{A})$	the column vector $(\{\boldsymbol{A}\}'_{.1}, \dots, \{\boldsymbol{A}\}'_{.n})'$ created by the columns of the matrix $\boldsymbol{A}$;
$\boldsymbol{A} \otimes \boldsymbol{B}$	the Kronecker (tensor) product of the matrices $\boldsymbol{A}$,$\boldsymbol{B}$;
$\boldsymbol{A}^-$	a generalized inverse of a matrix $\boldsymbol{A}$ (satisfying $\boldsymbol{A}\boldsymbol{A}^-\boldsymbol{A} = \boldsymbol{A}$);
$\boldsymbol{A}^+$	the Moore-Penrose generalized inverse of a matrix $\boldsymbol{A}$ (satisfying $\boldsymbol{A}\boldsymbol{A}^+\boldsymbol{A} = \boldsymbol{A}$, $\boldsymbol{A}^+\boldsymbol{A}\boldsymbol{A}^+ = \boldsymbol{A}^+$, $(\boldsymbol{A}\boldsymbol{A}^+)' = \boldsymbol{A}\boldsymbol{A}^+$, $(\boldsymbol{A}^+\boldsymbol{A})' = \boldsymbol{A}^+\boldsymbol{A}$);
$\boldsymbol{P}_A$	the ortogonal projector onto $\mathscr{R}(\boldsymbol{A})$;
$\boldsymbol{M}_A = \boldsymbol{I} - \boldsymbol{P}_A$	the ortogonal projector onto $\mathscr{R}^{\perp}(\boldsymbol{A}) = \mathscr{N}(\boldsymbol{A}')$;
$\boldsymbol{I}_k$	the $k \times k$ identity matrix;
$\mathbf{1}_k = (1, \dots, 1)' \in R^k$.	

If $\mathscr{R}(\boldsymbol{A}) \subset \mathscr{R}(\boldsymbol{S})$, $\boldsymbol{S}$p.s.d., then the symbol $\boldsymbol{P}_A^{S^-}$ denotes the projector projecting vectors in $\mathscr{R}(\boldsymbol{S})$ onto $\mathscr{R}(\boldsymbol{A})$ along $\mathscr{R}(\boldsymbol{S}\boldsymbol{A}^{\perp})$. A general representation of all such projectors $\boldsymbol{P}_A^{S^-}$ is given by

$$\boldsymbol{A}(\boldsymbol{A}'\boldsymbol{S}^-\boldsymbol{A})^-\boldsymbol{A}'\boldsymbol{S}^- + \boldsymbol{B}(\boldsymbol{I} - \boldsymbol{S}\boldsymbol{S}^-),$$

where $\boldsymbol{B}$ is arbitrary, cf. [4, (2.14)]. $\boldsymbol{M}_A^{S^-} = \boldsymbol{I} - \boldsymbol{P}_A^{S^-}$.

2 Partial linear regression model

Consider following partial model that can be realized in the j-th epoch of measurement

$$\boldsymbol{Y}_j = (\boldsymbol{X}_1, \boldsymbol{X}_2)\begin{pmatrix}\boldsymbol{\beta}_1\\ \boldsymbol{\beta}_{2j}\end{pmatrix} + \boldsymbol{\varepsilon}_j, \quad j = 1, \ldots, m, \tag{1}$$

where $n \times k$ matrix $\boldsymbol{X}_1$ and $n \times l$ matrix $\boldsymbol{X}_2$ are the design matrices, $\boldsymbol{\beta}_1 \in R^k$ is a vector of stable parameters (coordinates of the group of stable points), that are assumed to be *nuisance*, $\boldsymbol{\beta}_{2j} \in R^l$ is a vector of variable parameters (coordinates of the group of unstable points observed in the j-th epoch of measurement) that are supposed to be *useful.*

It is assumed that the *regularity conditions*: $r(\boldsymbol{X}_1, \boldsymbol{X}_2) = (k+l) < n$, $r(\boldsymbol{X}_1) = k$, $r(\boldsymbol{X}_2) = l$, $var(\boldsymbol{Y}_j) = \boldsymbol{\Sigma}_{\vartheta} = \sum_{i=1}^{p} \vartheta_i \boldsymbol{V}_i$, $\boldsymbol{\Sigma}_{\vartheta}$ *p.d.* $\forall \boldsymbol{\vartheta} \in \underline{\vartheta} \subset R^p$, $\underline{\vartheta} \subset R^p$ contains an open sphere, are fulfilled.

The model

$$\boldsymbol{Y}^{(m)} = \begin{pmatrix}\boldsymbol{Y}_1\\ \boldsymbol{Y}_2\\ \vdots\\ \boldsymbol{Y}_m\end{pmatrix} = \begin{pmatrix}\boldsymbol{X}_1, & \boldsymbol{X}_2, & \boldsymbol{0}, & \cdots & \boldsymbol{0}\\ \boldsymbol{X}_1, & \boldsymbol{0}, & \boldsymbol{X}_2, & \cdots & \boldsymbol{0}\\ \ldots & \ldots & \ldots & \ldots & \ldots\\ \boldsymbol{X}_1, & \boldsymbol{0}, & \boldsymbol{0}, & \cdots & \boldsymbol{X}_2\end{pmatrix}\begin{pmatrix}\boldsymbol{\beta}_1\\ \boldsymbol{\beta}_{21}\\ \vdots\\ \boldsymbol{\beta}_{2m}\end{pmatrix} + \boldsymbol{\varepsilon}^{(m)}, \tag{2}$$

$$var[\boldsymbol{Y}^{(m)}] = \boldsymbol{\Sigma}_{\vartheta}^{(m)} = \begin{pmatrix}\boldsymbol{\Sigma}_{\vartheta}, & \boldsymbol{0}, & \boldsymbol{0}, & \cdots & \boldsymbol{0}\\ \boldsymbol{0}, & \boldsymbol{\Sigma}_{\vartheta}, & \boldsymbol{0}, & \cdots & \boldsymbol{0}\\ \ldots & \ldots & \ldots & \ldots & \ldots\\ \boldsymbol{0}, & \boldsymbol{0}, & \boldsymbol{0}, & \cdots & \boldsymbol{\Sigma}_{\vartheta}\end{pmatrix}.$$

is said to be m-epoch linear regression model with a fixed number of stable (nuisance) parameters and with variable (useful) parameters, (cf. [1, p. 368]).

