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Testing Statistical Hypotheses in Deformation Measurement; One Generalization of the Scheffé Theorem *

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Abstract

A time course of the deformation is modelled by the multiepoch regression model. A test for the null hypothesis "a deformation does not occur" can result in the rejection of the null hypothesis and the problem arises which epochs differ significantly. An answer can be found by the help of a one generalization of the Scheffé theorem.

Key words: Multiepoch linear model, Scheffé theorem.

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Introduction

Let a deformation measurements be performed in m epochs and a state of the investigated object in each epoch be characterized by an l-dimensional vector of parameters; in the ith epoch this vector is denoted as $\beta_2^{(i)}$. Let the l-dimensional vector be a vector of 2D (or 3D) cartesian coordinate of a group of points. The coordinates of these points can change and thus they can be different in different epochs. Besides these points in an experiment there are points which coordinates

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do not change (stable points); the coordinates of them create a vector $\boldsymbol{\beta}_1$. The null hypothesis "the deformation does not occur" has been usually rejected and then it is necessary to decide between which epochs the estimators of coordinate vectors differ significantly and then to decide which points cause the rejecting the null hypothesis. The Scheffé theorem [4] is useful here, however it seems that a little different approach to it is more suitable in deformation measurements.

It is because of several facts. In practice the first task is to decide which epoch is the first that its coordinates are significantly changed with respect to the coordinates of the first epoch. Then it is necessary to identify a group of point which cause this fact. Then it is necessary to identify whether some functions of $((\boldsymbol{\beta}_1^{(1)})', \dots, (\boldsymbol{\beta}_2^{(m)})')$ differ significantly from zero (e.g. a distance between two points in the ith epoch and the same distance in the jth epoch), etc. Thus we should like to identify a whole group of parameters simultaneously which cause the rejection of the null hypothesis and not only several scalar functionals of them.

1 Notation, definitions and auxiliary statements

Let the model of the m-epoch measurements be

$$E(\mathbf{Y}) = \begin{pmatrix} \mathbf{X}_1, & \mathbf{X}_2, & \mathbf{0}, & \dots, & \mathbf{0}, & \mathbf{0} \\ \mathbf{X}_1, & \mathbf{0}, & \mathbf{X}_2, & \dots, & \mathbf{0}, & \mathbf{0} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \mathbf{X}_1, & \mathbf{0}, & \mathbf{0}, & \dots, & \mathbf{0}, & \mathbf{X}_2 \end{pmatrix} \begin{pmatrix} \boldsymbol{\beta}_1 \\ \boldsymbol{\beta}_2^{(1)} \\ \vdots \\ \boldsymbol{\beta}_2^{(m)} \end{pmatrix},$$

$$\mathrm{Var}(\mathbf{Y}) = \begin{pmatrix} \boldsymbol{\Sigma}, & \mathbf{0}, & \dots, & \mathbf{0}, & \mathbf{0} \\ \mathbf{0}, & \boldsymbol{\Sigma}, & \dots, & \mathbf{0}, & \mathbf{0} \\ \dots & \dots & \dots & \dots & \dots \\ \mathbf{0}, & \mathbf{0}, & \dots, & \boldsymbol{\Sigma}, & \mathbf{0} \\ \mathbf{0}, & \mathbf{0}, & \dots, & \mathbf{0}, & \boldsymbol{\Sigma} \end{pmatrix},$$

where $\mathbf{Y}$ is the mn-dimensional observation vector, $\mathbf{Y}' = (\mathbf{Y}_1', \dots, \mathbf{Y}_m')$, $\mathbf{Y}_i$ is the n-dimensional vector of the ith epoch, $i = 1, \dots, m$, $\mathbf{X}_1$ is a $n \times k$ design matrix belonging to the k-dimensional observation vector parameter $\boldsymbol{\beta}_1$ (an unknown coordinate vector of a group of points in 2D (or 3D) which are stable) with the rank $r(\mathbf{X}_1) = k < n$, $\mathbf{X}_2$ is a $n \times l$ design matrix belonging to the l-dimensional vector parameters $\boldsymbol{\beta}_2^{(s)}, s = 1, \dots, m$, with the rank $r(\mathbf{X}_2) = l < n$, $\boldsymbol{\Sigma}$ is the positive definite covariance matrix of $\mathbf{Y}_i$, $i = 1, \dots, m$. Further we assume $\boldsymbol{\beta}_1 \in R^k$ and $\boldsymbol{\beta}_2^{(i)} \in R^l, i = 1, \dots, m$, (i.e. no restrictions on the parameters are assumed). Here R^k means the k dimensional real vector space.

