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The Nonlinear Periodic Second Order Boundary Value Problem

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Abstract

This paper deals with a nonlinear periodic second order boundary value problem, which is at resonance. We will search assumptions for the right side of the equation, which will lead to the existence of a solution of the boundary value problem. We will use properties of the Fourier series, the method of a priori estimates and the topological degree arguments.

Key words: Periodic boundary value problem, topological degree, Fourier series, a priori estimates.

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1 Introduction

Let us consider a boudary value problem in the form

$$x'' + \omega x = f(t, x, x'), \tag{1}$$

$$x(0) = x(2\pi), \quad x'(0) = x'(2\pi), \tag{2}$$

where $\omega \in \mathbf{R}$, f is continuous function on the set $J \times \mathbf{R}^2$, $J = [0, 2\pi]$.

By a solution of (1), (2) we will mean every function u with the continuous second derivative, which fulfils (1) for all $t \in J$ and which satisfies the condition (2). First we will consider the homogeneous linear differential equation

$$x'' + \omega x = 0, \quad \text{where } \omega \in \mathbf{R}, \tag{3}$$

with the periodic condition (2). Let us recall, we say that the problem (1), (2) is at resonance, if the problem (3), (2) has a nontrivial solution.

We can distinguish three cases:

$\omega < 0$: In this case the homogeneous periodic problem (3), (2) has only the trivial solution and then the problem (1), (2) is not at resonance.

$\omega = 0$: In this case each constant $x \equiv c \in \mathbf{R}$ is a solution of the problem (3), (2). Therefore the problem (1), (2) is at resonance. This kind of a boudary value problem is described in [1], [2].

$\omega > 0$: We can write every solution of the differential equation (3) in the form

$$x(t) = A\cos\sqrt{\omega}t + B\sin\sqrt{\omega}t.$$

The problem (3), (2) has eigenvalues $\omega = 1, 4, 9, \ldots, k^2, \ldots$ where $k \in \mathbf{N}$, and there is a corresponding linear space of eigenfunctions $x(t) = A\cos kt + B\sin kt$ with the base $\{\sin kt, \cos kt\}$ for each eigenvalue k^2. In this case the problem (1), (2) is at resonance, as well.

In this paper we will study the resonance problem

$$x'' + m^2 x = f(t, x, x'), \tag{4}$$

$$x(0) = x(2\pi), \quad x'(0) = x'(2\pi), \tag{2}$$

where $m \in \mathbf{N}$.

We will present new results on the Leray–Schauder topological degree of an operator associated to (4), (2). These results generalize and extend those of [1] and imply the existence of a solution of (4), (2) in the case that f can cross eigenvalues higher than m^2.

2 Definitions and Lemmas

In this part we recall some notions and relations, which will be used later.

We will work with the following Banach spaces:

$C(J)$ is the space of continuous functions on J, where we define the norm $||x||_C = \max\{|x(t)| : t \in J\}$ for each $x \in C(J)$.

$C^k(J)$ is the space of functions, which have continuous k-th derivatives on J for $k \in \mathbf{N}$. In $C^k(J)$ we define the norm

$$||x||_{C^k} = \sum_{i=0}^{k} ||x^{(i)}||_C$$

for each function $x \in C^k(J)$.

$L_p(J)$ is the space of functions x with x^p Lebesgue integrable on J for $p \in [1, \infty)$. For each function $x \in L_p(J)$ we define in $L_p(J)$ the norm

$$||x||_p = \Big(\frac{1}{2\pi}\int_0^{2\pi} |x(t)|^p dt\Big)^{1/p}.$$

$W^{1,2}(J)$ is the space of functions x, derivatives x' of which are defined almost everywhere on J and $(x')^2$ are Lebesgue integrable on J. For $x \in W^{1,2}(J)$ we define the norm $\|x\|_{1,2} = (\|x\|_2 + \|x'\|_2)$ in $W^{1,2}(J)$.

For $p = 2$ we define the scalar product

$$(x,y)_2 = \frac{1}{2\pi}\int_0^{2\pi} x(t)y(t)\,dt$$

in the space $L_2(J)$ and then we get the Hilbert space.

Moreover we will work with the Hilbert space

$$H = \{x \in W^{1,2}; x(0) = x(2\pi)\},$$

where we define the norm $\|x\|_H = \left(\|x\|_2^2 + \|x'\|_2^2\right)^{\frac{1}{2}}$ for each $x \in H$ and the scalar product

$$(x,y)_H = \frac{1}{2\pi}\int_0^{2\pi} \left[x(t)y(t) + x'(t)y'(t)\right]dt$$

for each $x, y \in H$.

Definition 2.1 We say that a normed linear space X with the norm $\|\cdot\|_X$ is continuously imbedded to a normed linear space Y with the norm $\|\cdot\|_Y$, if

(a) $X \subset Y$,

(b) there exists a constant $k > 0$ such that each $u \in X$ fulfils $\|u\|_Y \le k\|u\|_X$.

Definition 2.2 We say that a Banach space X is compactly imbedded into a Banach space Y, if

(a) $X \subset Y$,

(b) every sequence $\{u_n\}_{n=1}^{\infty}$ of elements from X, which converges weakly in X to the element u_0, converges strongly in Y to u_0.