Here $\boldsymbol{Y}^{(m)}$ is a nm-dimensional observation vector after the m-th epoch of measurement, $\boldsymbol{Y}_j$ is a n-dimensional observation vector of the j-th epoch, $j = 1, \ldots, m$.

Model (2) is the simplest multiepoch linear regression model in which the design matrices, the dimensions of the observation vectors and the variance matrices $var(\boldsymbol{Y}_j)$ are the same in all epochs.

In what follows we deal with so called "small" partial linear regression model

$$\boldsymbol{Y}_j = \boldsymbol{X}_2 \boldsymbol{\beta}_{2j} + \boldsymbol{\varepsilon}_j, \tag{3}$$

$$var \boldsymbol{Y}_j = \boldsymbol{\Sigma}_{\vartheta}, \quad j = 1, \ldots, m,$$

where the nuisance parameters are neglected.

Notation 1

1. For the sake of simplicity we write $\boldsymbol{\Sigma}_0$ instead of $\boldsymbol{\Sigma}_{\vartheta_0}$.

2. A parametric *function* $\boldsymbol{f}'\boldsymbol{\beta}_{2j}$ is said to be *unbiasedly estimable* under the model (1) if there exists an estimator $\boldsymbol{g}'\boldsymbol{Y}_j$, $\boldsymbol{g} \in R^n$ such that $E[\boldsymbol{g}'\boldsymbol{Y}_j] = \boldsymbol{f}'\boldsymbol{\beta}_{2j}$,

$\forall \beta_1, \forall \beta_{2j}$, i.e. if there exists an unbiased linear estimator (LUE) of the function $f'\beta_{2j}$.

The statistic $g'Y_j$ is said to be the *efficient linear estimator of the function* $f'\beta_{2j}$ *at the point* $\vartheta_0 \in \underline{\vartheta}$ (ϑ_0-locally best) if it is a LUE of the function $f'\beta_{2j}$ and

$$var_{\Sigma_0}(g'Y_j) \le var_{\Sigma_0}(h'Y_j), \quad \forall h'Y_j \text{ LUE of } f'\beta_{2j},$$

($g'Y_j$ is the LBLUE [locally best unbiased estimator] of the function $f'\beta_{2j}$).

3. Let, according to [4], $\mathscr{E}_a$ and $\mathscr{E}$ denote the sets of all linear functions of β_{2j} which are unbiasedly estimable within the linear model (2) with nuisance parameters and within the linear model (3) without nuisance parameters, respectively. The index a will indicate, that the estimator is considered within the complete model, i.e. within the model with nuisance parameters.

Let $\mathscr{F}$ denote the set of all linear functions of β_1 which are unbiasedly estimable within the model (2).

Obviously (cf. [4, (2.1), (2.2)])

$$\begin{aligned} \mathscr{E} &= \{f'\beta_{2j} : f \in \mathscr{R}(X_2')\}, \\ \mathscr{E}_a &= \{f'\beta_{2j} : f \in \mathscr{R}(X_2' M_{X_1})\}, \\ \mathscr{F} &= \{f'\beta_1 : f \in \mathscr{R}(X_1' M_{X_2})\}. \end{aligned}$$

Assertion 1 *Consider the partial linear regression model (1) under the condition that the regularity assumptions are fulfilled. The Σ_0-LBLUEs of the parameter functions $f'\beta_1$ and $f'\beta_{2j}$ are given as*

$$\widehat{f'\beta_{1,\Sigma_0}}(Y_j)_a = f'[X_1'(M_{X_2}\Sigma_0 M_{X_2})^+ X_1]^{-1} X_1'(M_{X_2}\Sigma_0 M_{X_2})^+ Y_j, \quad (4)$$

if $f'\beta_1 \in \mathscr{F}$,

$$\widehat{f'\beta_{2j,\Sigma_0}}(Y_j)_a = f'[X_2'(M_{X_1}\Sigma_0 M_{X_1})^+ X_2]^{-1} X_2'(M_{X_1}\Sigma_0 M_{X_1})^+ Y_j, \quad (5)$$

if $f'\beta_{2j} \in \mathscr{E}_a$,

$$var_{\Sigma_0}[\widehat{f'\beta_{1,\Sigma_0}}(Y_j)_a] = f'[X_1'(M_{X_2}\Sigma_0 M_{X_2})^+ X_1]^{-1} f, \quad \textit{if } f'\beta_1 \in \mathscr{F}, \quad (6)$$

$$var_{\Sigma_0}[\widehat{f'\beta_{2j,\Sigma_0}}(Y_j)_a] = f'[X_2'(M_{X_1}\Sigma_0 M_{X_1})^+ X_2]^{-1} f, \quad \textit{if } f'\beta_{2j} \in \mathscr{E}_a, \quad (7)$$

where

$$(M_{X_i}\Sigma_0 M_{X_i})^+ = \Sigma_0^{-1} - \Sigma_0^{-1} X_i (X_i'\Sigma_0^{-1} X_i)^{-1} X_i'\Sigma_0^{-1}, \quad i = 1, 2.$$

Proof Cf. [1, Theorem 9.1.2.]

Remark 1 Within the model (3) under the assumption that the regularity conditions are fulfilled, the Σ_0-LBLUE of the parameter functions $f'\beta_{2j}$ are

$$\widehat{f'\beta_{2j,\Sigma_0}}(Y_j) = f'(X_2'\Sigma_0^{-1} X_2)^{-1} X_2'\Sigma_0^{-1} Y_j, \quad \text{if } f'\beta_{2j} \in \mathscr{E}, \quad (8)$$

$$var[\widehat{f'\beta_{2j,\Sigma_0}}(Y_j)] = f'(X_2'\Sigma_0^{-1} X_2)^{-1} f, \quad \text{if } f'\beta_{2j} \in \mathscr{E}. \quad (9)$$

Notation 2 Let, according to [4], $\mathscr{E}_0$ denote the subset of $\mathscr{E}_a$ consisting of all those functions of $\boldsymbol{f}'\boldsymbol{\beta}_{2j}$ for which the Σ_0-LBLUE within the model (1) posseses the same variance as the Σ_0-LBLUE within the model (3), i.e.

