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Incidence Structures of Independent Sets *

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Abstract

Independent sets in incidence structures are studied in this paper. By the help of the mappings norming independent sets we define incidence structures of independent sets. The substructures in them are described. The questions of reducibility of incidence structures in context with reducibility of corresponding structures of independent sets are also studied.

Key words: Incidence structures, independent sets, disjoint union of incidence structures.

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Definition 1 Let G and M be sets and $I \subseteq G \times M$. Then the triple $\mathcal{J} = (G, M, I)$ is called an *incidence structure*. If $A \subseteq G$, $B \subseteq M$ are non--empty sets, then we denote

$$A^{\uparrow} = \{m \in M \mid gIm \ \ \forall g \in A\}, \quad B^{\downarrow} = \{g \in G \mid gIm \ \ \forall m \in B\}.$$

For the empty set we put $\emptyset^{\uparrow} := M$, $\emptyset^{\downarrow} := G$. And moreover, we denote $A^{\uparrow\downarrow} := (A^{\uparrow})^{\downarrow}$, $B^{\downarrow\uparrow} := (B^{\downarrow})^{\uparrow}$, $g^{\uparrow} := \{g\}^{\uparrow}$, $m^{\downarrow} := \{m\}^{\downarrow}$ for $A \subseteq G$, $B \subseteq M$ and $g \in G$, $m \in M$.

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Definition 2 An incidence structure $\mathcal{J}_1 = (G_1, M_1, I_1)$ is *embedded* into an incidence structure $\mathcal{J} = (G, M, I)$ if $G_1 \subseteq G$, $M_1 \subseteq M$ and $I_1 \subseteq I \cap (G_1 \times M_1)$. If $I_1 = I \cap (G_1 \times M_1)$, then $\mathcal{J}_1$ is a *substructure* of $\mathcal{J}$.

If we put $\mathcal{P}_G = \{A \subseteq G \mid A = A^{\uparrow\downarrow}\}$, then the pair $\mathcal{G} = (G, \mathcal{P}_G)$ is a closure space in which $X^{\uparrow\downarrow}$ is a closure of any subset $X \subseteq G$ in $\mathcal{G}$. A set $A \subseteq G$ is *independent* in $\mathcal{G}$ if $a \notin (A - \{a\})^{\uparrow\downarrow}$ for all $a \in A$.

In what follows we denote $A_a := A - \{a\}$.

If $A \subseteq G$, then we put $X^A(a) := A_a^{\uparrow} - a^{\uparrow}$ for $a \in A$. Then $X^A(a) = \emptyset$ iff $A_a^{\uparrow} \subseteq a^{\uparrow}$ iff $a \in A_a^{\uparrow\downarrow}$. Hence the set A is independent in $\mathcal{G}$ if and only if $X^A(a) \neq \emptyset$ for all $a \in A$.

Let the subset $A \subseteq G$ be independent in $\mathcal{G}$. Then we put $\mathcal{X} = \{X^A(a) \mid a \in A\}$. For every choice $Q^A = \{m_a \in X^A(a) \mid X^A(a) \in \mathcal{X}\} \subseteq M$ from the set $\mathcal{X}$ we define a map $\alpha : A \to Q^A$ by the formula $\alpha(a) = m_a$. This map is called an *A-norming map.*

If we put $\mathcal{P}_M = \{B \subseteq M \mid B = B^{\downarrow\uparrow}\}$, then $\mathcal{M} = (M, \mathcal{P}_M)$ is a closure space again. A set $B \subseteq M$ is independent in $\mathcal{M}$ if $m \notin (B - \{m\})^{\downarrow\uparrow} = B_m^{\downarrow\uparrow}$ for all $m \in M$. If $m \in B$, then we put $Y^B(m) = B_m^{\downarrow} - m^{\downarrow}$. B is independent in $\mathcal{M}$ if and only if $Y^B(m) \neq \emptyset$ for all $m \in B$.