Lemma 2.1 *If a Banach space X is compactly imbedded into a Banach space Y, then X is imbedded into Y continously ([4], p. 205).*

Lemma 2.2 *The space $W^{1,2}(J)$ is compactly imbedded into the space $C(J)$ ([4], p. 206).*

Lemma 2.3 *The space H is compactly imbedded into the space $C(J)$ and there exists $k > 0$ such that each $u \in H$ fulfils*

(5) $$\|u\|_C < k\|u\|_H.$$

Proof We can also use the norm $\|\cdot\|_{1,2}$ in H, which is equivalent with the norm $\|\cdot\|_H$. (see [5], p. 348). Therefore the space H is also $W^{1,2}$ and then, with respect to Lemma 2.2, it is compactly imbedded into $C(J)$. Then from Lemma 2.1 and Definition 2.1, there is $k > 0$ such that (5) is fulfiled for each $u \in H$. □

Lemma 2.4 (Mean value theorem) *Let $m, M \in \mathbf{R}$ and $f, g \in C(J)$ satisfy $f(x) \geq 0$ and $m \leq g(x) \leq M$ for all $x \in J$. Then there exists a mean value $\widehat{g} \in [m, M]$ such, that*

$$\int_0^{2\pi} f(x)g(x)dx = \widehat{g} \int_0^{2\pi} f(x)dx.$$

Lemma 2.5 (Sobolev inequality) *Let $u \in H$ be 2π-periodic and let*

$$\int_0^{2\pi} u(t)dt = 0.$$

Then

$$\|u\|_C^2 \leq \frac{\pi}{6}\|u'\|_2^2. \tag{6}$$

([7], p. 25.)

Lemma 2.6 *Let X be a Banach space, $r > 0$ and $B(r) = \{u \in X; \|x\|_X < r\}$. Let I be the identical operator on $\overline{B(r)}$. Let F be a completely continuous operator, which is defined on $\overline{B(r)}$ with values in X and such that $Fu \neq u$ for each $u \in \partial B(r)$. Then there exists an integer number*

$$d[I - F; B(r)]$$

(which we call the Leray–Schauder topologic degree) such that:

a) $d[I; B(r)] = 1$

b) If $d[I - F; B(r)] \neq 0$, then there exists $u_0 \in B(r)$ such that $Fu_0 = u_0$.

c) If G is also a completely continuous map on $\overline{B(r)}$ with the values in X and if $(I - F - tG)u \neq 0$ is fulfiled for each $u \in \partial B(r)$ and for each $t \in [0, 1]$, then

$$d[I - F; B(r)] = d[I - G - F; B(r)]. \tag{7}$$

d) If the operator F is odd, then the degree $d[I - F; B(r)]$ is an odd number.

([4], p. 245, Th. 30.14.)

3 Fourier Series

In the whole section suppose that $m \in \mathbf{N}$ and that $x \in H$ is an arbitrary 2π-periodic function. Let us consider Fourier series

$$x(t) = a_0 + \sum_0^\infty a_k \cos kt + b_k \sin kt, \tag{8}$$

with Fourier coefficients with respect to the scalar product in $L_2(J)$. Let as put

(9)
$$\begin{cases} \overline{x}(t) = a_0 + \sum\limits_{k=0}^{m-1} a_k \cos kt + b_k \sin kt \\ x^0(t) = a_m \cos mt + b_m \sin mt \\ \widetilde{x}(t) = \sum\limits_{k=m+1}^{\infty} a_k \cos kt + b_k \sin kt \end{cases}$$

and

(10)
$$x^{\perp}(t) = x(t) - x^0(t).$$

Lemma 3.1 *There is a real number $\delta_1 > 0$ such that*

(11)
$$\delta_1 ||\overline{x}||_H^2 \leq m^2 ||\overline{x}||_2^2 - ||\overline{x}'||_2^2.$$

Proof Every $x \in H$ fulfils

$$||\overline{x}||_2^2 = \frac{1}{2\pi} \int_0^{2\pi} \Big(a_0 + \sum_{k=1}^{m-1} a_k \cos kt + b_k \sin kt\Big)^2 dt =$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \Big(a_0^2 + \sum_{k=1}^{m-1} a_k^2 \cos^2 kt + b_k^2 \sin^2 kt\Big) dt = a_0^2 + \sum_{k=1}^{m-1} \frac{a_k^2 + b_k^2}{2}$$

and

$$||\overline{x}'||_2^2 = \frac{1}{2\pi} \int_0^{2\pi} \Big(\sum_{k=1}^{m-1} -k a_k \sin kt + k b_k \cos kt\Big)^2 dt =$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \Big(\sum_{k=1}^{m-1} k^2 a_k^2 \sin^2 kt + k^2 b_k^2 \cos^2 kt\Big) dt = \sum_{k=1}^{m-1} k^2 \frac{a_k^2 + b_k^2}{2}.$$

Hence

$$||\overline{x}'||_2^2 \leq (m-1)^2 ||\overline{x}||_2^2.$$

Then for $\delta_1 = \frac{2m-1}{1+(m-1)^2}$ we get

$$(1 + \delta_1) ||\overline{x}'||_2^2 \leq (m^2 - \delta_1) ||\overline{x}||_2^2,$$

$$\delta_1 ||\overline{x}'||_H^2 = \delta_1 ||\overline{x}'||_2^2 + \delta_1 ||\overline{x}||_2^2 \leq m^2 ||\overline{x}||_2^2 - ||\overline{x}'||_2^2.$$ □

Lemma 3.2 *Let $\gamma \in [0, 2m+1)$. Then there exists $\delta_2 > 0$ such that*

(12)
$$||\widetilde{x}'||_2^2 - (m^2 + \gamma) ||\widetilde{x}||_2^2 \geq \delta_2 ||\widetilde{x}||_H^2.$$

Proof Like in the proof of Lemma 3.1 we can prove

$$||\widetilde{x}||_2^2 = \sum_{k=m+1}^{\infty} \frac{a_k^2 + b_k^2}{2} \quad \text{and} \quad ||\widetilde{x}'||_2^2 = \sum_{k=m+1}^{\infty} k^2 \frac{a_k^2 + b_k^2}{2}.$$

Then

$$\|\widetilde{x}'\|_2^2 \geq (m+1)^2\|\widetilde{x}\|_2^2. \tag{13}$$

Hence for $\delta_2 = \frac{2m+1-\gamma}{1+(m+1)^2}$ we get

$$(1-\delta_2)\|\widetilde{x}'\|_2^2 \geq (m^2+\gamma+\delta_2)\|\widetilde{x}\|_2^2,$$

$$\|\widetilde{x}'\|_2^2 - (m^2+\gamma)\|\widetilde{x}\|_2^2 \geq \delta_2\|\widetilde{x}'\|_2^2 + \delta_2\|\widetilde{x}\|_2^2 = \delta_2\|\widetilde{x}\|_H^2.$$ □

Lemma 3.3 *The following statements are valid:*

a) $m^2\|x^0\|_2^2 = \|x^{0\prime}\|_2^2$;

b) $\overline{x}\perp x^0$; $\overline{x}\perp\widetilde{x}$; $x^0\perp\widetilde{x}$; $\overline{x}'\perp\widetilde{x}'$;

c) $(\overline{x}'',\overline{x})_2 = -\|\overline{x}'\|_2^2$; $(x^{0\prime\prime},x^0)_2 = -\|x^{0\prime}\|_2^2$; $(\widetilde{x}'',\widetilde{x})_2 = -\|\widetilde{x}'\|_2^2$.