$$\mathscr{E}_0 = \{\boldsymbol{f}'\boldsymbol{\beta}_{2j} \in \mathscr{E}_a : var_{\Sigma_0}[\widehat{\boldsymbol{f}'\boldsymbol{\beta}_{2j,\Sigma_0}}(\boldsymbol{Y}_j)] = var_{\Sigma_0}[\widehat{\boldsymbol{f}'\boldsymbol{\beta}_{2j,\Sigma_0}}(\boldsymbol{Y}_j)_a]\}.$$

Assertion 2 *The class* $\mathscr{E}_0$ *is given by*

$$\begin{aligned}\mathscr{E}_0 &= \{\boldsymbol{f}'\boldsymbol{\beta}_{2j} : \boldsymbol{f} \in \mathscr{R}[\boldsymbol{X}_2'\boldsymbol{\Sigma}_0^{-1}\boldsymbol{X}_2\boldsymbol{M}_{X_2'\Sigma_0^{-1}X_1}]\} \\ &= \{\boldsymbol{f}'\boldsymbol{\beta}_{2j} : \boldsymbol{f} = \boldsymbol{X}_2'\boldsymbol{q},\ \boldsymbol{q} \in \mathscr{R}[\Sigma_0^{-1}\boldsymbol{X}_2\boldsymbol{M}_{X_2'\Sigma_0^{-1}X_1}]\}.\end{aligned}$$

Proof Cf. [4, Theorem 3.1, relation (3.3)].

3 Multiepoch linear regression model

Let us come back to the multiepoch model (2) of the measurement after the m-th epoch

$$\boldsymbol{Y}^{(m)} = \begin{pmatrix}\boldsymbol{Y}_1\\ \boldsymbol{Y}_2\\ \vdots\\ \boldsymbol{Y}_m\end{pmatrix} = \begin{pmatrix}\boldsymbol{X}_1, & \boldsymbol{X}_2, & \boldsymbol{0}, & \dots & \boldsymbol{0}\\ \boldsymbol{X}_1, & \boldsymbol{0}, & \boldsymbol{X}_2, & \dots & \boldsymbol{0}\\ \dots & \dots & \dots & \dots & \dots\\ \boldsymbol{X}_1, & \boldsymbol{0}, & \boldsymbol{0}, & \dots & \boldsymbol{X}_2\end{pmatrix}\begin{pmatrix}\boldsymbol{\beta}_1\\ \boldsymbol{\beta}_{21}\\ \vdots\\ \boldsymbol{\beta}_{2m}\end{pmatrix} + \boldsymbol{\varepsilon}^{(m)}, \qquad (10)$$

$$var(\boldsymbol{Y}^{(m)}) = \boldsymbol{\Sigma}_\vartheta^{(m)} = \boldsymbol{I}_m \otimes \boldsymbol{\Sigma}_\vartheta.$$

Model (10) can be written in the equivalent form

$$\boldsymbol{Y}^{(m)} = (\mathbf{X}_1^{(m)}, \mathbf{X}_2^{(m)})\begin{pmatrix}\boldsymbol{\beta}_1\\ \boldsymbol{\beta}_2^{(m)}\end{pmatrix} + \boldsymbol{\varepsilon}^{(m)},$$

where

$$\mathbf{X}_1^{(m)} = \begin{pmatrix}\boldsymbol{X}_1\\ \boldsymbol{X}_1\\ \vdots\\ \boldsymbol{X}_1\end{pmatrix} = \mathbf{1}_m \otimes \boldsymbol{X}_1, \qquad \mathbf{X}_2^{(m)} = \begin{pmatrix}\boldsymbol{X}_2, & \boldsymbol{0}, & \dots & \boldsymbol{0}\\ \boldsymbol{0}, & \boldsymbol{X}_2, & \dots & \boldsymbol{0}\\ \dots & \dots & \dots & \dots\\ \boldsymbol{0}, & \boldsymbol{0}, & \dots & \boldsymbol{X}_2\end{pmatrix} = \boldsymbol{I}_m \otimes \boldsymbol{X}_2,$$

$$\boldsymbol{\beta}_2^{(m)} = \begin{pmatrix}\boldsymbol{\beta}_{21}\\ \boldsymbol{\beta}_{22}\\ \vdots\\ \boldsymbol{\beta}_{2m}\end{pmatrix}.$$

We also deal with the “small” model

$$\boldsymbol{Y}^{(m)} = \mathbf{X}_2^{(m)}\boldsymbol{\beta}_2^{(m)} + \boldsymbol{\varepsilon}^{(m)} = (\boldsymbol{I}_m \otimes \boldsymbol{X}_2)\boldsymbol{\beta}_2^{(m)} + \boldsymbol{\varepsilon}^{(m)}, \qquad (11)$$

$$var(\boldsymbol{Y}^{(m)}) = \boldsymbol{\Sigma}_\vartheta^{(m)} = \boldsymbol{I}_m \otimes \boldsymbol{\Sigma}_\vartheta,$$

where the nuisance parameters $\boldsymbol{\beta}_1$ are neglected.

Theorem 1 *Within the model (10) of the measurement after the m-th epoch in which the regularity conditions* $r(\mathbf{X}_1^{(m)}, \mathbf{X}_2^{(m)}) = k+ml$, $r(\boldsymbol{X}_1) = k$, $r(\boldsymbol{X}_2) = l$, $r[var(\boldsymbol{Y}^{(m)})] = mn$, $r[var(\boldsymbol{Y}_i)] = n$, $i = 1,\dots,m$, *are fulfilled, the* $\Sigma_0^{(m)}$*-LBLUEs of the parameters* $\boldsymbol{\beta}_1$ *and* $\boldsymbol{\beta}_2$, $\boldsymbol{\beta}_{2i}$, $i = 1,\dots,m$ *are*

$$\widehat{\boldsymbol{\beta}^{(m)}_{1,\Sigma_0^{(m)}}}(\boldsymbol{Y}^{(m)})_a = [m\boldsymbol{X}_1'(\boldsymbol{M}_{X_2}\Sigma_0\boldsymbol{M}_{X_2})^+\boldsymbol{X}_1]^{-1}\boldsymbol{X}_1'(\boldsymbol{M}_{X_2}\Sigma_0\boldsymbol{M}_{X_2})^+(\sum_{i=1}^{m}\boldsymbol{Y}_i), \tag{12}$$

