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Lidmila Vránová

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Irreducible Elements of Posets

LIDMILA VRÁNOVÁ

Department of Mathematics, PdF Masaryk University
Poříčí 31, 603 00 Brno, Czech Republic,
e-mail: vranoval@post.cz

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Abstract

Join- (meet-, resp.) irreducible elements of posets are defined and their connection with the weakly dense subsets is studied. Especially, the weak separability of posets of finite length is found.

Key words: Poset, join- and meet-irreducible element, weakly dense subset, weak separability.

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Introduction

B. Šešelja and A. Tepavčević [5] defined meet-irreducible elements in a general poset: an element x of a poset $(G, \leq)$ is *meet-irreducible* if $x = \inf\{y, z\}$ implies $x = y$ or $x = z$. They proved that every element of a finite poset is the infimum of meet-irreducible elements. In this note we give some modification of the concept of irreducible elements and prove an analogical result under more general assumptions. Further, the role of irreducible elements in weakly dense subsets is studied.

We now remember some denotations and facts which will be used in the following text. If $\mathbb{G} = (G, \leq)$ is a poset and $x, y \in G$ then $x \parallel y$ means that x, y are *incomparable* and $x \prec y$ means that y is *a cover* of x, i.e. $x < y$ and $x < z < y$ holds for no $z \in G$. The set of all minimal elements of $\mathbb{G}$ is denoted $\operatorname{Min} \mathbb{G}$; the least element of $\mathbb{G}$ (if it exists) is denoted 0. A poset $\mathbb{G}$ satisfies *the descending chain condition* (DCC) if each its subchain has the least element.

The ascending chain condition (ACC) is defined dually. $\mathbb{G}$ is *a poset of locally finite length* if for each $a, b \in \mathbb{G}$, $a \leq b$ each chain in $\mathbb{G}$ with the least element a and the greatest element b is finite. Cardinality of a set M is denoted $|M|$.

In [3] weakly dense subsets of a poset are defined and studied. A subset H of a poset $\mathbb{G} = (G, \leq)$ is *weakly l-dense* in $\mathbb{G}$ if

$$x, y \in G,\ x \not\leq y \Rightarrow \text{there exists } h \in H \text{ such that } \ h \leq x,\ h \not\leq y; \tag{1}$$

H is *weakly u-dense* in $\mathbb{G}$ if

$$x, y \in G,\ x \not\leq y \Rightarrow \text{there exists } h \in H \text{ such that } y \leq h,\ x \not\leq h, \tag{2}$$

i.e. if H is weakly l-dense in the dual of $\mathbb{G}$;

$$H \text{ is } \textit{weakly dense} \text{ in } \mathbb{G} \text{ if it is both weakly l-dense and weakly u-dense in } \mathbb{G}. \tag{3}$$

Further, put

$$\text{wl-sep } \mathbb{G} = \min\{|H|;\ H \subseteq G \text{ is weakly l-dense in } \mathbb{G}\}, \tag{4}$$

$$\text{wu-sep } \mathbb{G} = \min\{|H|;\ H \subseteq G \text{ is weakly u-dense in } \mathbb{G}\}, \tag{5}$$

$$\text{w-sep } \mathbb{G} = \min\{|H|;\ H \subseteq G \text{ is weakly dense in } \mathbb{G}\}; \tag{6}$$

these cardinals are called *weak l-separability* of $\mathbb{G}$, *weak u-separability* of $\mathbb{G}$ and *weak separability* of $\mathbb{G}$, resp.

Let us note that if H is weakly l-dense in $\mathbb{G}$ and $x \in \operatorname{Min}\mathbb{G}$, $x \neq 0$, then $x \in H$. The following Theorem is proved in [3]: H is weakly l-dense in $\mathbb{G} = (G, \leq)$ if and only if

$$x = sup\{h \in H,\ h \leq x\} \tag{7}$$

for each $x \in G$. The dual assertion for weakly u-dense subsets is also valid.