Proof

a)

$$\|x^0\|_2^2 = \frac{1}{2\pi}\int_0^{2\pi}(a_m\cos mt + b_m\sin mt)^2dt$$
$$= \frac{1}{2\pi}\int_0^{2\pi}(a_m^2\cos^2 mt + b_m^2\sin^2 mt)dt = \frac{a_m^2+b_m^2}{2},$$

$$\|x^{0\prime}\|_2^2 = \frac{1}{2\pi}\int_0^{2\pi}m^2(-a_m\sin mt + b_m\cos mt)^2dt = m^2\frac{a_m^2+b_m^2}{2} = m^2\|x^0\|_2^2.$$

b)

$$(\overline{x},x^0)_2 = \frac{1}{2\pi}\int_0^{2\pi}\left[a_0+\sum_{k=1}^{m-1}(a_k\cos kt + b_k\sin kt)\right](a_m\cos mt + b_m\sin mt)\,dt$$

$$= \frac{1}{2\pi}\int_0^{2\pi}a_0(a_m\cos mt + b_m\sin mt)\,dt + \frac{1}{2\pi}\sum_{k=1}^{m-1}\left[\int_0^{2\pi}a_ka_m\cos kt\cos mt\,dt\right.$$
$$+\int_0^{2\pi}b_ka_m\sin kt\cos mt\,dt + \int_0^{2\pi}a_kb_m\cos kt\sin mt\,dt$$
$$\left.+\int_0^{2\pi}b_kb_m\sin kt\sin mt)\,dt\right] = 0$$

i.e. $\overline{x}\perp x^0$. We can prove $\widetilde{x}\perp x^0$, $\overline{x}\perp\widetilde{x}$ and $\overline{x}'\perp\widetilde{x}'$ analogously.

c)

$$(\overline{x}'',\overline{x})_2 =$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\left[-\sum_{k=1}^{m-1}(-k^2a_k\cos kt + -k^2b_k\sin kt)\right]\left[a_0+\sum_{k=1}^{m-1}(a_k\cos kt + b_k\sin kt)\right]dt$$
$$= -\frac{1}{2\pi}\sum_{k=1}^{m-1}k^2\int_0^{2\pi}(a_k\cos kt + b_k\sin kt)^2dt = -\sum_{k=1}^{m-1}k^2\frac{a_k^2+b_k^2}{2} = -\|\overline{x}'\|_2^2.$$

We can prove the statements $(\widetilde{x}'', \widetilde{x})_2 = -||\widetilde{x}'||_2^2$ and $(x^{0''}, x^0)_2 = -||x^{0'}||_2^2$ in the same way. □

Lemma 3.4 *Let $\gamma \in [0, 2m+1)$. Then there exists $\delta > 0$ depending on γ such that the inequality*

$$-||\overline{x}'||_2^2 + ||\widetilde{x}'||_2^2 + m^2||\overline{x}||_2^2 - (m^2+\gamma)||\widetilde{x}||_2^2 \geq \delta ||x^\perp||_H^2 \tag{14}$$

is valid.

Proof From (8), (9), (10) and Lemma 3.3 it follows

$$\begin{aligned} ||x^\perp||_2^2 &= ||x - x^0||_2^2 = ||\overline{x} + \widetilde{x}||_2^2 = (\overline{x} + \widetilde{x}, \overline{x} + \widetilde{x})_2 \\ &= ||\overline{x}||_2^2 + 2(\overline{x}, \widetilde{x})_2 + ||\widetilde{x}||_2^2 = ||\overline{x}||_2^2 + ||\widetilde{x}||_2^2, \end{aligned}$$

$$\begin{aligned} ||x^{\perp\prime}||_2^2 &= ||x' - x^{0\prime}||_2^2 = ||\overline{x}' + \widetilde{x}'||_2^2 = (\overline{x}' + \widetilde{x}', \overline{x}' + \widetilde{x}')_2 \\ &= ||\overline{x}'||_2^2 + 2(\overline{x}', \widetilde{x}')_2 + ||\widetilde{x}'||_2^2 = ||\overline{x}'||_2^2 + ||\widetilde{x}'||_2^2. \end{aligned}$$

Hence

$$||x^\perp||_H^2 = ||x - x^0||_2^2 + ||x' - x^{0\prime}||_2^2 = ||\overline{x}||_2^2 + ||\widetilde{x}||_2^2 + ||\overline{x}'||_2^2 + ||\widetilde{x}'||_2^2 = ||\overline{x}||_H^2 + ||\widetilde{x}||_H^2.$$

Using (11) and (12) we get

$$\begin{aligned} -||\overline{x}'||_2^2 + ||\widetilde{x}'||_2^2 + m^2||\overline{x}||_2^2 - (m^2+\gamma)||\widetilde{x}||_2^2 &\geq \delta_1 ||\overline{x}||_H^2 + \delta_2 ||\widetilde{x}||_H^2 \\ &\geq \delta ||\overline{x}||_H^2 + \delta ||\widetilde{x}||_H^2 = \delta ||x^\perp||_H^2, \end{aligned}$$

where

$$\delta = \min\{\delta_1; \delta_2\} = \min\left\{\frac{2m-1}{1+(m-1)^2}, \frac{2m+1-\gamma}{1+(m+1)^2}\right\}. \tag{15}$$ □