Let B be independent in $\mathcal{M}$. Then we put $\mathcal{Y} = \{Y^B(m) \mid m \in B\}$. For every choice $Q^B = \{a_m \in Y^B(m) \mid Y^B(m) \in \mathcal{Y}\} \subseteq G$ we consider a map $\beta : B \to Q^B$ given by the formula $\beta(m) = a_m$. It will be called a *B-norming map.*

Theorem 1 *Let $A \subseteq G$ ($B \subseteq M$) be an independent set in $\mathcal{G}$ ($\mathcal{M}$). Then each norming map $\alpha : A \to Q^A$ ($B \to Q^B$) is injective and the set Q^A (Q^B) is independent in $\mathcal{M}$ ($\mathcal{G}$). (See [3].)*

If $\alpha : A \to B$ is a map norming an independent set A of $\mathcal{G}$, then $\alpha^{-1} : B \to A$ is a map norming an independent set B of $\mathcal{M}$. Moreover, from $\alpha(a) = m_a$ for $a \in A$ we get $a \in Y^B(m_a)$.

Theorem 2 *Let $\mathcal{J}_1 = (G_1, M_1, I_1)$ be a substructure of an incidence structure $\mathcal{J} = (G, M, I)$ and $\mathcal{G}_1 = (G_1, \mathcal{P}_{G_1})$, $\mathcal{M}_1 = (M_1, \mathcal{P}_{M_1})$ be corresponding closure spaces in $\mathcal{J}_1$. A set $A \subseteq G_1$ ($B \subseteq M_1$) is independent in $\mathcal{G}_1$ ($\mathcal{M}_1$) if and only if $X^A(a) \cap M_1 \neq \emptyset$ for all $a \in A$ ($Y^B(m) \cap G_1 \neq \emptyset$ for all $m \in B$).*

Remark 1 If a set A (B) is independent in $\mathcal{G}_1$ ($\mathcal{M}_1$), then it is independent in $\mathcal{G}$ ($\mathcal{M}$).

Definition 3 Let us consider an incidence structure $\mathcal{J} = (G, M, I)$ and a natural number $p \geq 2$. Let G^p and M^p be the sets of all independent sets of $\mathcal{G}$ and $\mathcal{M}$ of cardinality p, respectively. Then $\mathcal{J}^p = (G^p, M^p, I^p)$ is an *incidence structure of independent sets* of $\mathcal{J}$, where AI^pB iff there exists an A-norming map $\alpha : A \to B$ for $A \in G^p, B \in M^p$.

Remark 2 If $G^p = M^p = \emptyset$, then $\mathcal{J}^p = (\emptyset, \emptyset, \emptyset)$. In this case we will write $\mathcal{J}^p = \emptyset$. Since each subset of an independent set is independent, from $G^p \neq \emptyset$ we obtain $G^q \neq \emptyset$ for $2 \leq q \leq p$.

Remark 3 Let $A \in G^p$. Then $X^A(a) \neq \emptyset$ for all $a \in A$ and there exists a set $B \in M^p$ and a norming map $\alpha : A \to B$. Similarly for an arbitrary subset $B \in M^p$. Thus $A^\uparrow \neq \emptyset$, $B^\downarrow \neq \emptyset$ for all $A \in G^p$, $B \in M^p$.

Theorem 3 *If $\mathcal{J}_1 = (G_1, M_1, I_1)$ is a substructure of $\mathcal{J} = (G, M, I)$, then $\mathcal{J}_1^p = (G_1^p, M_1^p, I_1^p)$ is a substructure of $\mathcal{J}^p$.*

Proof Let $A \in G_1^p$. It means that A is independent in $\mathcal{G}_1$ and $|A| = p$. By Theorem 2, A is also independent in $\mathcal{G}$ and thus $A \in G^p$. Hence $G_1^p \subseteq G^p$. Similarly $M_1^p \subseteq M^p$.