1 Irreducible elements

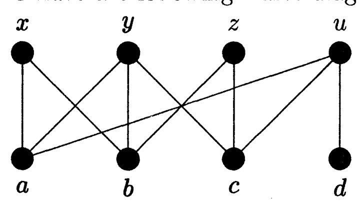
Definition 1.1 Let $\mathbb{G} = (G, \leq)$ be a poset and $x \in G$. We say that x is *a join-irreducible element* of $\mathbb{G}$ if

$$H \subseteq G,\ H \text{ finite},\ x = \sup H \Rightarrow x \in H. \tag{8}$$

The set of all join-irreducible elements of $\mathbb{G}$ will be denoted $J(\mathbb{G})$.

Let us note that the least element of $\mathbb{G}$ (if it exists) is not join-irreducible for $0 = \sup\emptyset$. Further, if x is a minimal and not the least element of $\mathbb{G}$ then, trivially, $x \in J(\mathbb{G})$. Also, if $0 \prec x$ then $x \in J(\mathbb{G})$.

Example 1.2 Let $\mathbb{G}$ have the following Hasse diagram



Then $J(\mathbb{G}) = \{a, b, c, d, x, z\}$; $y, u \notin J(\mathbb{G})$ for $y = \sup\{a, b, c\}$, $u = \sup\{c, d\}$.

Theorem 1.3 *Let $\mathbb{G} = (G, \leq)$ be a poset satisfying DCC. Then for each $x \in G$ there exists a finite subset $G_x \subseteq J(\mathbb{G})$ such that $x = \sup G_x$.*

Proof If $x \in J(\mathbb{G})$ then the existence of G_x with the desired property is trivial. Assume that the theorem does not hold and denote $M = \{x \in G$; there exists no finite $G_x \subseteq J(\mathbb{G})$ with $x = \sup G_x\}$. Thus $M \neq \emptyset$ and $M \cap J(\mathbb{G}) = \emptyset$. As $(M, \leq)$ satisfies the DCC, it must exist an $x \in \mathrm{Min}(M, \leq)$. Then $x \notin J(\mathbb{G})$; hence there exists a finite $H \subseteq G$ such that $x = \sup H$ and $x \notin H$. From this $y < x$ for any $y \in H$ implying $y \notin M$; thus for any $y \in H$ a finite subset $G_y \subseteq J(\mathbb{G})$ exists such that $y = \sup G_y$. If we put $G_x = \bigcup_{y \in H} G_y$ then G_x is a finite set, $G_x \subseteq J(\mathbb{G})$ and $x = \sup G_x$. It implies $x \notin M$ contradicting $x \in \mathrm{Min}(M, \leq)$. □

We shall need the following simple Lemmas.

Lemma 1.4 *Let $(G, \leq)$ be a poset and $x \in G$. If there exists a subset $H \subseteq G$ such that $x = \sup H$ and $x \notin H$, then $x = \sup\{y \in G;\ y < x\}$.*

Proof Trivial. □

Lemma 1.5 *Let $\mathbb{G} = (G, \leq)$ be a poset of a locally finite length and $x \in G$. If there exists a subset $H \subseteq G$ such that $supH = x$ and $x \notin H$, then $x = \sup H_0$ where $H_0 = \{y \in G;\ y \prec x\}$.*

Proof It is $y \leq x$ for any $y \in H_0$. Suppose the existence of an element $z \in G$, $x \not\leq z$ such that $y \leq z$ for any $y \in H_0$. By 1.4. we have $x = \sup\{u \in G;\ u < x\}$. As $\mathbb{G}$ has locally finite length, for any $u \in G$, $u < x$ there exists $u_0 \in G$ such that $u \leq u_0 \prec x$. Then $u_0 \leq z$, thus $u \leq z$ and this contradicts the fact $sup\{u \in G;\ u < x\} = x$. □

Lemma 1.6 *Let $\mathbb{G} = (G, \leq)$ be a poset having locally finite length and let $x \in J(\mathbb{G})$, $x \notin \mathrm{Min}\,\mathbb{G}$. Then just one of the following facts occurs:*

1) it exists the only element $y \in G$ such that $y \prec x$,

2) it exists an element $y \in G$ such that $x \parallel y$ and $z \in G$, $z < x \Rightarrow z < y$.