$$var_{\Sigma_0^{(m)}}[\widehat{\boldsymbol{\beta}^{(m)}_{1,\Sigma_0^{(m)}}}(\boldsymbol{Y}^{(m)})_a] = [m\boldsymbol{X}_1'(\boldsymbol{M}_{X_2}\Sigma_0\boldsymbol{M}_{X_2})^+\boldsymbol{X}_1]^{-1}, \tag{13}$$

$$\widehat{\boldsymbol{\beta}^{(m)}_{2,\Sigma_0^{(m)}}}(\boldsymbol{Y}^{(m)})_a = \{[\boldsymbol{I}_m \otimes (\boldsymbol{X}_2'\boldsymbol{\Sigma}_0^{-1}\boldsymbol{X}_2)^{-1}\boldsymbol{X}_2'\boldsymbol{\Sigma}_0^{-1}]$$

$$- [\mathbf{1}_m\mathbf{1}_m' \otimes (\boldsymbol{X}_2'\boldsymbol{\Sigma}_0^{-1}\boldsymbol{X}_2)^{-1}\boldsymbol{X}_2'\boldsymbol{\Sigma}_0^{-1}\boldsymbol{X}_1[m\boldsymbol{X}_1'(\boldsymbol{M}_{X_2}\boldsymbol{\Sigma}_0\boldsymbol{M}_{X_2})^+\boldsymbol{X}_1]^{-1}$$

$$\times \boldsymbol{X}_1'(\boldsymbol{M}_{X_2}\boldsymbol{\Sigma}_0\boldsymbol{M}_{X_2})^+]\}\boldsymbol{Y}^{(m)}, \tag{14}$$

$$var_{\Sigma_0^{(m)}}[\widehat{\boldsymbol{\beta}^{(m)}_{2,\Sigma_0^{(m)}}}(\boldsymbol{Y}^{(m)})_a] = [\boldsymbol{I}_m \otimes (\boldsymbol{X}_2'\boldsymbol{\Sigma}_0^{-1}\boldsymbol{X}_2)^{-1}]$$

$$+ [\mathbf{1}_m\mathbf{1}_m' \otimes (\boldsymbol{X}_2'\boldsymbol{\Sigma}_0^{-1}\boldsymbol{X}_2)^{-1}\boldsymbol{X}_2'\boldsymbol{\Sigma}_0^{-1}\boldsymbol{X}_1[m\boldsymbol{X}_1'(\boldsymbol{M}_{X_2}\boldsymbol{\Sigma}_0\boldsymbol{M}_{X_2})^+\boldsymbol{X}_1]^{-1}$$

$$\times \boldsymbol{X}_1'\boldsymbol{\Sigma}_0^{-1}\boldsymbol{X}_2(\boldsymbol{X}_2'\boldsymbol{\Sigma}_0^{-1}\boldsymbol{X}_2)^{-1}], \tag{15}$$

$$\widehat{\boldsymbol{\beta}^{(m)}_{2i,\Sigma_0^{(m)}}}(\boldsymbol{Y}^{(m)})_a = (\boldsymbol{X}_2'\boldsymbol{\Sigma}_0^{-1}\boldsymbol{X}_2)^{-1}\boldsymbol{X}_2'\boldsymbol{\Sigma}_0^{-1}$$

$$\times \left[\boldsymbol{Y}_i - \boldsymbol{X}_1[m\boldsymbol{X}_1'(\boldsymbol{M}_{X_2}\boldsymbol{\Sigma}_0\boldsymbol{M}_{X_2})^+\boldsymbol{X}_1]^{-1}\boldsymbol{X}_1'(\boldsymbol{M}_{X_2}\boldsymbol{\Sigma}_0\boldsymbol{M}_{X_2})^+[\sum_{i=1}^{m}\boldsymbol{Y}_i]\right], \tag{16}$$

$i = 1,\dots,m,$

$$var_{\Sigma_0^{(m)}}[\widehat{\boldsymbol{\beta}^{(m)}_{2i,\Sigma_0^{(m)}}}(\boldsymbol{Y}^{(m)})_a] = (\boldsymbol{X}_2'\boldsymbol{\Sigma}_0^{-1}\boldsymbol{X}_2)^{-1} + (\boldsymbol{X}_2'\boldsymbol{\Sigma}_0^{-1}\boldsymbol{X}_2)^{-1}\boldsymbol{X}_2'\boldsymbol{\Sigma}_0^{-1}\boldsymbol{X}_1$$

$$\times [m\boldsymbol{X}_1'(\boldsymbol{M}_{X_2}\boldsymbol{\Sigma}_0\boldsymbol{M}_{X_2})^+\boldsymbol{X}_1]^{-1}\boldsymbol{X}_1'\boldsymbol{\Sigma}_0^{-1}\boldsymbol{X}_2(\boldsymbol{X}_2'\boldsymbol{\Sigma}_0^{-1}\boldsymbol{X}_2)^{-1}, \tag{17}$$

$i = 1,\dots,m,$

$$cov_{\Sigma_0^{(m)}}[\widehat{\boldsymbol{\beta}^{(m)}_{2r,\Sigma_0^{(m)}}}(\boldsymbol{Y}^{(m)})_a, \widehat{\boldsymbol{\beta}^{(m)}_{2s,\Sigma_0^{(m)}}}(\boldsymbol{Y}^{(m)})_a] = (\boldsymbol{X}_2'\boldsymbol{\Sigma}_0^{-1}\boldsymbol{X}_2)^{-1}\boldsymbol{X}_2'\boldsymbol{\Sigma}_0^{-1}\boldsymbol{X}_1$$

$$\times [m\boldsymbol{X}_1'(\boldsymbol{M}_{X_2}\boldsymbol{\Sigma}_0\boldsymbol{M}_{X_2})^+\boldsymbol{X}_1]^{-1}\boldsymbol{X}_1'\boldsymbol{\Sigma}_0^{-1}\boldsymbol{X}_2(\boldsymbol{X}_2'\boldsymbol{\Sigma}_0^{-1}\boldsymbol{X}_2)^{-1}, \tag{18}$$