Assume that AI_1^pB for $A \in G_1^p$, $B \in M_1^p$. There exists a norming map $\alpha : a \mapsto m_a$ in $\mathcal{J}_1$, where $m_a \in {}^\uparrow A_a - {}^\uparrow a$ (we write the operators $\uparrow, \downarrow$ to the left in $\mathcal{J}_1$). Since ${}^\uparrow A_a - {}^\uparrow a = X^A(a) \cap M_1$, we get $m_a \in X^A(a)$ and α is also norming map in $\mathcal{J}$. This yields AI^pB.

Let AI^pB for $A \in G_1^p$, $B \in M_1^p$. Then there exists a map $\alpha : a \mapsto m_a$ norming the set A in $\mathcal{J}$, where $m_a \in X^A(a) \cap M_1$. Thus $m_a \in {}^\uparrow A_a - {}^\uparrow a$. Therefore α is a norming map in $\mathcal{J}_1$ and AI_1^pB. □

Theorem 4 *Let $\mathcal{J}_1^p = (G_1^p, M_1^p, I_1^p)$ be a substructure of $\mathcal{J}^p = (G^p, M^p, I^p)$ such that ${}^\uparrow A \neq \emptyset$, ${}^\downarrow B \neq \emptyset$ for all $A \in G_1^p$, $B \in M_1^p$ ($\uparrow, \downarrow$ are operators in $\mathcal{J}_1^p$). If $\mathcal{J}' = (G', M', I')$ is a substructure in $\mathcal{J}$ such that*

$$G' = \bigcup_{A \in G_1^p} A \quad \textit{and} \quad M' = \bigcup_{B \in M_1^p} B,$$

then $\mathcal{J}_1^p$ is a substructure of $\mathcal{J}'^p = (G'^p, M'^p, I'^p)$.

Proof Consider $A \in G_1^p$. Because of ${}^\uparrow A \neq \emptyset$ there exists $B \in M_1^p$ such that AI_1^pB. Since $\mathcal{J}_1^p$ is a substructure in $\mathcal{J}^p$, we get AI^pB. Hence there exists a norming map $\alpha : A \to B$ in $\mathcal{J}$ assigning to every $a \in A$ an element $m_a \in X^A(a) \cap M'$. This implies $X^A(a) \cap M' \neq \emptyset$ and (by Theorem 2) A is independent in $\mathcal{J}'$. Thus $A \in G'^p$ and we obtain $G_1^p \subseteq G'^p$. Similarly $M_1^p \subseteq M'^p$.

Suppose that $A \in G_1^p$, $B \in M_1^p$, i.e. $A \subseteq G'$, $B \subseteq M'$ If AI_1^pB, then AI^pB and there exists a norming map $\alpha : A \to B$ in $\mathcal{J}$ which is at the same time norming in $\mathcal{J}'$. Thus AI'^pB. Conversely, consider AI'^pB. According to Theorem 3, $\mathcal{J}'^p$ is a substructure in $\mathcal{J}^p$ which implies AI^pB. Because of $\mathcal{J}_1^p$ is a substructure in $\mathcal{J}^p$, we obtain AI_1^pB. □

Remark 4 Let the assumptions from Theorem 4 be satisfied. If $\mathcal{J}^+ = (G^+, M^+, I^+)$ is a substructure in $\mathcal{J}$ such that $\mathcal{J}^{+p} = \mathcal{J}_1^p$, then $G^{+p} = G'^p$ and $M^{+p} = M'^p$.

Example 1 Let us show an example of a substructure $\mathcal{J}_1^p$ in $\mathcal{J}^p$ such that there exists no incidence structure $\mathcal{J}^+$ embedded into $\mathcal{J}$ with the property $\mathcal{J}^{+p} = \mathcal{J}_1^p$.