In the following $\mathbf{A}^+$ denotes the Moore–Penrose generalized inverse of the matrix $\mathbf{A}$ (i.e. $\mathbf{A}\mathbf{A}^+\mathbf{A} = \mathbf{A}$, $\mathbf{A}^+\mathbf{A}\mathbf{A}^+ = \mathbf{A}^+$, $\mathbf{A}\mathbf{A}^+ = (\mathbf{A}\mathbf{A}^+)'$, $\mathbf{A}^+\mathbf{A} = (\mathbf{A}^+\mathbf{A})'$ cf. [3]).

The symbol $\mathbf{M}_X$ means the projection matrix $\mathbf{I} - \mathbf{P}_X$, where $\mathbf{I}$ is the indentity matrix and $\mathbf{P}_X$ is the projection matrix (in the Euclidean norm) on the subspace

$\mathcal{M}(\mathbf{X}) = \{\mathbf{Xu} : \mathbf{u} \in R^k\}$. If the norm in R^n is $||\mathbf{x}|| = \sqrt{\mathbf{x}'\mathbf{V}^{-1}\mathbf{x}}$, $\mathbf{x} \in R^k$, where $\mathbf{V}$ is a positive definite matrix, then the symbol $\mathbf{P}_X^{V^{-1}}$ means the projection matrix on $\mathcal{M}(\mathbf{X})$ in this norm.

Lemma 1.1 *Let $(\boldsymbol{\beta}_2^{(.)})' = ((\boldsymbol{\beta}_2^{(1)})', \ldots, (\boldsymbol{\beta}_2^{(m)})')$. Then the best linear unbiased estimator of $(\boldsymbol{\beta}_1', (\boldsymbol{\beta}_2^{(.)})')'$ is*

$$\hat{\boldsymbol{\beta}}_1 = \left(\mathbf{X}_1'(\mathbf{M}_{X_2}\boldsymbol{\Sigma}\mathbf{M}_{X_2})^+\mathbf{X}_1\right)^{-1}\mathbf{X}_1'(\mathbf{M}_{X_2}\boldsymbol{\Sigma}\mathbf{M}_{X_2})^+\overline{\mathbf{Y}}$$

and

$$\hat{\boldsymbol{\beta}}_2^{(k)} = (\mathbf{X}_2'\boldsymbol{\Sigma}^{-1}\mathbf{X}_2)^{-1}\mathbf{X}_2'\boldsymbol{\Sigma}^{-1}(\mathbf{Y}_k - \mathbf{X}_1\hat{\boldsymbol{\beta}}_1),$$

$k = 1, \ldots, m$, *where* $\overline{\mathbf{Y}} = (1/m)\sum_{i=1}^m \mathbf{Y}_i$.

Further

$$\mathrm{Var}(\hat{\boldsymbol{\beta}}_1) = (1/m)\left(\mathbf{X}_1'(\mathbf{M}_{X_2}\boldsymbol{\Sigma}\mathbf{M}_{X_2})^+\mathbf{X}_1\right)^{-1},$$

$$\begin{aligned}\mathbf{W}_{k,k} = \mathrm{Var}(\hat{\boldsymbol{\beta}}_2^{(k)}) &= (\mathbf{X}_2'\boldsymbol{\Sigma}^{-1}\mathbf{X}_2)^{-1} \\ &+ (1/m)(\mathbf{X}_2'\boldsymbol{\Sigma}^{-1}\mathbf{X}_2)^{-1}\mathbf{X}_2'\boldsymbol{\Sigma}^{-1}\mathbf{X}_1\left(\mathbf{X}_1'(\mathbf{M}_{X_2}\boldsymbol{\Sigma}\mathbf{M}_{X_2})^+\mathbf{X}_1\right)^{-1} \\ &\times \mathbf{X}_1'\boldsymbol{\Sigma}^{-1}\mathbf{X}_2(\mathbf{X}_2'\boldsymbol{\Sigma}^{-1}\mathbf{X}_2)^{-1},\end{aligned}$$