$\forall r, s = 1,\dots,m,\ r \neq s.$

Proof According to the Theorem 1.1.1 in [1] using the following Rohde's formula for inverse of partitioned p.d. matrix (cf. [2, Lemma 13, p. 68])

$$\begin{pmatrix} A, & B \\ B', & C \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} A^{-1} + A^{-1}B(C - B'A^{-1}B)^{-1}B'A^{-1}, & -A^{-1}B(C - B'A^{-1}B)^{-1} \\ -(C - B'A^{-1}B)^{-1}B'A^{-1}, & (C - B'A^{-1}B)^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} (A - BC^{-1}B')^{-1}, & -(A - BC^{-1}B')^{-1}BC^{-1} \\ -C^{-1}B'(A - BC^{-1}B')^{-1}, & C^{-1} + C^{-1}B'(A - BC^{-1}B')^{-1}BC^{-1} \end{pmatrix},$$

we get

$$\begin{pmatrix} \widehat{\beta^{(m)}_{1,\Sigma_0^{(m)}}} \\ \widehat{\beta^{(m)}_{2,\Sigma_0^{(m)}}} \end{pmatrix} = \left[\begin{pmatrix} \mathbf{1}'_m \otimes X'_1 \\ I_m \otimes X'_2 \end{pmatrix} (I_m \otimes \Sigma_0^{-1})(\mathbf{1}_m \otimes X_1, I_m \otimes X_2) \right]^{-1}$$

$$\times \begin{pmatrix} \mathbf{1}'_m \otimes X'_1 \\ I_m \otimes X'_2 \end{pmatrix} (I_m \otimes \Sigma_0^{-1}) Y^{(m)}$$

$$= \begin{bmatrix} A; & -A(\mathbf{1}'_m \otimes X'_1\Sigma_0^{-1}X_2B) \\ -(\mathbf{1}_m \otimes BX'_2\Sigma_0^{-1}X_1)A; & (I_m \otimes B) + (\mathbf{1}_m\mathbf{1}'_m \otimes BX'_2\Sigma_0^{-1}X_1AX'_1\Sigma_0^{-1}X_2B) \end{bmatrix}$$

$$\times \begin{pmatrix} \mathbf{1}'_m \otimes X'_1 \\ I_m \otimes X'_2 \end{pmatrix} (I_m \otimes \Sigma_0^{-1}) Y^{(m)},$$

where $A = [mX'_1(M_{X_2}\Sigma_0 M_{X_2})^+ X_1]^{-1}$, $B = (X'_2\Sigma_0^{-1}X_2)^{-1}$.

Thus

$$\widehat{\beta^{(m)}_{1,\Sigma_0^{(m)}}} =$$

$$= \Big\{ [mX'_1(M_{X_2}\Sigma_0 M_{X_2})^+ X_1]^{-1}(\mathbf{1}'_m \otimes X'_1\Sigma_0^{-1}) - [mX'_1(M_{X_2}\Sigma_0 M_{X_2})^+ X_1]^{-1}$$

$$\times [\mathbf{1}'_m \otimes X'_1\Sigma_0^{-1}X_2(X'_2\Sigma_0^{-1}X_2)^{-1}](I_m \otimes X'_2\Sigma_0^{-1}) \Big\} Y^{(m)}$$

$$= \left\{ [mX'_1(M_{X_2}\Sigma_0 M_{X_2})^+ X_1]^{-1}(\mathbf{1}'_m \otimes X'_1(M_{X_2}\Sigma_0 M_{X_2})^+) \right\} Y^{(m)}$$

$$= \left\{ [mX'_1(M_{X_2}\Sigma_0 M_{X_2})^+ X_1]^{-1} X'_1(M_{X_2}\Sigma_0 M_{X_2})^+ \right\} (\sum_{i=1}^{m} Y_i).$$

$$var[\widehat{\beta^{(m)}_{1,\Sigma_0^{(m)}}}] = [mX'_1(M_{X_2}\Sigma_0 M_{X_2})^+ X_1]^{-1}[\mathbf{1}'_m \otimes X'_1(M_{X_2}\Sigma_0 M_{X_2})^+](I_m \otimes \Sigma_0)$$

$$\times [\mathbf{1}_m \otimes (M_{X_2}\Sigma_0 M_{X_2})^+ X_1][mX'_1(M_{X_2}\Sigma_0 M_{X_2})^+ X_1]^{-1}$$

$$= [mX'_1(M_{X_2}\Sigma_0 M_{X_2})^+ X_1]^{-1}.$$

We have proved assertions (12), (13). The rest of the proof is obvious, the assertions (16), (17), (18) follow immediately from (14) and (15). □

Remark 2 The $\Sigma_0^{(m)}$-LBLUEs of the vector parameter β_{2m} within the multivariate model (11) (without nuisance parameters) are

$$\widehat{\beta_{2,\Sigma_0^{(m)}}^{(m)}}(\boldsymbol{Y}^{(m)}) = [\boldsymbol{I}_m \otimes (\boldsymbol{X}_2'\boldsymbol{\Sigma}_0^{-1}\boldsymbol{X}_2)^{-1}\boldsymbol{X}_2'\boldsymbol{\Sigma}_0^{-1}]\boldsymbol{Y}^{(m)}, \tag{19}$$

$$var_{\Sigma_0^{(m)}}[\widehat{\beta_{2,\Sigma_0^{(m)}}^{(m)}}(\boldsymbol{Y}^{(m)})] = \boldsymbol{I}_m \otimes (\boldsymbol{X}_2'\boldsymbol{\Sigma}_0^{-1}\boldsymbol{X}_2)^{-1}. \tag{20}$$

Thus the $\Sigma_0^{(m)}$-LBLUEs of the parameter β_{2i}, $i = 1, \ldots, m$, within the model (11) are the same as in the partial model (3) without nuisance parameters (cf. Remark 1).

Notation 3

1. Let $\mathscr{E}_a^{(m)}$ and $\mathscr{E}^{(m)}$ denote the set of all linear functions of $\beta_2^{(m)}$, which are unbiasedly estimable under the model (10) and (11), respectively. The index a will indicate that the estimator is considered in the complete model.