Lemma 3.5 *Let $\gamma \in [0, 2m-1)$. Then there exists $\delta > 0$ depending on γ such that*

$$\frac{1}{2\pi}\int_0^{2\pi} [x'' + (m^2+\gamma)x][\overline{x} + x^0 - \widetilde{x}]dt \geq \delta ||x^\perp||_H^2. \tag{16}$$

Proof We will rearrange the left side of (16) using (14) and Lemma 3.3.

$$\begin{aligned} &\frac{1}{2\pi}\int_0^{2\pi} [x'' + (m^2+\gamma)x][\overline{x} + x^0 - \widetilde{x}]dt = \\ &= \frac{1}{2\pi}\int_0^{2\pi} [x''(\overline{x} + x^0 - \widetilde{x}) + (m^2+\gamma)x(\overline{x} + x^0 - \widetilde{x})]dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} [\overline{x}''\overline{x} - \widetilde{x}''\widetilde{x} + m^2\overline{x}^2 - m^2\widetilde{x}^2 + \gamma x(\overline{x} + x^0 - \widetilde{x})]dt \\ &\geq -||\overline{x}'||_2^2 + ||\widetilde{x}'||_2^2 + m^2||\overline{x}||_2^2 - (m^2+\gamma)||\widetilde{x}||_2^2 \geq \delta ||x^\perp||_H^2, \end{aligned}$$

where δ is the constant given by (15). □

Lemma 3.6 *Let $\epsilon \in (0, 2m-1)$, $\epsilon_0 \in (0, \epsilon)$, $\gamma_1, \gamma_2 \in C(J)$ be such that*

$$(17) \qquad \begin{cases} 0 \le \gamma_1(t) < 2m+1-\epsilon, 0 \le \gamma_2(t) \quad \textit{for all } t \in J, \\ \int_0^{2\pi} \gamma_2(t)dt \le \frac{12(\epsilon-\epsilon_0)}{1+(m+1)^2}. \end{cases}$$

Then there exists $\delta^* \in (0, \epsilon_0)$ satisfying

$$(18) \qquad \frac{1}{2\pi}\int_0^{2\pi} [x'' + (m^2+\gamma_1(t)+\gamma_2(t))x][\overline{x} + x^0 - \widetilde{x}]\,dt \ge \delta^* \|x^\perp\|_H^2.$$

Proof Following the proof of Lemma 3.5 we have

$$\frac{1}{2\pi}\int_0^{2\pi} [x'' + (m^2+\gamma_1(t)+\gamma_2(t))x][\overline{x} + x^0 - \widetilde{x}]dt \ge$$

$$\ge -\|\overline{x}'\|_2^2 + m^2\|\overline{x}\|_2^2 + \|\widetilde{x}'\|_2^2 - m^2\|\widetilde{x}\|_2^2 - \frac{1}{2\pi}\int_0^{2\pi} \widetilde{x}^2(\gamma_1(t)+\gamma_2(t))dt.$$

By Lemma 2.4 there is a mean value $\widehat{\gamma_1} \in [0, 2m+1-\epsilon]$ fulfilling

$$(19) \qquad \frac{1}{2\pi}\int_0^{2\pi} \widetilde{x}^2\gamma_1(t)dt = \frac{\widehat{\gamma_1}}{2\pi}\int_0^{2\pi} \widetilde{x}^2 dt \le (2m+1-\epsilon)\|\widetilde{x}\|_2^2.$$

Further, using (17) and the Sobolev inequality (6), we get

$$(20) \qquad \frac{1}{2\pi}\int_0^{2\pi} \widetilde{x}^2\gamma_2(t)dt \le \|\widetilde{x}\|_C^2 \frac{1}{2\pi}\frac{12(\epsilon-\epsilon_0)}{1+(m+1)^2} \le \frac{\epsilon-\epsilon_0}{1+(m+1)^2}\|\widetilde{x}'\|_2^2.$$

Then, by (19), (20) and (14) with $\gamma = 2m+1-\epsilon$, we obtain

$$\frac{1}{2\pi}\int_0^{2\pi} [x'' + (m^2+\gamma_1(t)+\gamma_2(t))x][\overline{x} + x^0 - \widetilde{x}]dt \ge$$

$$\ge -\|\overline{x}'\|_2^2 + m^2\|\overline{x}\|_2^2 + \|\widetilde{x}'\|_2^2 - (m^2+2m+1-\epsilon)\|\widetilde{x}\|_2^2 - \frac{\epsilon-\epsilon_0}{1+(m+1)^2}\|\widetilde{x}\|_2^2$$

$$\ge \delta\|x^\perp\|_H^2 - \frac{\epsilon-\epsilon_0}{1+(m+1)^2}\|\widetilde{x}\|_2^2 \ge \left[\delta - \frac{\epsilon-\epsilon_0}{1+(m+1)^2}\right]\|x^\perp\|_H^2$$

$$= \frac{\epsilon_0}{1+(m+1)^2}\|x^\perp\|_H^2.$$

Here we have used the fact that

$$\delta = \min\left\{\frac{2m-1}{1+(m+1)^2}, \frac{\epsilon}{1+(1+m)^2}\right\} = \frac{\epsilon}{1+(1+m)^2}.$$

So, for $\delta^* = \frac{\epsilon_0}{1+(m+1)^2} \in (0, \epsilon_0)$ the inequality (18) is true. □