$k = 1, \ldots, m$,

$$\begin{aligned}\mathbf{W}_{k,l} &= \mathrm{cov}(\hat{\boldsymbol{\beta}}_2^{(k)}, \hat{\boldsymbol{\beta}}_2^{(l)}) \\ &= -(1/m)\left(\mathbf{X}_1'(\mathbf{M}_{X_2}\boldsymbol{\Sigma}\mathbf{M}_{X_2})^+\mathbf{X}_1\right)^{-1}\mathbf{X}_1'\boldsymbol{\Sigma}^{-1}\mathbf{X}_2(\mathbf{X}_2'\boldsymbol{\Sigma}^{-1}\mathbf{X}_2)^{-1},\end{aligned}$$

$k, l = 1, \ldots, m$.

Proof It follows from [1]. □

Remark 1.2 If $\mathbf{Y} \sim N_{nm}(\mathbf{X}\boldsymbol{\beta}, \mathrm{Var}(\mathbf{Y}))$, where

$$\mathbf{X} = \begin{pmatrix} \mathbf{X}_1, & \mathbf{X}_2, & \mathbf{0}, & \ldots, & \mathbf{0}, & \mathbf{0} \\ \mathbf{X}_1, & \mathbf{0}, & \mathbf{X}_2, & \ldots, & \mathbf{0}, & \mathbf{0} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \mathbf{X}_1, & \mathbf{0}, & \mathbf{0}, & \ldots, & \mathbf{0}, & \mathbf{X}_2 \end{pmatrix}$$

and

$$\boldsymbol{\beta}' = (\boldsymbol{\beta}_1', (\boldsymbol{\beta}_2^{(1)})', \ldots, (\boldsymbol{\beta}_2^{(m)})'),$$

then

$$\begin{pmatrix} \hat{\boldsymbol{\beta}}_2^{(1)} \\ \vdots \\ \hat{\boldsymbol{\beta}}_2^{(m)} \end{pmatrix} \sim N_{lm}\left((\mathbf{I} \otimes \mathbf{X}_2)\boldsymbol{\beta}_2^{(.)}, \mathbf{W}\right).$$

where

$$\mathbf{W} = \begin{pmatrix} \mathbf{W}_{1,1}, & \mathbf{W}_{1,2}, & \ldots, & \mathbf{W}_{1,m} \\ \mathbf{W}_{2,1}, & \mathbf{W}_{2,2}, & \ldots, & \mathbf{W}_{2,m} \\ \ldots & \ldots & \ldots & \ldots \\ \mathbf{W}_{m,1}, & \mathbf{W}_{m,2}, & \ldots, & \mathbf{W}_{m,m} \end{pmatrix}.$$

Lemma 1.3 *Let* $\mathbf{Y} \sim N_{nm}(\mathbf{X}\boldsymbol{\beta}, \mathrm{Var}(\mathbf{Y}))$ *and the null hypothesis on* $\boldsymbol{\beta}_2^{(.)}$ *be*

$$H_0: \qquad \mathbf{H}\boldsymbol{\beta}_2^{(.)} + \mathbf{h} = \mathbf{0}$$

and an alternative hypothesis be

$$H_a: \qquad \mathbf{H}\boldsymbol{\beta}_2^{(.)} + \mathbf{h} \neq \mathbf{0},$$

where $q \times (ml)$ *matrix* $\mathbf{H}$ *is of the rank* $r(\mathbf{H}) = q$. *Then*