Obviously

$$\begin{aligned}\mathscr{E}^{(m)} &= \{\boldsymbol{f}'\beta_2^{(m)} : \boldsymbol{f} \in \mathscr{R}(\boldsymbol{I}_m \otimes \boldsymbol{X}_2')\},\\ \mathscr{E}_a^{(m)} &= \{\boldsymbol{f}'\beta_2^{(m)} : \boldsymbol{f} \in \mathscr{R}[(\boldsymbol{I}_m \otimes \boldsymbol{X}_2')\boldsymbol{M}_{1_m\otimes X_1}]\}.\end{aligned}$$

2. Let, as above, $\mathscr{E}_0{}^{(m)}$ denote the subset of $\mathscr{E}_a^{(m)}$ consisting of all those functions of $\boldsymbol{f}'\beta_2^{(m)}$ for which the Σ_0-LBLUE within the model (10) posseses the same variance as the Σ_0-LBLUE within the model (11):

$$\mathscr{E}_0^{(m)} = \{\boldsymbol{f}'\beta_2^{(m)} \in \mathscr{E}_a^{(m)} : var_{\Sigma_0^{(m)}}[\widehat{\boldsymbol{f}'\beta_{2,\Sigma_0^{(m)}}^{(m)}}(\boldsymbol{Y}^{(m)})]$$

$$= var_{\Sigma_0^{(m)}}[\widehat{\boldsymbol{f}'\beta_{2,\Sigma_0^{(m)}}^{(m)}}(\boldsymbol{Y}^{(m)})_a]\}.$$

Theorem 2 *The class* $\mathscr{E}_0^{(m)}$ *is given by*

$$\mathscr{E}_0^{(m)} = \{\boldsymbol{f}'\beta_2^{(m)} : \boldsymbol{f} = (\boldsymbol{I} \otimes \boldsymbol{X}_2')q,\ q \in \mathscr{R}[(\boldsymbol{I} \otimes \boldsymbol{\Sigma}_0^{-1}\boldsymbol{X}_2)\boldsymbol{M}_{1_m\otimes X_2'\Sigma_0^{-1}X_1}\}$$

$$= \{\boldsymbol{f}'\beta_2^{(m)} : \boldsymbol{f} \in \mathscr{R}[(\boldsymbol{I} \otimes \boldsymbol{X}_2'\boldsymbol{\Sigma}_0^{-1}\boldsymbol{X}_2)\boldsymbol{M}_{1_m\otimes X_2'\Sigma_0^{-1}X_1}\}.$$

Proof The equality of variances

$$var_{\Sigma_0^{(m)}}[\widehat{\boldsymbol{f}'\beta_{2,\Sigma_0^{(m)}}^{(m)}}(\boldsymbol{Y}^{(m)})] = var_{\Sigma_0^{(m)}}[\widehat{\boldsymbol{f}'\beta_{2,\Sigma_0^{(m)}}^{(m)}}(\boldsymbol{Y}^{(m)})_a]\}, \quad \boldsymbol{f}'\beta_2 \in \mathscr{E}_a^{(m)},$$

is fulfiled if and only if (cf. (15), (20))

$$\boldsymbol{f}'[\boldsymbol{1}_m\boldsymbol{1}_m' \otimes (\boldsymbol{X}_2'\boldsymbol{\Sigma}_0^{-1}\boldsymbol{X}_2)^{-1}\boldsymbol{X}_2'\boldsymbol{\Sigma}_0^{-1}\boldsymbol{X}_1[m\boldsymbol{X}_1'(\boldsymbol{M}_{X_2}\boldsymbol{\Sigma}_0\boldsymbol{M}_{X_2})^{+}\boldsymbol{X}_1]^{-1}$$

$$\times \boldsymbol{X}_1'\boldsymbol{\Sigma}_0^{-1}\boldsymbol{X}_2(\boldsymbol{X}_2'\boldsymbol{\Sigma}_0^{-1}\boldsymbol{X}_2)^{-1}]\boldsymbol{f} = 0, \quad \boldsymbol{f} \in \mathscr{R}[(\boldsymbol{I}_m \otimes \boldsymbol{X}_2')\boldsymbol{M}_{1_m\otimes X_1}]. \tag{21}$$

As it is possible to write

$$[m\boldsymbol{X}_1'(\boldsymbol{M}_{X_2}\boldsymbol{\Sigma}_0\boldsymbol{M}_{X_2})^{+}\boldsymbol{X}_1]^{-1} = \boldsymbol{A}^{-\frac{1}{2}}\boldsymbol{A}^{-\frac{1}{2}},$$

$\boldsymbol{A}^{-\frac{1}{2}}$ p.d., thus (21) holds if and only if

$$\boldsymbol{q}'(\boldsymbol{I}_m \otimes \boldsymbol{X}_2)[\boldsymbol{1}_m \otimes (\boldsymbol{X}_2'\boldsymbol{\Sigma}_0^{-1}\boldsymbol{X}_2)^{-1}\boldsymbol{X}_2'\boldsymbol{\Sigma}_0^{-1}\boldsymbol{X}_1] = \boldsymbol{0},$$

where $\boldsymbol{q} \in \mathscr{R}(\boldsymbol{M}_{1_m \otimes X_1})$, and it is equivalent to

$$\begin{aligned}\boldsymbol{q}\perp\mathscr{R}\,&\big[(\boldsymbol{I}_m \otimes \boldsymbol{X}_2)[(\boldsymbol{I}_m \otimes \boldsymbol{X}_2')(\boldsymbol{I}_m \otimes \boldsymbol{\Sigma}_0^{-1})(\boldsymbol{I}_m \otimes \boldsymbol{X}_2)]^{-1}\\ &\times (\boldsymbol{I}_m \otimes \boldsymbol{X}_2')(\boldsymbol{I}_m \otimes \boldsymbol{\Sigma}_0^{-1})(\boldsymbol{1}_m \otimes \boldsymbol{X}_1)\big]\\ &= \mathscr{R}[\boldsymbol{P}_{I_m\otimes X_2}^{I_m \otimes \Sigma_0^{-1}}(\boldsymbol{1}_m \otimes \boldsymbol{X}_1)] \quad \& \quad \boldsymbol{q} \in \mathscr{R}(\boldsymbol{M}_{1_m \otimes X_1}),\end{aligned}$$