I	m_1	m_2	m_3
g_1		–	–
g_2	–		–
g_3	–	–	
g_4	–		–
g_5		–	–

Table 1

I^3	M
A_1	–
A_2	–
A_3	–
A_4	–

Table 2

An incidence structure $\mathcal{J} = (G, M, I)$, where $G = \{g_1, \ldots, g_5\}$ and $M = \{m_1, m_2, m_3\}$ is given by the incidence table (Table 1). Let us consider an incidence structure $\mathcal{J}^3 = (G^3, M^3, I^3)$. Then $G^3 = \{A_1, A_2, A_3, A_4\}$, $M^3 = \{M\}$, where $A_1 = \{g_1, g_2, g_3\}$, $A_2 = \{g_1, g_3, g_4\}$, $A_3 = \{g_2, g_3, g_5\}$, $A_4 = \{g_3, g_4, g_5\}$ and $A_1, A_2, A_3, A_4\ I^3\ M$ (see Table 2).

If we denote $G_1^3 = \{A_1, A_4\}$, $M_1^3 = \{M\}$, then $\mathcal{J}_1^3 = (G_1^3, M_1^3, I_1^3)$ is a substructure in $\mathcal{J}^3$, where $A_1, A_4\ I_1^3\ M$. Let us assume that $\mathcal{J}^+ = (G^+, M^+, I^+)$ is an incidence structure embedded into $\mathcal{J}$ such that $\mathcal{J}^{+3} = \mathcal{J}_1^3$. Thus $G^{+3} = G_1^3$ and $A_1, A_4 \in G^{+3}$, $A_1 \cup A_4 \subseteq G^+$.

From this $G^+ = G$ and $M^+ = M$. Since $\mathcal{J}^+$ is embedded into $\mathcal{J}$, we obtain $I^+ \subseteq I$. If $I^+ = I$, then $\mathcal{J}^+ = \mathcal{J}$ and $\mathcal{J}^{+3} = \mathcal{J}^3$. Hence $\mathcal{J}_1^3 = \mathcal{J}^3$ and that is a contradiction.

Assume that $I^+ \neq I$. Then there exist elements $g_i \in G$, $m_j \in M$ such that $g_i I m_j$ but not $g_i I^+ m_j$. Obviously $g_i \in A_1$ or $g_i \in A_4$. However, it means that A_1 or A_4 is not independent in $\mathcal{G}^+ = (G^+, \mathcal{P}_{G^+})$. Therefore both $A_1 \notin G^{+3}$ or $A_4 \notin G^{+3}$ and from that $\mathcal{J}^{+3} \neq \mathcal{J}_1^3$ follows. Obviously ${}^{\uparrow}A_i \neq \emptyset$ for all $i \in \{1, 2, 3, 4\}$ and ${}^{\downarrow}M \neq \emptyset$. For a substructure $\mathcal{J}'$ described in Theorem 4 we get $\mathcal{J}' = \mathcal{J}$ and $\mathcal{J}_1^3$ is a substructure in $\mathcal{J}'^3$.

Definition 4 An incidence structure $\mathcal{J} = (G, M, I)$ is said to be a *disjoint union* of its substructures $\mathcal{J}_t = (G_t, M_t, I_t)$, $t \in T$, if $\overline{G} = \{G_t \mid t \in T\}$, $\overline{M} = \{M_t \mid t \in T\}$ and $\overline{I} = \{I_t \mid t \in T\}$ are decompositions of the sets G, M, I. We will write $\mathcal{J} = \dot{\bigcup}_{t \in T} \mathcal{J}_t$.

An incidence structure $\mathcal{J}$ is called *reducible* if there exists a disjoint union $\mathcal{J} = \dot{\bigcup}_{t \in T} \mathcal{J}_t$ for $|T| > 1$. In other case $\mathcal{J}$ is *irreducible.* If $\mathcal{J} = \dot{\bigcup}_{t \in T} \mathcal{J}_t$, then the substructures $\mathcal{J}_t$ are *decompositions components* of $\mathcal{J}$.