Proof By 1.5 the relation $x = \sup\{z \in G;\ z \prec x\}$ is not valid. Consequently, an element $y \in G$ exists such that $x \not\leq y$ and $z \in G$, $z \prec x \Rightarrow z \leq y$. If $y < x$ then necessarily $y \prec x$ and it is the only element with this property for $z \in G$, $z \prec x$ implies $z \leq y$, thus $z = y$. Let $y \parallel x$ and suppose $z \in G$, $z < x$. Then an element $z_0 \in G$ exists such that $z \leq z_0 \prec x$, thus $z_0 \leq y$ and $z \leq y$. The equality is impossible for $y \parallel x$; hence $z < y$. □

Dually we can define a meet-irreducible element of a poset $\mathbb{G} = (G, \leq)$: $x \in G$ is *meet-irreducible* if

$$H \subseteq G,\ H \text{ finite},\ x = \inf H \Rightarrow x \in H. \tag{9}$$

The set of all meet-irreducible elements of $\mathbb{G}$ will be denoted $M(\mathbb{G})$. The dual assertions to 1.3, 1.4, 1.5 and 1.6 are also valid.

2 Irreducible elements and weakly dense subsets

Theorem 2.1 *Let $\mathbb{G} = (G, \leq)$ be a poset satisfying the DCC. Then the set $J(\mathbb{G})$ is weakly l-dense in $\mathbb{G}$.*

Proof Let $x, y \in G$, $x \not\leq y$. By 1.3 there exists a finite subset $G_x = \{x_1, \dots, x_n\} \subseteq J(\mathbb{G})$ such that $\sup G_x = x$. It follows $x_i \leq x$ for all $i = 1, \dots, n$. If it would be $x_i \leq y$ for all $i = 1, \dots, n$ then $\sup\{x_1, \dots, x_n\} = x \leq y$, a contradiction. Thus it must exist $i \in \{1, \dots, n\}$ such that $x_i \leq x$, $x_i \not\leq y$. Hence $J(\mathbb{G})$ is weakly l-dense in $\mathbb{G}$. □

Corollary 2.2 *Let $\mathbb{G} = (G, \leq)$ be a poset satisfying the DCC. Then $x = \sup\{y \in J(\mathbb{G});\ y \leq x\}$ for each $x \in G$.*

Proof follows from 2.1 and from (7). □

Theorem 2.3 *Let $\mathbb{G} = (G, \leq)$ be a poset of locally finite length. Then $J(\mathbb{G}) \subseteq H$ for any weakly l-dense subset H of $\mathbb{G}$.*

Proof Let $x \in J(\mathbb{G})$. If $x \in \operatorname{Min} \mathbb{G}$, then $x \neq 0$ and, therefore, $x \in H$. Suppose $x \notin \operatorname{Min} \mathbb{G}$; then, by 1.6., an element $y \in G$ exists such that either 1) $y \prec x$ and y is the only element with this property, or 2) $y \parallel x$ and $z \in G$, $z < x \Rightarrow z < y$. In the case 1) we have $x \not\leq y$ and therefore an element $h \in H$ exists such that $h \leq x$, $h \not\leq y$. This is possible only for $h = x$ implying $x \in H$. In the case 2) it is also $x \not\leq y$ so that there must exist an element $h \in H$ such that $h \leq x$, $h \not\leq y$. This implies $h = x$, thus $x \in H$, as well. □

Theorem 2.4 *Let $\mathbb{G}$ be a poset of locally finite length satisfying the DCC. Then $J(\mathbb{G})$ is the least (with respect to set inclusion) weakly l-dense subset of $\mathbb{G}$.*