it means that

$$\boldsymbol{q}\perp\mathscr{R}(\boldsymbol{I}_m \otimes \boldsymbol{X}_2)\cap\{\mathscr{R}[(\boldsymbol{I}_m \otimes \boldsymbol{\Sigma}_0)(\boldsymbol{I}_m \otimes \boldsymbol{X}_2)^{\perp}]+\mathscr{R}(\boldsymbol{1}_m \otimes \boldsymbol{X}_1)\}\ \&\ \boldsymbol{q} \in \mathscr{R}(\boldsymbol{M}_{1_m \otimes X_1}),$$

where following assertion (cf. [4, Lemma 2.1])

$$\boldsymbol{S}\text{ p.s.d. matrix, } \mathscr{R}(\boldsymbol{A},\boldsymbol{B}) \subset \mathscr{R}(\boldsymbol{S}) \Rightarrow \mathscr{R}(\boldsymbol{P}_A^{S^-}B) = \mathscr{R}(\boldsymbol{A})\cap[\mathscr{R}(\boldsymbol{S}\boldsymbol{A}^{\perp})+\mathscr{R}(\boldsymbol{B})],$$

has been taken into account. Thus

$$\begin{aligned}\boldsymbol{q} \in \mathscr{R}^{\perp}(\boldsymbol{I}_m \otimes \boldsymbol{X}_2) + \{\mathscr{R}^{\perp}[(\boldsymbol{I}_m \otimes \boldsymbol{\Sigma}_0)(\boldsymbol{I}_m \otimes \boldsymbol{X}_2)^{\perp}] \cap \mathscr{R}^{\perp}(\boldsymbol{1}_m \otimes \boldsymbol{X}_1)\}\\ \&\ \boldsymbol{q} \in \mathscr{R}(\boldsymbol{M}_{1_m \otimes X_1}).\end{aligned} \tag{22}$$

Using [4, (2.16)]:

$$\boldsymbol{S}\text{ p.s.d., } \mathscr{R}(\boldsymbol{A}) \subset \mathscr{R}(\boldsymbol{S}) \Rightarrow \mathscr{R}^{\perp}(\boldsymbol{S}\boldsymbol{A}^{\perp}) = \mathscr{R}(\boldsymbol{S}^{-}\boldsymbol{A}, \boldsymbol{I} - \boldsymbol{S}^{-}\boldsymbol{S}),$$

(22) can be rewritten in the form

$$\begin{aligned}\boldsymbol{q} \in \mathscr{R}^{\perp}(\boldsymbol{I}_m \otimes \boldsymbol{X}_2) + \{\mathscr{R}[(\boldsymbol{I}_m \otimes \boldsymbol{\Sigma}_0^{-1})(\boldsymbol{I}_m \otimes \boldsymbol{X}_2)] \cap \mathscr{R}^{\perp}(\boldsymbol{1}_m \otimes \boldsymbol{X}_1)\}\\ \&\ \boldsymbol{q} \in \mathscr{R}^{\perp}(\boldsymbol{1}_m \otimes \boldsymbol{X}_1).\end{aligned}$$

For all $\boldsymbol{q} \in \mathscr{R}^{\perp}(\boldsymbol{I}_m \otimes \boldsymbol{X}_2)$ we have $\boldsymbol{f} = (\boldsymbol{I}_m \otimes \boldsymbol{X}_2')\boldsymbol{M}_{I_m \otimes X_2}v = \boldsymbol{0}$, i.e.

$$\boldsymbol{q} \in \mathscr{R}[(\boldsymbol{I}_m \otimes \boldsymbol{\Sigma}_0^{-1})(\boldsymbol{I}_m \otimes \boldsymbol{X}_2)] \cap \mathscr{R}^{\perp}(\boldsymbol{1}_m \otimes \boldsymbol{X}_1),$$

thus

$$\boldsymbol{q} = (\boldsymbol{I}_m \otimes \Sigma_0^{-1}\boldsymbol{X}_2)t \quad \& \quad \boldsymbol{q} = (\boldsymbol{1}_m \otimes \boldsymbol{X}_1)^{\perp}v.$$

As

$$\begin{aligned}\boldsymbol{0} = (\boldsymbol{1}_m' \otimes \boldsymbol{X}_1')q = (\boldsymbol{1}_m' \otimes \boldsymbol{X}_1')(\boldsymbol{I}_m \otimes \boldsymbol{\Sigma}_0^{-1}\boldsymbol{X}_2)\boldsymbol{t}\\ \Rightarrow \boldsymbol{t}\perp\mathscr{R}[(\boldsymbol{I}_m \otimes \boldsymbol{X}_2')(\boldsymbol{I}_m \otimes \boldsymbol{\Sigma}_0^{-1})(\boldsymbol{1}_m \otimes \boldsymbol{X}_1)],\end{aligned}$$

we have

$$\boldsymbol{f} = (\boldsymbol{I}_m \otimes \boldsymbol{X}_2')(\boldsymbol{I}_m \otimes \boldsymbol{\Sigma}_0^{-1})(\boldsymbol{I}_m \otimes \boldsymbol{X}_2)\boldsymbol{M}_{\boldsymbol{1}_m \otimes X_2'\Sigma_0^{-1}X_1}\boldsymbol{v}, \quad \boldsymbol{v} \in R^{ml},$$

which suffices for completing the proof. □

4 The case of the model for a deformation measurement

In the multiepoch models for deformation measurements it is important to estimate the difference of the useful parameters between adjacent epochs. There is a question whether we can omit the nuisance parameters by computing this difference. We shall see from the following that this question can be answered in affirmative.