Theorem 5 *Let $\mathcal{J} = \dot{\bigcup}_{t \in T} \mathcal{J}_t$, $|T| > 1$. We will write the operators $\uparrow, \downarrow$ to the right in $\mathcal{J}$ and to the left in $\mathcal{J}_t$. Then the following statements are valid:*

1. *If a set $A \subseteq G$ is not contained in any subset G_t, then $A^{\uparrow} = \emptyset$ and $A^{\uparrow\downarrow} = G$.*
2. *Let $A \subseteq G_t$ for certain $t \in T$.*
   (a) *If $A \neq \emptyset$, then $A^{\uparrow} = {}^{\uparrow}A$. Moreover, $A^{\uparrow\downarrow} = {}^{\downarrow\uparrow}A$ if and only if $A^{\uparrow} \neq \emptyset$.*
   (b) *If $A = \emptyset$, then $A^{\uparrow\downarrow} = {}^{\downarrow\uparrow}A$ if and only if ${}^{\downarrow}M_t = \emptyset$.*

Analogous statements are valid for subset $B \subseteq M$. (See [2].)

Theorem 6 *Let an incidence structure $\mathcal{J} = (G, M, I)$ be reducible. If for a natural number $p > 2$ there exists at least two different components of some decomposition of $\mathcal{J}$ containing independent sets of cardinality p, then the incidence structure $\mathcal{J}^p = (G^p, M^p, I^p)$ is reducible.*

Proof Let A be a subset of G of cardinality $p > 2$. Then A is independent in $\mathcal{G}$ if and only if there exists $t \in T$ such that $A \subseteq G_t$ and A is independent in $\mathcal{G}_t = (G_t, \mathcal{P}_{G_t})$: Consider $t \in T$ and a substructure $\mathcal{J}_t = (G_t, M_t, I_t)$ in $\mathcal{J}$. Let $A \subseteq G_t$ be an independent set in $\mathcal{G}_t$. Then A is independent in $\mathcal{G}$ by Theorem 2 and Remark 1.

Conversely, let A be independent in $\mathcal{G}$. At the same time we suppose that A is not contained in any subset G_t, $t \in T$. Since $p > 2$, there exists such element $a \in A$ that the set A_a is not contained in any subset G_t too. According to (1) from Theorem 5 we get $A_a^{\uparrow\downarrow} = G$ and $a \in A_a^{\uparrow\downarrow}$. That is a contradiction. Hence $A \subseteq G_t$ for certain $t \in T$. We know that $X^A(a) = A_a^{\uparrow} - a^{\uparrow} \neq \emptyset$ for all $a \in A$ because A is independent in $\mathcal{G}$. Since $A_a \neq \emptyset$, we obtain $A_a^{\uparrow} = {}^{\uparrow}A_a$ by Theorem 5 (we write the operator $\uparrow$ to the left in $\mathcal{J}_t$) and $X^A(a) \subseteq M_t$. Thus A is independent in $\mathcal{G}_t$ by Theorem 2.

Similarly we can prove: Let B be a subset of M, $|B| = p > 2$. Then B is independent in $\mathcal{M}$ if and only if there exists $t \in T$ such that $B \subseteq M_t$ and B is independent in $\mathcal{M}_t = (M_t, \mathcal{P}_{M_t})$.