$$T(\mathbf{Y}) = (\mathbf{H}\hat{\boldsymbol{\beta}}_2^{(.)} + \mathbf{h})'[\mathbf{H}\,\mathrm{Var}(\hat{\boldsymbol{\beta}}_2^{(.)})\mathbf{H}']^{-1}(\mathbf{H}\hat{\boldsymbol{\beta}}_2^{(.)} + \mathbf{h}) \sim \chi_q^2(\delta),$$

where $\chi_q^2(\delta)$ *means the noncentral chi-square distribution with* q *degrees of freedom and the parameter of the noncentrality* δ *is*

$$\delta = (\mathbf{H}(\boldsymbol{\beta}_2^{(.)})^* + \mathbf{h})'[\mathbf{H}\,\mathrm{Var}(\hat{\boldsymbol{\beta}}_2^{(.)})\mathbf{H}']^{-1}(\mathbf{H}(\boldsymbol{\beta}_2^{(.)})^* + \mathbf{h}),$$

and $(\boldsymbol{\beta}_2^{(.)})^*$ *means the actual value of the vector* $\boldsymbol{\beta}_2^{(.)}$

Proof Cf. [2], section 3.5, p. 155. □

Remark 1.4 It is obvious how to use Lemma 1.3 for testing the hypothesis H_0. If $T(\mathbf{Y}) > \chi_q^2(0, 1-\alpha)$, then the hypothesis H_0 is rejected with the risk α. Here $\chi_q^2(0, 1-\alpha)$ is the $(1-\alpha)$-quantile of the central chi-square distribution with q degrees of freedom.

Let the vector $\mathbf{h}$ from Lemma 1.3 be $\mathbf{0}$ and the matrix $\mathbf{H}$ be

$$\mathbf{H} = \begin{pmatrix} \mathbf{I}, & -\mathbf{I}, & \mathbf{0}, \dots, & \mathbf{0}, & \mathbf{0} \\ \mathbf{0}, & \mathbf{I}, & -\mathbf{I}, \dots, & \mathbf{0}, & \mathbf{0} \\ \dots & \dots & \dots & \dots & \dots \\ \mathbf{0}, & \mathbf{0}, \dots, & & \mathbf{0}, \mathbf{I}, & -\mathbf{I} \end{pmatrix},$$

where $\mathbf{I}$ is the $l \times l$ identity matrix. Then the null hypothesis can be formulated "the deformation does not occur". Within a deformation measurement frequently this null hypothesis is rejected and therefore it is necessary to identify the parameter or the parameters which cause the rejection.

One answer can be obtained from the following Scheffé theorem

Scheffé theorem *Let* $\boldsymbol{\eta} \sim N_s(\boldsymbol{\mu}, \mathbf{V})$ *where* $\mathbf{V}$ *is a positive definite matrix. Let* $\mathcal{N}$ *be an* $r(< s)$*-dimesional subspace of* R^s. *Then*

$$P\left\{\forall\{\mathbf{p} \in \mathcal{N}\} : |\mathbf{p}'\boldsymbol{\eta} - \mathbf{p}'\boldsymbol{\mu}| < \sqrt{\chi_r^2(0, 1-\alpha)}\sqrt{\mathbf{p}'\mathbf{V}\mathbf{p}}\right\} = 1 - \alpha.$$

Proof Cf. [4], section 3.5. □

If $\hat{\boldsymbol{\beta}}_2^{(.)} \sim N_{ml}(\boldsymbol{\beta}_2^{(.)}, \mathbf{W})$ and $\mathbf{p}_{j,r|i} \in R^{ml}$ is a vector with the property $\mathbf{p}'_{j,r|i}\boldsymbol{\beta}_2^{(.)} = \{\boldsymbol{\beta}_2^{(j)}\}_i - \{\boldsymbol{\beta}_2^{(r)}\}_i$ (the difference between i-th components), then the nonrejection of the null hypothesis $\mathbf{H}\beta_2^{(\cdot)} = \mathbf{0}$ results in a fact that the mentioned difference must be smaller then $\sqrt{\chi^2_{l(m-1)}(0, 1-\alpha)}\sqrt{\mathbf{p}'_{j,r|i}\mathbf{W}\mathbf{p}_{j,r|i}}$ for all i, j and r. If the hypothesis is rejected, then there must exist a difference which exeeds the value $\sqrt{\chi^2_{l(m-1)}(0, 1-\alpha)}\sqrt{\mathbf{p}'_{j,r|i}\mathbf{W}\mathbf{p}_{j,r|i}}$. Thus it is identified the ith coordinate (i.e. we know the point) and the numbers j and r of the epochs which cause the rejection. Of course, such coordinates and epochs can be more than one.

