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A Note on Existence of Bounded Solutions of an n-th Order ODE on the Real Line

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Abstract

The criteria for entirely bounded, periodic and antiperiodic weak solutions of a scalar quasi-linear differential equation of an n-th order are developed via an asymptotic boundary value problem. The estimates of entirely bounded solutions (and their derivatives) of associated linearized equations are employed for this purpose.

Key words: Asymptotic boundary value problems, boundedness, periodicity.

1991 Mathematics Subject Classification: 34B15, 34C11, 34C25

1 Introduction

In this paper, we deal with the scalar differential equation of the n-th order in the form

$$x^{(n)} + \sum_{j=1}^{n} a_j x^{(n-j)} = f(t, x, \ldots, x^{(n-1)}), \tag{1}$$

where $a_1, \ldots, a_n$ are real constants and $f : \mathbb{R}^{n+1} \to \mathbb{R}$ is a given function.

By a weak solution of the equation (1) we mean any function which has $(n-1)$-st locally absolutely continuous derivative and satisfies (1) a.e. in $\mathbb{R}$.

We say that a function $\varphi : \mathbb{R} \to \mathbb{R}$ is entirely bounded if the inequality

$$\sup_{t\in\mathbb{R}} |\varphi(t)| < +\infty$$

holds for φ.

There are only few results concerning bounded solutions of higher-order differential equations (see e.g. [2], [3] and the references therein). In the beginning of this century, P. Bohl proved the boundedness theorem for systems of ODEs (see e.g. [7]). From this theorem, one can immediately obtain the following consequence for the equation (1):

Theorem 1 *Assume that each root of the polynomial $\lambda^n + \sum_{j=1}^{n} a_j \lambda^{n-j}$ has a nonzero real part. Suppose that f is continuous on $\mathbb{R}^{n+1}$, i.e. $f \in C(\mathbb{R}^{n+1}, \mathbb{R})$, and satisfies the following conditions*

(a)
$$\sup_{t\in\mathbb{R}} |f(t,0)| < +\infty\,,$$

(b) there exists a sufficiently small constant L such that, for every $t \in \mathbb{R}$, the inequality
$$|f(t,u) - f(t,v)| \le L\ ||u - v||$$
holds for every $u, v \in \mathbb{R}^n$.

Then the equation (1) admits at least one entirely bounded solution.

Note that the Lipschitz condition (b) indicates Banach's contractive principle as a main tool for proving the Bohl theorem. It is therefore natural to expect that this condition can be weakened via results which are based on another fixed point principles, e.g. the Tychonoff theorem. The main aim of our investigation consists in presentation the more general conditions for the existence of an entirely bounded weak solution of the equation (1) of that kind. Our results can be directly deduced from the existence of appropriate a priori bounds of the solutions of an associated linearized equation related to (1). Therefore, our attention will be paid to the linear equation.

Let us remark that a similar approach related to systems of the first order can be found in [4].

The following very special case of the result in [6] will be essential for our aim.

Proposition 1 *Consider the problem*

$$x^{(n)} + \sum_{j=1}^{n} a_j x^{(n-j)} = f(t, x, \dots, x^{(n-1)}), \quad x \in S, \tag{2}$$

where $a_1, \dots, a_n$ are real constants, $f : \mathbb{R}^{n+1} \to \mathbb{R}$ is a Carathéodory function and S is a nonempty closed convex subset of the Fréchet space $C^{(n-1)}(\mathbb{R}, \mathbb{R})$ which is bounded in $C(\mathbb{R}, \mathbb{R})$. Then problem (2) admits at least one entirely bounded weak solution, provided the following conditions are satisfied:

(*i*) *for any* $q \in S$ *there exists a unique entirely bounded weak solution of the linearized problem*

$$x^{(n)} + \sum_{j=1}^{n} a_j x^{(n-j)} = f(t, q(t), \ldots, q^{(n-1)}(t)), \quad x \in S, \tag{3}$$

(*ii*) *there exists locally Lebesgue integrable function* $\alpha : \mathbb{R} \to \mathbb{R}$ *such that the inequality*

$$|f(t, q(t), \ldots, q^{(n-1)}(t))| \leq \alpha(t), \quad a.e. \ in \ \mathbb{R},$$

holds for every $q \in S$.