Consider a subset $T' \subseteq T$ such that $k \in T'$ iff $G_k^p \neq \emptyset$. If $A \in G^p$, then $A \in G_k^p$ for certain $k \in T'$ and $\overline{G^p} = \{G_k^p \mid k \in T'\}$ is a decomposition of G^p. We show that $\overline{M^p} = \{M_k^p \mid k \in T'\}$ is also a decomposition of M^p: Take an arbitrary $k \in T'$. Then (by the assumption) there exists a set $A \in G_k^p$ and (by Remark 3) ${}^{\uparrow}A \neq \emptyset$ in $\mathcal{J}_k^p$. Moreover, there exists $B \in M_k^p$ and thus $M_k^p \neq \emptyset$. If $B \in M^p$, then there exists $t \in T$ such that $B \subseteq M_t$ and $B \in M_t^p$. This yields (by Remark 3 again) that ${}^{\downarrow}B \neq \emptyset$ in $\mathcal{J}_t^p$ and there exists $A \in G_t^p$. It means that $G_t^p \neq \emptyset$ and $t \in T'$. Obviously $\overline{I^p} = \{I_k^p \mid k \in T'\}$ is a decomposition of I. We have obtained that $\mathcal{J}^p = \dot{\bigcup}_{k \in T'} \mathcal{J}_k^p$. Since $|T'| > 1$ (by our assumption), the incidence structure $\mathcal{J}^p$ is reducible. □

Example 2 Let us show an example of an irreducible incidence structure $\mathcal{J} = (G, M, I)$ such that the structure $\mathcal{J}^p$, $p > 2$, is reducible: An incidence structure $\mathcal{J} = (G, M, I)$, where $G = \{g_1, \ldots, g_5\}$, $M = \{m_1, \ldots, m_4\}$ is given by its incidence table (Table 3) and incidence graph (Figure 1).

I	m_1	m_2	m_3	m_4
g_1	–		–	
g_2	–	–		
g_3		–	–	
g_4	–		–	–
g_5		–		–

Table 3

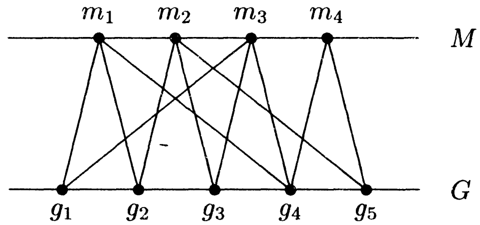

Figure 1

Obviously, $\mathcal{J}$ is irreducible. Consider the incidence structure of independent sets $\mathcal{J}^3 = (G^3, M^3, I^3)$. Then $G^3 = \{A_1, A_2, A_3, A_4\}$ and $M^3 = \{B_1, B_2, B_3\}$, where $A_1 = \{g_1, g_2, g_3\}$, $A_2 = \{g_2, g_3, g_4\}$, $A_3 = \{g_3, g_4, g_5\}$, $A_4 = \{g_2, g_4, g_5\}$, $B_1 = \{m_1, m_2, m_3\}$, $B_2 = \{m_2, m_3, m_4\}$, $B_3 = \{m_1, m_2, m_4\}$. Obviously $A_1 \; I^3 \; B_1$, $A_2 \; I^3 \; B_1$, $A_3 \; I^3 \; B_2$ and $A_4 \; I^3 \; B_3$. See the incidence table of the structure $\mathcal{J}^3$ (Table 4).

I^3	B_1	B_2	B_3
A_1	–		
A_2	–		
A_3		–	
A_4			–

Table 4

Let us consider substructures $\mathcal{J}_1^3 = (\{A_1, A_2\}, \{B_1\}, I_1)$, $\mathcal{J}_2^3 = (\{A_3\}, \{B_2\}, I_2)$, $\mathcal{J}_3^3 = (\{A_4\}, \{B_3\}, I_3)$ in $\mathcal{J}^3$. Then $\mathcal{J}^3 = \mathcal{J}_1^3 \dot\cup \mathcal{J}_2^3 \dot\cup \mathcal{J}_3^3$ and the incidence structure $\mathcal{J}^p$ is reducible.

Remark 5 Let $\mathcal{J}$ be an incidence structure. If the incidence structure $\mathcal{J}^p$ is irreducible, then the structures $\mathcal{J}^{p-1}$, $\mathcal{J}^{p+1}$ can be reducible.

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