4 A priori Estimates

Now, let us consider the equation (4) and suppose that

$$f(t,x,y) = g_1(t,x,y) + g_2(t,x),$$

where g_1, g_2 are continuous. Then (4) has the form

$$x'' + m^2 x = g_1(t,x,x') + g_2(t,x). \tag{21}$$

Lemma 4.1 *Suppose that $\gamma_1, \gamma_2 \in C(J)$ satisfy (17) and that for $M_1, M_2, B \in (0,\infty)$ the conditions*

$$|g_1(t,x,y)| \le M_1 \quad na\ J \times R^2, \tag{22}$$

$$|g_2(t,x) + (\gamma_1(t) + \gamma_2(t))x| \le M_2 \quad \text{for all } t \in J \text{ and } |x| \ge B \tag{23}$$

are valid. Then there exists $a \in (0,\infty)$ such that each solution of the problem (21),(2) satisfies the inequality

$$\|x^\perp\|_H \le a + \sqrt{a^2 + 2a\|x^0\|_H}. \tag{24}$$

Proof Put

$$\begin{cases} \gamma^* = \max\{|\gamma_1(t)| + |\gamma_2(t)| : t \in J\}, \\ g^* = \max\{|g_2(t,x)| : t \in J, |x| \le B\}, \\ M^* = \max\{M_2, g^* + \gamma^* B\}. \end{cases} \tag{25}$$

Then

$$|g_2(t,x) + (\gamma_1(t) + \gamma_2(t))x| \le M^* \quad \text{on } J \times R. \tag{26}$$

Let x be a solution of (21),(2). Then

$$x'' + (m^2 + \gamma_1(t) + \gamma_2(t))x = g_1(t,x,x') + g_2(t,x) + (\gamma_1(t) + \gamma_2(t))x. \tag{27}$$

If we multiply (27) by $\frac{1}{2\pi}(\overline{x} + x^0 - \widetilde{x})$ and integrate, we get by (18)

$$\delta^* \|x^\perp\|_H^2 \le \frac{M_1 + M^*}{2\pi} \int_0^{2\pi} |\overline{x} + x^0 - \widetilde{x}| dt.$$

Let us put $M = M_1 + M^*$ and $a = \frac{Mk}{\delta^*}$, where k is the constant from (5). Then

$$\begin{aligned} \delta^* \|x^\perp\|_H &\le \frac{M}{2\pi} \int_0^{2\pi} |\overline{x} + x^0 - \widetilde{x}| dt \le M \sqrt{\frac{1}{2\pi} \int_0^{2\pi} |\overline{x} + x^0 - \widetilde{x}|^2 dt} \\ &= M \sqrt{\|\overline{x} + x^0 - \widetilde{x}\|_2^2} = M \sqrt{\|\overline{x} + x^0\|_2^2 + \|\widetilde{x}\|_2^2} \\ &\le M(\|x^\perp\|_2 + \|x^0\|_2) \le M(\|x^\perp\|_H + \|x^0\|_H), \end{aligned}$$

$$||x^\perp||_H^2 \le \frac{Mk}{\delta^*}||x^\perp||_H + \frac{Mk}{\delta^*}||x^0||_H,$$
$$||x^\perp||_H^2 - 2a||x^\perp||_H - 2a||x^0||_H \le 0,$$
$$(||x^\perp||_H - a)^2 \le 2a||x^0||_H + a^2,$$
$$a - \sqrt{a^2 + 2a||x^0||_H} \le 0 \le ||x^\perp||_H \le a + \sqrt{a^2 + 2a||x^0||_H}.$$

Since the norm $||x^\perp||_H$ is nonnegative then the negative solution

$$a - \sqrt{a^2 + 2a||x^0||_H}$$

is not important for the restriction of $||x^\perp||_H$. Hence we can write

$$||x^\perp||_H \le a + \sqrt{a^2 + 2a||x^0||_H}. \qquad \square$$

Lemma 4.2 *Let the conditions (17), (22) and (23) be fulfiled. Further, let $\{x_n\}_{n=1}^\infty$ be a sequence of solutions of the problem (21),(2) and*

$$\lim_{n\to\infty} ||x_n||_H = \infty. \tag{28}$$

Then there exists (prospectively for a convergent subsequence) a function v such that

$$\frac{x_n^0}{||x_n^0||_H} \to v \quad \text{in } C(J), \tag{29}$$

$$\frac{x_n}{||x_n||_H} \to v \quad \text{in } C(J), \tag{30}$$

where v is an eigenfunction of (3),(2) with $\omega = m^2$.

Proof Firstly we prove that the sequence $\{\frac{x_n}{||x^0||_H}\}_{n=1}^\infty$ is bounded in H and hence there is a subsequence, which is convergent in $C(J)$. Since

$$||x_n||_H \le ||x_n^\perp||_H + ||x_n^0||_H \le ||x_n^0||_H + a + \sqrt{a^2 + 2a||x_n^0||_H},$$

then using (28) we obtain $\lim_{n\to\infty} ||x_n^0||_H = \infty$. The inequality

$$\frac{||x_n||_H}{||x_n^0||_H} \le \frac{||x_n^0||_H + ||x_n^\perp||_H}{||x_n^0||_H} \le 1 + \frac{a}{||x_n^0||_H} + \sqrt{\left(\frac{a}{||x_n^0||_H}\right)^2 + 2\frac{a}{||x_n^0||_H}}$$

implies $\lim_{n\to\infty} \frac{||x_n||_H}{||x_n^0||_H} = 1$, i.e. the sequence is bounded in H and hence with respect to Lemma 2.3 there is a subsequence, which is convergent in $C(J)$.

Then there exists a function $v \in C(J)$ such, that

$$\lim_{n\to\infty} \frac{x_n}{||x_n^0||_H} = v. \tag{31}$$

From

$$\lim_{n\to\infty} \frac{x_n^\perp}{\|x_n^0\|_H} \le \lim_{n\to\infty} \frac{\|x_n^\perp\|_C}{\|x_n^0\|_H} \le k \lim_{n\to\infty} \frac{\|x_n^\perp\|_H}{\|x_n^0\|_H} = 0$$

we get

$$\lim_{n\to\infty} \frac{x_n^0}{\|x_n^0\|_H} = \lim_{n\to\infty} \frac{x_n - x_n^\perp}{\|x_n^0\|_H} = v - 0 = v,$$

$$\lim_{n\to\infty} \frac{x_n}{\|x_n\|_H} = \lim_{n\to\infty} \frac{\frac{x_n}{\|x_n^0\|_H}}{\frac{\|x_n\|_H}{\|x_n^0\|_H}} = \frac{v}{1} = v.$$