2 Estimates for bounded solutions of linear equations

As we could see in Proposition 1, estimates of entirely bounded solutions (and their derivatives) of the equation in (3) will be crucial. Consider therefore the hyperbolic equation (i.e. the real parts of roots of the associated polynomial (6) below are nonzero)

$$x^{(n)} + \sum_{j=1}^{n} a_j x^{(n-j)} = p(t)\,, \tag{4}$$

where $p : \mathbb{R} \to \mathbb{R}$ is a measurable function such that

$$\operatorname*{ess\,sup}_{t \in \mathbb{R}} |p(t)| =: P < +\infty\,. \tag{5}$$

Writing roots of the characteristic polynomial

$$\lambda^n + \sum_{j=1}^{n} a_j \lambda^{n-j} \tag{6}$$

in the form $\lambda_j = \alpha_j + i\beta_j$, $\alpha_j \in \mathbb{R}$, $\alpha_j \neq 0$, $\beta_j \in \mathbb{R}$ for $j = 1, \ldots, n$, we may denote

$$\Lambda_j := (+\infty)\alpha_j\,, \quad j = 1, \ldots, n.$$

In the sequel, we proceed by the similar way as in [5].

Lemma 1 *The function* $\varphi : \mathbb{R} \to \mathbb{R}$ *given by the formula*

$$\varphi(t) = e^{\lambda_1 t} \int_{\Lambda_1}^{t} e^{(\lambda_2 - \lambda_1)t_1} \int_{\Lambda_2}^{t_1} e^{(\lambda_3 - \lambda_2)t_2} \int_{\Lambda_3}^{t_2} \cdots$$

$$\cdots \int_{\Lambda_{n-1}}^{t_{n-2}} e^{(\lambda_n - \lambda_{n-1})t_{n-1}} \int_{\Lambda_n}^{t_{n-1}} e^{-\lambda_n t_n} p(t_n)\, dt_n \ldots dt_1 \tag{7}$$

represents a weak solution of the equation (4).

Proof Obviously, φ has a locally absolutely continuous $(n-1)$-st derivative.

By the straightforward computation, one can verify that for $n = 1$ the function

$$\varphi(t) = e^{\lambda_1 t} \int_{\Lambda_1}^{t} e^{-\lambda_1 t_1} p(t_1)\, dt_1$$

satisfies the differential equation $x' + a_1 x = p(t)$ (with the coefficient $a_1 = -\lambda_1$) a.e. in $\mathbb{R}$.

Now suppose that the assertion of Lemma 1 holds for $n = k - 1 \geq 1$ and show its validity for $n = k$.

It follows from the hypothesis that the function φ given by (7), for $n = k$, and taking the form

$$\varphi(t) = e^{\lambda_1 t} \int_{\Lambda_1}^{t} \dots \int_{\Lambda_{k-1}}^{t_{k-2}} e^{-\lambda_{k-1} t_{k-1}} \left(e^{\lambda_k t_{k-1}} \int_{\Lambda_k}^{t_{k-1}} e^{-\lambda_k t_k} p(t_k)\, dt_k \right) dt_{k-1} \dots dt_1 ,$$

obeys the differential equation of the $(k-1)$-st order

$$x^{(k-1)} + \sum_{j=1}^{k-1} b_j x^{(k-1-j)} = e^{\lambda_k t} \int_{\Lambda_k}^{t} e^{-\lambda_k t_k} p(t_k)\, dt_k , \tag{8}$$

with the coefficients uniquely determined by the relation

$$\lambda^{k-1} + \sum_{j=1}^{k-1} b_j \lambda^{k-1-j} = \prod_{j=1}^{k-1} (\lambda - \lambda_j). \tag{9}$$

The function $\varphi^{(k-1)}$ is locally absolutely continuous. Therefore, φ satisfies the equation

$$x^{(k)} + \sum_{j=1}^{k-1} b_j x^{(k-j)} = \lambda_k e^{\lambda_k t} \int_{\Lambda_k}^{t} e^{-\lambda_k t_k} p(t_k)\; dt_k + p(t) \tag{10}$$

a.e. in $\mathbb{R}$. Subtracting the λ_k multiple of (8) from (10) we obtain that φ is a weak solution of the differential equation

$$x^{(k)} + (b_1 - \lambda_k) x^{(k-1)} + \sum_{j=2}^{k-1} (b_j - \lambda_k b_{j-1}) x^{(k-j)} - \lambda_k b_{k-1} = p(t). \tag{11}$$

A trivial verification shows that

$$\lambda^k + (b_1 - \lambda_k)\lambda^{k-1} + \sum_{j=2}^{k-1} (b_j - \lambda_k b_{j-1}) \lambda^{k-j} - \lambda_k b_{k-1} = \prod_{j=1}^{k} (\lambda - \lambda_j)$$

holds for the characteristic polynomial of (11), which completes the proof. □

Lemma 2 *The equation (4) has exactly one entirely bounded solution. This solution is given by the formula (7) and satisfies the inequality*

$$\sup_{t\in\mathbb{R}} |\varphi(t)| \leq \frac{P}{|\alpha_1 \ldots \alpha_n|}. \tag{12}$$