This approach in $2D$ or $3D$ terminology has a disadvantege that it identifies only one coordinate of the critical point. It seems to be much more suitable to identify the point in the whole, i.e. the group of two ($2D$) or three ($3D$) coordinates simultaneously. The aim of the following section is to solve this problem.

2 A generalization

Lemma 2.1 *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be* $t \times s$ *and* $(t-s) \times s$, *respectively, matrices such that* $\begin{pmatrix}\mathbf{A}\\ \mathbf{B}\end{pmatrix}$ *is regular. Then* $\mathcal{M}(\mathbf{B}) = \{\mathbf{Bu} : \mathbf{u} \in R^s\} = \mathcal{M}(\mathbf{BK}_A)$, *where* $\mathbf{K}_A = \mathcal{K}er(\mathbf{A}) = \{\mathbf{u} : \mathbf{Au} = \mathbf{0}\}$.

Proof With respect to the equality

$$r\begin{pmatrix}\mathbf{A}\\ \mathbf{B}\end{pmatrix} = r(\mathbf{BK}_A) + r(\mathbf{A})$$

(cf. Lemma 7.1.2 in [3], p. 138) and our assumption on the regularity of the matrix $\begin{pmatrix}\mathbf{A}\\ \mathbf{B}\end{pmatrix}$, we have

$$r(\mathbf{BK}_A) = s - t \quad \& \quad \mathcal{M}(\mathbf{BK}_A) \subset \mathcal{M}(\mathbf{B}) \Rightarrow \mathcal{M}(\mathbf{B}) = \mathcal{M}(\mathbf{BK}_A). \qquad \square$$

The symbol $\mathbf{P}^{V^{-1}}_{\mathcal{K}er(A)}$ means the projection matrix on the subspace $\mathcal{K}er(\mathbf{A})$ in the $\mathbf{V}^{-1}$-norm ($||\mathbf{v}||_{V^{-1}} = \sqrt{\mathbf{v}\mathbf{V}^{-1}\mathbf{v}}$).

Theorem 2.2 *Let* $\boldsymbol{\eta} \sim N_s(\boldsymbol{\mu}, \mathbf{V})$, *where* $\mathbf{V}$ *is a positive definite matrix. Let* $\mathcal{A}$ *be a class of* $t \times s$ *matrices* $\mathbf{A}$ *of the rank* $r(\mathbf{A}) = t < s$. *Then*

$$P\{(\boldsymbol{\eta}-\boldsymbol{\mu})'\mathbf{V}^{-1}(\boldsymbol{\eta}-\boldsymbol{\mu}) \leq \chi^2_s(0, 1-\alpha)\} = 1-\alpha$$

$$\Leftrightarrow P\{\forall\{\mathbf{A} \in \mathcal{A}\} : [\mathbf{A}(\boldsymbol{\eta}-\boldsymbol{\mu})]'(\mathbf{AVA}')^{-1}\mathbf{A}(\boldsymbol{\eta}-\boldsymbol{\mu}) \leq \chi^2_s(0, 1-\alpha)\} = 1-\alpha.$$