Since $\{\frac{x_n^0}{\|x^0\|_H}\}$ is the sequence of functions from a linear space S with the base $\{\cos mt, \sin mt\}$, then $v \in S$. □

Lemma 4.3 *Let (17), (22) and (23) be fulfiled and moreover*

(32) $$\int_0^{2\pi} \gamma_1(t)dt > 0.$$

Then there exists $r^* \in (0, \infty)$ *such that*

(33) $$\|x\|_H < r^*$$

for each solution x *of (21),(2).*

Proof On the contrary, we suppose that (33) is not valid, i.e. there is a sequence $\{x_n\}_{n=1}^\infty$ of solutions of (21),(2), which fulfils (28). Thus x_n fulfils the equation

$$x_n'' + m^2 x_n + (\gamma_1(t) + \gamma_2(t))x_n = g_1(t, x_n, x_n') + g_2(t, x_n) + (\gamma_1(t) + \gamma_2(t))x_n$$

for each $n \in N$. If we multiply these equations by $\frac{x_n^0}{\|x_n^0\|_H}$ and integrate, we get

$$\int_0^{2\pi} \left(\frac{x_n'' x_n^0}{\|x_n^0\|_H} + m^2 \frac{x_n x_n^0}{\|x_n^0\|_H} + (\gamma_1(t) + \gamma_2(t)) \frac{x_n x_n^0}{\|x_n^0\|_H} \right) dt =$$

$$= \int_0^{2\pi} [g_1(t, x_n, x_n') + g_2(t, x_n) + (\gamma_1(t) + \gamma_2(t))x_n] \frac{x_n^0}{\|x_n^0\|_H} dt.$$

Since the sequences $\{\frac{x_n}{\|x_n^0\|_H}\}_{n=1}^\infty$ and $\{\frac{x_n^0}{\|x_n^0\|_H}\}_{n=1}^\infty$ are uniformly convergent on J, we can exchange the order of limit and integration. Then with respect to (29) and (30)

$$\lim_{n\to\infty} \int_0^{2\pi} \left(\frac{x_n'' x_n^0}{\|x_n^0\|_H} + m^2 \frac{x_n x_n^0}{\|x_n^0\|_H} + (\gamma_1(t) + \gamma_2(t)) \frac{x_n x_n^0}{\|x_n^0\|_H} \right) dt =$$

$$= \lim_{n\to\infty} \|x_n^0\|_H \lim_{n\to\infty} \int\limits_0^{2\pi} (\gamma_1(t)+\gamma_2(t)) \frac{x_n}{\|x_n^0\|_H} \frac{x_n^0}{\|x_n^0\|_H} dt$$

$$= \lim_{n\to\infty} \|x_n^0\|_H \int\limits_0^{2\pi} \lim_{n\to\infty} (\gamma_1(t)+\gamma_2(t)) \frac{x_n}{\|x_n^0\|_H} \frac{x_n^0}{\|x_n^0\|_H} dt$$

$$= \lim_{n\to\infty} \|x_n^0\|_H \int\limits_0^{2\pi} (\gamma_1(t)+\gamma_2(t)) v^2 dt.$$

Since $v \in S$ i.e. $v^2(t) > 0$ for almost all $t \in J$ and $\gamma_1(t)$ is nonnegative on J and positive on I, where $I \subset J$ is the set with the positive Lebesgue measure, then $\gamma_1(t)v^2(t)$ is nonnegative on J and positive on the set with the positive Lebesgue measure.

Using (28) we get

$$\lim_{n\to\infty} \int_0^{2\pi} \left(\frac{x_n'' x_n^0}{\|x_n^0\|_H} + m^2 \frac{x_n x_n^0}{\|x_n^0\|_H} + (\gamma_1(t)+\gamma_2(t)) \frac{x_n x_n^0}{\|x_n^0\|_H} \right) dt =$$
$$= \lim_{n\to\infty} \|x_n^0\|_H \int_0^{2\pi} (\gamma_1(t)+\gamma_2(t)) v^2 dt = \infty.$$

Simultaneously, putting M from the proof of Lemma 4.1 we have

$$\lim_{n\to\infty} \int_0^{2\pi} [g_1(t,x_n,x_n') + g_2(t,x_n) + (\gamma_1(t)+\gamma_2(t))x_n] \frac{x_n^0}{\|x_n^0\|_H} dt \le 2\pi M \|v\|_H,$$

which leads to a contradiction. □

Since we use the method of the topological degree in the next part, we need to study the system of equations with a parameter

$$x'' + (m^2 + c_1)x = \lambda[g_1(t,x,x') + g_2(t,x) + c_1 x], \quad \lambda \in [0,1], \tag{34}$$

where $c_1 = 2m + 1 - \epsilon$, $\epsilon \in (0, 2m-1)$.

Lemma 4.4 *Let the asumptions (17), (22), (23) and (32) be fulfilled. Then there exists $r^* \in (0,\infty)$ such that for any $\lambda \in [0,1]$ each solution x of the problem (34),(2) fulfils (33).*

Proof Let for some $\lambda \in [0,1]$ a function x be a solution of (34),(2). Then x fulfils

$$\begin{aligned} x'' + (m^2 + c_1)x + \lambda(\gamma_1(t) + \gamma_2(t) - c_1)x &= \\ = \lambda[g_1(t,x,x') + g_2(t,x) + (\gamma_1(t)+\gamma_2(t))x]&. \end{aligned} \tag{35}$$

The left-hand side of (35) can be written in the form $x'' + (m^2 + \widetilde{\gamma}_1(t) + \widetilde{\gamma}_2(t))x$, where $\widetilde{\gamma}_1(t) = (1-\lambda)c_1 + \lambda\gamma_1(t)$, $\widetilde{\gamma}_2(t) = \lambda\gamma_2(t)$. We can see that $\widetilde{\gamma}_1, \widetilde{\gamma}_2$ satisfy (17).