Proof follows from 2.1 and 2.3. □

Corollary 2.5 *Let $\mathbb{G}$ be a poset of locally finite length satisfying the DCC. Then wl-sep $\mathbb{G} = |J(\mathbb{G})|$.*

Analogical results hold for the set $M(\mathbb{G})$ of meet-irreducible elements of $\mathbb{G}$; especially, we have

Theorem 2.6 *Let $\mathbb{G}$ be a poset of locally finite length which satisfies the ACC. Then $M(\mathbb{G})$ is the least weakly u-dense subset of $\mathbb{G}$.*

Corollary 2.7 *If $\mathbb{G}$ is a poset of locally finite length satisfying the ACC, then wu-sep $\mathbb{G} = |M(\mathbb{G})|$.*

With combining of 2.4 and 2.6 we get further

Theorem 2.8 *Let* $\mathbb{G}$ *be a poset each subchain of which is finite. Then the set* $J(\mathbb{G}) \cup M(\mathbb{G})$ *is the least weakly dense subset of* $\mathbb{G}$*; consequently,*

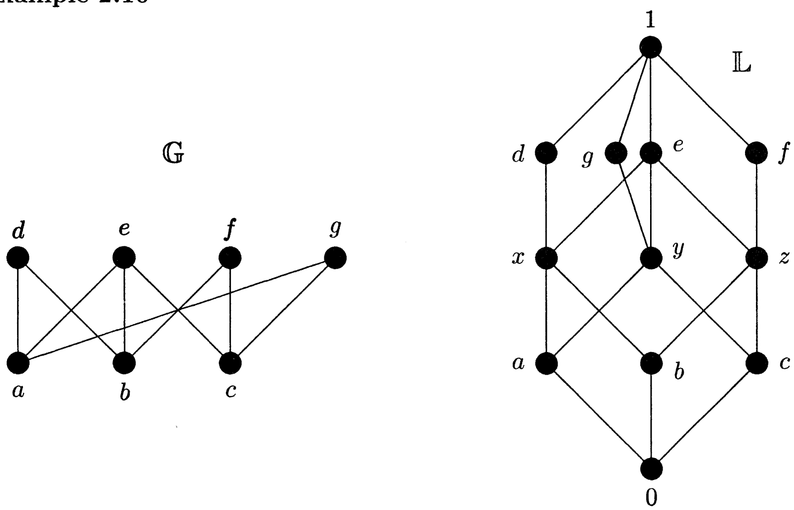
$$wl\text{-}sep\ \mathbb{G} = |J(\mathbb{G}) \cup M(\mathbb{G})|.$$

Let $\mathbb{G}$ be a finite poset, $\mathbb{L}$ its MacNeille completion. Then $J(\mathbb{L}) = J(\mathbb{G})$, $M(\mathbb{L}) = M(\mathbb{G})$ (see, e.g. [2]). Thus we have

Corollary 2.9 *Let* $\mathbb{G}$ *be a finite poset and* $\mathbb{L}$ *be its MacNeille completion. Then*

$$\begin{aligned} wl\text{-}sep\ \mathbb{L} &= wl\text{-}sep\ \mathbb{G} = |J(\mathbb{G})| \\ wu\text{-}sep\ \mathbb{L} &= wu\text{-}sep\ \mathbb{G} = |M(\mathbb{G})| \\ w\text{-}sep\ \mathbb{L} &= w\text{-}sep\ \mathbb{G} = |J(\mathbb{G}) \cup M(\mathbb{G})|. \end{aligned}$$

Example 2.10



$J(\mathbb{L}) = J(\mathbb{G}) = \{a, b, c, d, f, g\}$
$M(\mathbb{L}) = M(\mathbb{G}) = \{d, e, f, g\}$
Thus wl-sep $\mathbb{L} = 6$, wu-sep $\mathbb{L} = 4$, w-sep $\mathbb{L} = 7$.

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