Proof Let $\mathbf{B}$ be an arbitrary $(s-t) \times s$ matrix with the property that $\begin{pmatrix}\mathbf{A}\\ \mathbf{B}\end{pmatrix}$ is regular. Further

$$\mathbf{v}'\mathbf{V}^{-1}\mathbf{v} = \left[\begin{pmatrix}\mathbf{A}\\ \mathbf{B}\end{pmatrix}\mathbf{v}\right]'\left[\begin{pmatrix}\mathbf{A}\\ \mathbf{B}\end{pmatrix}\mathbf{V}(\mathbf{A}', \mathbf{B}')\right]^{-1}\begin{pmatrix}\mathbf{A}\\ \mathbf{B}\end{pmatrix}\mathbf{v}$$

$$= \left[\mathbf{T}\begin{pmatrix}\mathbf{A}\\ \mathbf{B}\end{pmatrix}\mathbf{v}\right]'\left[\mathbf{T}\begin{pmatrix}\mathbf{A}\\ \mathbf{B}\end{pmatrix}\mathbf{V}(\mathbf{A}', \mathbf{B}')\mathbf{T}'\right]^{-1}\mathbf{T}\begin{pmatrix}\mathbf{A}\\ \mathbf{B}\end{pmatrix}\mathbf{v},$$

where

$$\mathbf{T} = \begin{pmatrix} \mathbf{I}, & \mathbf{0} \\ -\mathbf{BVA}'(\mathbf{AVA}')^{-1}, & \mathbf{I} \end{pmatrix}.$$

Thus we obtain

$$\begin{aligned} &\mathbf{v}'\mathbf{V}^{-1}\mathbf{v} = \\ &= (\mathbf{Av})'(\mathbf{AVA}')^{-1}\mathbf{Av} + \{\mathbf{B}[\mathbf{I} - \mathbf{VA}'(\mathbf{AVA}')^{-1}\mathbf{A}]\mathbf{v}\}' \\ &\quad \times[\mathbf{BVB}' - \mathbf{BVA}'(\mathbf{AVA}')^{-1}\mathbf{AVB}']^{-1}\mathbf{B}[\mathbf{I} - \mathbf{VA}'(\mathbf{AVA}')^{-1}\mathbf{A}]\mathbf{v} \\ &= (\mathbf{Av})'(\mathbf{AVA}')^{-1}\mathbf{Av} + (\mathbf{BP}^{V^{-1}}_{\mathcal{K}er(A)}\mathbf{v})'[\mathbf{B}(\mathbf{M}_A\mathbf{V}^{-1}\mathbf{M}_A)^{+}\mathbf{B}']^{-1}\mathbf{BP}^{V^{-1}}_{\mathcal{K}er(A)}\mathbf{v}. \end{aligned}$$

Since the term $(\mathbf{BP}^{V^{-1}}_{\mathcal{K}er(A)}\mathbf{v})'[\mathbf{B}(\mathbf{M}_A\mathbf{V}^{-1}\mathbf{M}_A)^{+}\mathbf{B}']^{-1}\mathbf{BP}^{V^{-1}}_{\mathcal{K}er(A)}\mathbf{v}$ is non-negative, obviously

$$\mathbf{v}'\mathbf{V}^{-1}\mathbf{v} \le c^2 \quad \Rightarrow \quad (\mathbf{Av})'(\mathbf{AVA}')^{-1}\mathbf{Av} \le c^2.$$

If $\mathbf{u} \in \mathcal{M}(\mathbf{A}) = \{\mathbf{Ax} : \mathbf{x} \in R^s\} = R^t$ satisfies the inequality

$$\mathbf{u}'(\mathbf{AVA}')^{-1}\mathbf{u} \le c^2,$$

then there exists a vector $\mathbf{v}$ such that

$$\mathbf{v}'\mathbf{V}^{-1}\mathbf{v} \le c^2, \qquad \mathbf{u} = \mathbf{Av}.$$

It is given by the following.