So, by Lemma 3.6, we can find $\delta^* > 0$ such that (18) is true. Therefore, following the proof of Lemma 4.1, we can find $a > 0$ such that (24) is valid for any $\lambda \in [0,1]$ and any solution of (34), (2).

Now, consider a sequence of parameters $\{\lambda_n\}_{n=1}^{\infty} \subset [0,1]$ and a corresponding sequence of solutions $\{x_n\}_{n=1}^{\infty}$ of (34), (2). We can choose a convergent subsequence from $\{\lambda_n\}_{n=1}^{\infty}$ and hence we can suppose without loss of generality that there is $\lambda_0 \in [0,1]$ such that

$$\lim_{n\to\infty} \lambda_n = \lambda_0.$$

Following the proof of Lemma 4.3 we assume that (28) is valid. Then we can prove (29) and (30) in the same way as in the proof of Lemma 4.3. Substituting λ_n and x_n in (34), multiplying by $\frac{x_n^0}{\|x_n^0\|_H}$ and integrating, we have

$$\int_0^{2\pi} \left[\frac{x_n'' x_n^0}{\|x_n^0\|_H} + m^2 \frac{x_n x_n^0}{\|x_n^0\|_H} + c_1(1-\lambda_n)\frac{x_n x_n^0}{\|x_n^0\|_H} + \lambda_n(\gamma_1(t)+\gamma_2(t))\frac{x_n x_n^0}{\|x_n^0\|_H}\right] dt =$$

$$= \int_0^{2\pi} \lambda_n[g_1(t,x_n,x_n') + g_2(t,x_n) + (\gamma_1(t)+\gamma_2(t))x_n]\frac{x_n^0}{\|x_n^0\|_H}\, dt$$

$$\leq (M_1 + M^*) \int_0^{2\pi} \frac{x_n^0}{\|x_n^0\|_H}\, dt,$$

where M^* is given by (25). Therefore, by (29), (30),

$$\lim_{n\to\infty} \int_0^{2\pi} \left[(1-\lambda_n)c_1 \frac{x_n x_n^0}{\|x_n^0\|_H} + \lambda_n(\gamma_1(t)+\gamma_2(t))\frac{x_n x_n^0}{\|x_n^0\|_H}\right] dt =$$

$$= \lim_{n\to\infty} \|x_n\|_H \left[c_1(1-\lambda_0)\int_0^{2\pi} v^2(t)\, dt + \lambda_0 \int_0^{2\pi} (\gamma_1(t)+\gamma_2(t))v^2(t)\, dt\right]$$

$$\leq (M_1 + M^*) \int_0^{2\pi} |v(t)|\, dt.$$

Since $\int_0^{2\pi}(1-\lambda_0)c_1 v^2(t)dt + \lambda_0\int_0^{2\pi}(\gamma_1(t)+\gamma_2(t))v^2(t)dt = D > 0$, we have $\lim_{n\to\infty}\|x_n\|_H D = \infty$, a contradiction. □

We can write the problem (34),(2) in the form of an operator equation

$$Lx = \lambda N x, \quad \lambda \in [0,1], \tag{36}$$

where

$L : domL \to C(J), x \mapsto x'' + (m^2 + c_1)x$, $domL = \{x \in C^2(J); x$ fulfils (2)$\}$,
$N : C^1(J) \to C(J); x \mapsto g_1(\cdot, x(\cdot), x'(\cdot)) + g_2(\cdot, x(\cdot)) + c_1 x(\cdot)$,
$KerL = \{x \in domL; Lx = 0\}$.

Since $m^2 + c_1 \in (m^2, (m+1)^2)$, then $KerL = \{0\}$ and hence there exists the inverse operator $L^{-1} : C(J) \to C^2(J)$. We can write equation (36) in the equivalent form

$$(I - \lambda i L^{-1} N)x = 0, \quad \lambda \in [0,1], \tag{37}$$

where $I : C^1(J) \to C^1(J)$ is the identical operator and $i : C^2(J) \to C^1(J)$, $x \mapsto x$ is the operator of compact imbedding.

Lemma 4.5 *Let the assumptions (17), (22), (23) and (32) be fulfilled. Then there exists $\rho^* \in (0, \infty)$ such, that*

$$||x||_{C^1} < \rho^* \tag{38}$$

is valid for any $\lambda \in [0,1]$ and all solutions of (37).

Proof Lemma 4.4 implies that every solution of (37) for $\lambda \in [0,1]$ is bounded by r^* in H. Then

$$||x||_C \le k||x||_H < kr^*,$$

where k is the constant from (5).

The conditions (2) imply the existence of $t_0 \in J$ such that $x'(t_0) = 0$. Integrating (34) we get

$$x'(t) - x'(t_0) = \int_{t_0}^{t} \lambda\big[g_1(t,x,x') + g_2(t,x) - c_1(1-\lambda)x - m^2 x\big]\, dt$$

and hence

$$||x'||_C < 2\pi[M_1 + \max\{|g_2(t,x)| : t \in J, |x| \le kr^*\} + (m+1)^2 kr^*] = M_3.$$

Therefore (38) is valid for $\rho^* = M_3 + kr^*$. □

5 Main Results

Theorem 5.1 *Let the assuptions (17), (22), (23) and (32) be fulfilled. Then there is ρ^* such, that*

$$d[I - iL^{-1}N, K(\rho^*)] = 1,$$

where $K(\rho^*) = \{x \in C^1(J); ||x||_{C^1} < \rho^*\}$.

Proof We will use the properties of the Leray-Schauder topological degree.

It will be sufficient, if $L^{-1} : C(J) \to C^2(J)$ is bounded. We can write the operator $L^{-1} : C(J) \to C^2(J)$ in the form

$$L^{-1}x(t) = \int_0^{2\pi} G(t,s)x(s)\, ds,$$

where

$$G(t,s) = \begin{cases} \frac{\cos r(s-t-\pi)}{2r\sin \pi r} & \text{for } 0 \le t \le s \le 2\pi \\ \frac{\cos r(s-t+\pi)}{2r\sin \pi r} & \text{for } 0 \le s \le t \le 2\pi \end{cases}$$

is the Green function of (34),(2) for $\lambda = 0$ and where $r = \sqrt{m^2 + c_1} \in (m, m+1)$. Since $r \notin \mathbf{N}$ then $\sin \pi r \neq 0$ and hence there is $K \in \mathbf{R}$ such that $|G(t,s)| < K$.