Let us consider (for the sake of simplicity) the model (10) for $m = 3$, i.e. the model

$$\boldsymbol{Y}^{(3)} = \begin{pmatrix} \boldsymbol{X}_1, & \boldsymbol{X}_2, & \mathbf{0}, & \mathbf{0} \\ \boldsymbol{X}_1, & \mathbf{0}, & \boldsymbol{X}_2, & \mathbf{0} \\ \boldsymbol{X}_1, & \mathbf{0}, & \mathbf{0}, & \boldsymbol{X}_2 \end{pmatrix} \begin{pmatrix} \boldsymbol{\beta}_1 \\ \boldsymbol{\beta}_{21} \\ \boldsymbol{\beta}_{22} \\ \boldsymbol{\beta}_{23} \end{pmatrix} + \boldsymbol{\varepsilon}^{(3)} = (\mathbf{X}_1^{(3)}, \mathbf{X}_2^{(3)}) \begin{pmatrix} \boldsymbol{\beta}_1 \\ \boldsymbol{\beta}_2^{(3)} \end{pmatrix} + \boldsymbol{\varepsilon}^{(3)}. \tag{23}$$

Let $\mathbf{P} = (\mathbf{0}, \boldsymbol{I}, -\boldsymbol{I}, \mathbf{0})$ be a $l \times (k+3l)$-dimensional matrix. Thus the difference of the useful parameters

$$\boldsymbol{\beta}_{21} - \boldsymbol{\beta}_{22} = \mathbf{P}\boldsymbol{\beta}, \quad \boldsymbol{\beta} = \begin{pmatrix} \boldsymbol{\beta}_1 \\ \boldsymbol{\beta}_{21} \\ \boldsymbol{\beta}_{22} \\ \boldsymbol{\beta}_{23} \end{pmatrix}.$$

As

$$\mathscr{R}[(\mathbf{0}, \boldsymbol{I}, -\boldsymbol{I}, \mathbf{0})'] \subset \mathscr{R}[(\mathbf{X}_1^{(3)}, \mathbf{X}_2^{(3)})'],$$

the Theorem 5.3.1. in [2] yields that the difference of the Σ_0-LBLUEs

$$\mathbf{P}\hat{\boldsymbol{\beta}} = \hat{\boldsymbol{\beta}_{21}} - \hat{\boldsymbol{\beta}_{22}}$$

is the Σ_0-LBLUE of the difference of the parameters $\mathbf{P}\boldsymbol{\beta} = \boldsymbol{\beta}_{21} - \boldsymbol{\beta}_{22}$.

Using assertion (16) for $i = 1, 2$, we get one very interesting result.

Corollary 1 *Within the three-epoch model (23) the estimator of the difference of the useful parameters between adjacent epochs does not depend on the matrix* $\boldsymbol{X}_1$*:*

$$\begin{aligned} (\widehat{\boldsymbol{\beta}_{21} - \boldsymbol{\beta}_{22}})_a &= \widehat{\boldsymbol{\beta}^{(3)}_{21,\Sigma_0^{(3)}}}(\boldsymbol{Y}^{(3)})_a - \widehat{\boldsymbol{\beta}^{(3)}_{22,\Sigma_0^{(3)}}}(\boldsymbol{Y}^{(3)})_a \\ &= (\boldsymbol{X}_2'\boldsymbol{\Sigma}_0^{-1}\boldsymbol{X}_2)^{-1}\boldsymbol{X}_2'\boldsymbol{\Sigma}_0^{-1}(\boldsymbol{Y}_1 - \boldsymbol{Y}_2). \end{aligned}$$

It means that the estimator of the difference of the useful parameters between adjacent epochs $\boldsymbol{\beta}_{21}$–$\boldsymbol{\beta}_{22}$ is the same within the "small" and within the "large" model, the difference is estimable irrespective of the nuisance parameters.

The obtained result suggests the idea to investigate the following transformation of the three-epoch model (23):

$$\boldsymbol{Z} = \boldsymbol{T}\boldsymbol{Y}^{(3)}, \quad \text{where } \boldsymbol{T} = \begin{pmatrix} \boldsymbol{I}, & -\boldsymbol{I}, & \mathbf{0} \\ \mathbf{0}, & \boldsymbol{I}, & -\boldsymbol{I} \end{pmatrix},$$

i.e.

$$\boldsymbol{Z} = \begin{pmatrix} \boldsymbol{Y}_1 - \boldsymbol{Y}_2 \\ \boldsymbol{Y}_2 - \boldsymbol{Y}_3 \end{pmatrix} \sim \left\{ \begin{pmatrix} \boldsymbol{X}_2, & \mathbf{0} \\ \mathbf{0}, & \boldsymbol{X}_2 \end{pmatrix} \begin{pmatrix} \boldsymbol{\beta}_{21} - \boldsymbol{\beta}_{22} \\ \boldsymbol{\beta}_{22} - \boldsymbol{\beta}_{23} \end{pmatrix}; \begin{pmatrix} 2\boldsymbol{\Sigma}_0, & -\boldsymbol{\Sigma}_0 \\ -\boldsymbol{\Sigma}_0, & 2\boldsymbol{\Sigma}_0 \end{pmatrix} \right\}. \tag{24}$$

We can easily see that the estimator of the parameter $\boldsymbol{\beta}_{21} - \boldsymbol{\beta}_{22}$ in the model (24) and the estimator of the parametric function $\mathbf{P}\boldsymbol{\beta}$ in the model (23), where $\mathbf{P} = (\mathbf{0}, \boldsymbol{I}, -\boldsymbol{I}, \mathbf{0})'$ are the same. Both models are identical in this sense.

The same idea can be obviously used for the m-epoch model as well.

The procedure set out in Corollary uses the matrix $\boldsymbol{\Sigma}_0$ that is often diagonal. In the model (24) it is necessary to use the covariance matrix that does not have this property. Nevertheless the two procedures mentioned [the procedure using Corollary and the second one using (24)] provide us two independent possibilities how to determine the deformation shifts. This is a good check of the correctness of numerical evaluations.

The question which of these two procedures is numerical less complicated has not been answered yet.

In this paper only the problem of determination of the estimators of the differences of coordinates of the unstable points between adjacent epochs has been solved. The aim is not only to learn about differences but to investigate the process of the deformation in whole with the task to establish the picture of the deformations in the course of time (not only to establish the trajectory of one unstable point). It means to design the joint confidence ellipsoids (ellipses) for the position of the point in particular successive epochs or to design the relative confidence ellipses between points within the framework of the same epoch etc. That is now possible within the framework of the "small" model.

The paper solving the above mentioned questions is being prepared.

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