Let $\mathbf{v}$ be a solution of the equation $\mathbf{Av} = \mathbf{u}$. The class of all solutions is given by the class

$$\mathcal{V}_u = \{\mathbf{v} : \mathbf{v} = \mathbf{v}_0 + \mathbf{P}^{V^{-1}}_{\mathcal{K}er(A)}\mathbf{t}, \mathbf{t} \in R^s\},$$

where $\mathbf{v}_0$ is a particular solution. With respect to our assumption

$$\forall\{\mathbf{v} \in \mathcal{V}_u\}(\mathbf{Av})'(\mathbf{AVA}')^{-1}\mathbf{Av} \le c^2.$$

If the last inequality holds with "=", then the vector

$$\mathbf{v} = \mathbf{v}_0 - \mathbf{P}^{V^{-1}}_{\mathcal{K}er(A)}\mathbf{v}_0,$$

satisfies

$$\mathbf{v}'\mathbf{V}^{-1}\mathbf{v} = (\mathbf{Av})'(\mathbf{AVA}')^{-1}\mathbf{Av} = c^2$$

(notice that $\mathbf{P}^{V^{-1}}_{\mathcal{K}er(A)} = \mathbf{I} - \mathbf{VA}'(\mathbf{AVA}')^{-1}\mathbf{A}$).

If $\mathbf{u} \in \mathcal{M}(\mathbf{A}) = R^t$ and

$$\mathbf{u}'(\mathbf{AVA}')^{-1}\mathbf{u} = d^2 < c^2,$$

then for any $\mathbf{v} \in \mathcal{V}_u$ satisfying the equality

$$\left(\mathbf{P}^{V^{-1}}_{\mathcal{K}er(A)}\mathbf{v}\right)'\mathbf{B}'[\mathbf{B}(\mathbf{M}_A\mathbf{V}^{-1}\mathbf{M}_A)^{+}\mathbf{B}']^{-1}\mathbf{BP}^{V^{-1}}_{\mathcal{K}er(A)}\mathbf{v} = c^2 - d^2,$$

it holds

$$\mathbf{v}'\mathbf{V}^{-1}\mathbf{v} < c^2.$$

Such vectors $\mathbf{v}$ must exist, since (with respect to Lemma 2.1)

$$R^{s-t} = \mathcal{M}(\mathbf{B}) = \mathcal{M}(\mathbf{BK}_A) = \mathbf{BP}^{V^{-1}}_{\mathcal{K}er(A)}\mathcal{V}_u.$$

Thus for any $\mathbf{A} \in \mathcal{A}$,

$$\{\mathbf{Av} : \mathbf{v}'\mathbf{V}^{-1}\mathbf{v} \leq c^2\} = \{\mathbf{u} : \mathbf{u}'(\mathbf{AVA}')^{-1}\mathbf{u} \leq c^2\}.$$

Let

$$\mathcal{E} = \{\mathbf{v} : \mathbf{v}'\mathbf{V}^{-1}\mathbf{v} \leq c^2\}$$

and

$$\mathcal{E}_A = \{\mathbf{u} : \mathbf{u}'(\mathbf{AVA}')^{-1}\mathbf{u} \leq c^2\} \quad \text{for} \quad \mathbf{A} \in \mathcal{A};$$

i.e. $\{\mathbf{Av} : \mathbf{v} \in \mathcal{E}\} = \mathcal{E}_A$.

If $\mathbf{x} \notin \mathcal{E}$, then there exists $\mathbf{A} \in \mathcal{A}$ such that $\mathbf{Ax} \notin \mathcal{E}_A$.

Let $\mathbf{x}'\mathbf{V}^{-1}\mathbf{x} > c^2$. Let

$$\mathbf{A} = \begin{pmatrix} \mathbf{x}'\mathbf{V}^{-1} \\ \mathbf{N} \end{pmatrix},$$

where $\mathbf{N}$ is an $(s-1) \times s$ matrix with the property $\mathbf{Nx} = \mathbf{0}$. Then

$$(\mathbf{AVA}')^{-1} = \begin{pmatrix} (\mathbf{x}'\mathbf{V}^{-1}\mathbf{x})^{-1}, & \mathbf{0} \\ \mathbf{0}, & (\mathbf{NVN}')^{-1} \end{pmatrix}$$

and

$$(\mathbf{Ax})'(\mathbf{AVA}')^{-1}\mathbf{Ax} = \mathbf{x}'\mathbf{V}^{-1}\mathbf{x} > c^2. \qquad \square$$