Furthermore

$$\frac{\partial G(t,s)}{\partial t} = \begin{cases} -\frac{\sin r(s-t-\pi)}{2\sin \pi r} & \text{for } 0 \le t \le s \le 2\pi \\ -\frac{\sin r(s-t+\pi)}{2\sin \pi r} & \text{for } 0 \le s \le t \le 2\pi \end{cases}$$

and

$$\frac{\partial^2 G(t,s)}{\partial t^2} = \begin{cases} -r\frac{\cos r(s-t-\pi)}{2\sin \pi r} & \text{for } 0 \le t \le s \le 2\pi \\ -r\frac{\cos r(s-t+\pi)}{2\sin \pi r} & \text{for } 0 \le s \le t \le 2\pi \end{cases},$$

than there exist konstants $K_1, K_2 < \infty$ such that

$$\left|\frac{\partial G(t,s)}{\partial t}\right| < K_1 \quad \text{and} \quad \left|\frac{\partial^2 G(t,s)}{\partial t^2}\right| < K_2.$$

Therefore

$$||L^{-1}x||_{C^2} = ||L^{-1}x||_C + ||(L^{-1}x)'||_C + ||(L^{-1}x)''||_C$$

$$\le \max_{t\in J} \int_0^{2\pi} |G(t,s)||x(s)|ds + \max_{t\in J} \int_0^{2\pi} \left|\frac{\partial G(t,s)}{\partial t}\right||x(s)|ds +$$

$$+ \max_{t\in J} \int_0^{t} \left|\frac{\partial^2 G(t,s)}{\partial t^2}\right||x(s)|ds + \max_{t\in J}\left[\left|\frac{\partial G(t,t-)}{\partial t} - \frac{\partial G(t,t+)}{\partial t}\right|\right]|x(t)| +$$

$$+ \max_{t\in J} \int_t^{2\pi} \left|\frac{\partial^2 G(t,s)}{\partial t^2}\right||x(s)|\, ds \le [2\pi(K + K_1 + K_2) + 1]||x||_C.$$

We can see that the map L^{-1} is bounded.

Further we prove that N is continuous i.e. if for every $\epsilon > 0$ there is $\delta > 0$ such that for each $x, y \in C^1(J)$, which fulfil

$$||x - y||_{C^1} = \max_{t\in J}\{|x(t) - y(t)| + |x'(t) - y'(t)|, t \in J\} < \delta,$$

the expression $||Nx - Ny||_C < \epsilon$ is valid.

Put $g(t,x,y) = g_1(t,x,y) + g_2(t,x) + c_1 x$. Since $g(t,x,y)$ is continuous on $J \times \mathbf{R}^2$ and $x, y \in C^1(J)$ then the function $g(t,x(t),x'(t)) - g(t,y(t),y'(t))$ is also continuous on J and it has a maximum in J. Hence there exists $t_0 \in J$ such that

$$\max_{t\in J} |g(t,x(t),x'(t)) - g(t,y(t),y'(t))| = |g(t_0,x(t_0),x_0'(t)) - g(t_0,y(t_0),y'(t_0))|.$$

Further for every $\epsilon > 0$ we can search $\delta > 0$ such that if $|x(t_0) - y(t_0)| + |x'(t_0) - y'(t_0)| < \delta$ then

$$|g(t_0, x(t_0), x_0'(t)) - g(t_0, y(t_0), y'(t_0))| < \epsilon.$$

Therefore if $\|x - y\|_{C^1} < \delta$ then $|x(t_0) - y(t_0)| + |x'(t_0) - y'(t_0)| < \delta$ is valid, thus

$$|g(t_0, x(t_0), x_0'(t)) - g(t_0, y(t_0), y'(t_0))| < \epsilon$$

and hence $\|Nx - Ny\|_C < \epsilon$.

Let us prove that the map $iL^{-1}N : C^1(J) \to C^1(J)$ is compact. Since $L^{-1} : C(J) \to C^2(J)$ is a linear bounded operator and $i : C^2(J) \to C^1(J)$ is compact then $iL^{-1} : C(J) \to C^1(J)$ is compact. The operator $N : C(J) \to C^1(J)$ is continuous. Therefore the operator $iL^{-1}N : C^1(J) \to C^1(J)$ is compact.

Lemma 4.5 implies that there exists $\rho^* \in (0, \infty)$ such that for any $\lambda \in [0, 1]$ every solution of (37) lies in the interior of $K(\rho^*)$. Thus for any $\lambda \in [0, 1]$ and $x \in \partial K(\rho^*)$ we get $x \neq \lambda i L^{-1} N x$.

Then the map $F = iL^{-1}N$ fulfils the assumptions of Lemma 2.6. Let us put $G = -F$. From Lemma 2.6 $d[I, K(\rho^*)] = 1$ and using (7) we get

$$\begin{aligned} d[I - iL^{-1}N, K(\rho^*)] &= d[I - F, K(\rho^*)] = d[I - F - (1 - \lambda)G, K(\rho^*)] \\ &= d[I - \lambda F, K(\rho^*)] = d[I, K(\rho^*)] = 1. \end{aligned}$$ □

Theorem 5.2 *Let the assumption (17), (22), (23) and (32) be fulfilled. Then the problem (21),(2) has at least one solution.*

Proof In view of Theorem 5.1 we have $d[I - iL^{-1}N, K(\rho^*)] = 1$ and with respect to the Lemma 2.6 there is $u \in K(\rho^*)$ such that $u = iL^{-1}Nu$, i.e. u is a fixed point of $iL^{-1}N$. Since the equation (37) with $\lambda = 1$ is equivalent to the problem (21),(2), then $u(t)$ is a solution of (21),(2). □

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