Remark 2.3 Let $i = 1, \ldots, m-1, \quad j = 2, \ldots, m \quad j > i$ and $\mathbf{A}$ be $l \times (m-1)l$ matrix of the form

$$\mathbf{A}_{i,j} = (\mathbf{0}_1, \ldots, \mathbf{0}_{i-1}, \mathbf{1}' \otimes \mathbf{I}, \mathbf{0}_{j+1}, \ldots, \mathbf{0}_{m-1}),$$

where $\mathbf{1} = (1, \ldots, 1)' \in R^{j-i+1}$ and $\mathbf{I}$ is $l \times l$ identity matrix. Then

$$\mathbf{A}_{i,j}\mathbf{H}\boldsymbol{\beta}_2^{(\cdot)} = \boldsymbol{\beta}_2^{(j)} - \boldsymbol{\beta}_2^{(i)}, \quad i - 1, \ldots, m-1, \quad j = 2, \ldots, m \quad i < j.$$

Corollary 2.4 *Let in the model from Lemma 1.1 and Remark 1.2 there exist a pair $(i,j), i \in \{1, \ldots, m-1\}, j \in \{2, \ldots, m\}$ with the property*

$$(\hat{\boldsymbol{\beta}}_2^{(j)} - \hat{\boldsymbol{\beta}}_2^{(i)})'(\mathbf{A}_{i,j}\mathbf{HWH}'\mathbf{A}'_{i,j})^{-1}(\hat{\boldsymbol{\beta}}_2^{(j)} - \hat{\boldsymbol{\beta}}_2^{(i)}) > \chi^2_{l(m-1)}(0, 1-\alpha).$$

Then the null hypothesis H_0 is rejected with the risk which does not exeed the value α ("the ith epoch differs significantly from the jth epoch").

The matrix $\mathbf{A}_{i,j}\mathbf{HWH}'\mathbf{A}'_{i,j}$ can be expressed as $\mathbf{W}_{i,i} + \mathbf{W}_{j,j} - \mathbf{W}_{i,j} - \mathbf{W}_{j,i}$.

Proof With respect to Theorem 2.2

$$P\{(\mathbf{H}\hat{\beta}_2^{(.)})'(\mathbf{HWH}')^{-1}\mathbf{H}\hat{\beta}_2^{(.)} > \chi^2_{l(m-1)}(0,1-\alpha)|H_0\}$$
$$> P\{(\mathbf{A}_{i,j}\mathbf{H}\hat{\beta}_2^{(.)})'(\mathbf{A}_{i,j}\mathbf{HWH}'\mathbf{A}'_{i,j})^{-1}\mathbf{A}_{i,j}\mathbf{H}\hat{\beta}_2^{(.)} > \chi^2_{l(m-1)}(0,1-\alpha)|H_0\}$$
$$= P\{(\hat{\beta}_2^{(j)} - \hat{\beta}_2^{(i)})'(\mathbf{W}_{i,i} + \mathbf{W}_{j,j} - \mathbf{W}_{i,j} - \mathbf{W}_{j,i})^{-1}(\hat{\beta}_2^{(j)} - \hat{\beta}_2^{(i)})$$
$$> \chi^2_{l(m-1)}(0,1-\alpha)|H_0\}. \qquad \square$$

From the previous text it is obvious that a procedure for an identifying some break in the time course of investigated deformations can be based on several functionals simultaneously (i.e on the basis of a suitably chosen matrix **A**).

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References

[1] Kubáčková, L.: *The locally best estimators of the first and second order parameters in epoch regression models.* Applications of Mathematics **37** (1992), 1–12.

[2] Rao C. R.: *Linear Statistical Inference and Its Applications.* J. Wiley, New York–London–Sydney, 1965.

[3] Rao C. R., Mitra K. S.: *Generalized Inverse of Matrices and Its Application.* J. Wiley, New York–London–Sydney–Toronto, 1971.

[4] Scheffé H.: *The Analysis of Variance.* J. Wiley, New York, 1959.