Proof Uniqueness follows immediately from the hyperbolicity of (4). For $n = 1$, we have

$$|\varphi(t)| \leq |e^{\lambda_1 t}| \left| \int_{\Lambda_1}^{t} |e^{-\lambda_1 t_1}| \, |p(t_1)| dt_1 \right|$$
$$\leq \frac{P}{|\alpha_1|} e^{\alpha_1 t} \lim_{s\to\Lambda_1} \left| e^{-\alpha_1 t} - e^{-\alpha_1 s} \right| = \frac{P}{|\alpha_1|}, \quad t \in \mathbb{R}. \tag{13}$$

Now let (12) hold for $n = k - 1 \geq 1$. The function

$$\varphi(t) = e^{\lambda_1 t} \int_{\Lambda_1}^{t} \ldots \int_{\Lambda_{k-1}}^{t_{k-2}} e^{-\lambda_{k-1} t_{k-1}} \left(e^{\lambda_k t_{k-1}} \int_{\Lambda_k}^{t_{k-1}} e^{-\lambda_k t_k} p(t_k)\, dt_k \right) dt_{k-1} \ldots dt_1$$

is then an entirely bounded solution of the equation (8). Applying the inductive assumption and (13), we can write

$$\sup_{t\in\mathbb{R}} |\varphi(t)| \leq \frac{\sup_{t\in\mathbb{R}} \left| e^{\lambda_k t} \int_{\Lambda_k}^{t} e^{-\lambda_k t_k} p(t_k) dt_k \right|}{|\alpha_1 \ldots \alpha_{k-1}|} \leq \frac{1}{|\alpha_1 \ldots \alpha_{k-1}|} \frac{P}{|\alpha_k|},$$

which is the desired estimate. □

Consequence 1 *If the characteristic polynomial (6) has only real nonzero roots, then the estimate*

$$\sup_{t\in\mathbb{R}} |\varphi(t)| \leq \frac{P}{|a_n|} \tag{14}$$

holds for φ in (7).

Proof This result follows immediately from the Vieta formula

$$a_n = (-1)^n \prod_{j=1}^{n} \lambda_j \,.$$ □

Consequence 2 *Assume that the forcing term p in the hyperbolic equation (4) is a T-periodic function. Then (4) admits exactly one T-periodic weak solution. This solution is given by the formula (7). If p is a T-antiperiodic function, then the function φ in (7) represents exactly one T-antiperiodic weak solution of the equation (4).*

Proof Consider the unique entirely bounded solution φ of (4). If p is a T-periodic, then the function ψ, $\psi(t) := \varphi(t + T)$ represents also the entirely bounded weak solution of (4), and we conclude that $\psi = \varphi$, i.e. T-periodicity of φ. Writing $\chi(t) := -\varphi(t + T)$, we derive by the same manner the second assertion. □

3 Estimates of derivatives

As we could see, there exists exactly one entirely bounded solution of the equation (4). Consequently, φ given by the formula (7) can take the form

$$\varphi(t) = e^{\lambda_{i_1} t} \int_{\Lambda_{i_1}}^{t} e^{(\lambda_{i_2}-\lambda_{i_1})t_1} \int_{\Lambda_{i_2}}^{t_1} e^{(\lambda_{i_3}-\lambda_{i_2})t_2} \int_{\Lambda_{i_3}}^{t_2} \cdots$$
$$\cdots \int_{\Lambda_{i_{n-1}}}^{t_{n-2}} e^{(\lambda_{i_n}-\lambda_{i_{n-1}})t_{n-1}} \int_{\Lambda_{i_n}}^{t_{n-1}} e^{-\lambda_{i_n} t_n} p(t_n)\, dt_n \ldots dt_1\,, \tag{15}$$

where $(i_1, i_2, \ldots, i_n)$ is any permutation of $(1, 2, \ldots, n)$. Denoting briefly the right-hand side of (15) by $[i_1, i_2, \ldots, i_n]$, we can write (12) in the form

$$\sup_{t\in\mathbb{R}} |[i_1, i_2, \ldots, i_n]| \leq \frac{P}{|\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_n}|}\,. \tag{16}$$

Lemma 3

$$\frac{d([i_1, i_2, \ldots, i_n])}{dt} = \lambda_{i_1}[i_1, i_2, \ldots, i_n] + [i_2, \ldots, i_n]. \tag{17}$$

Proof The formula (17) follows from the rule for derivating the product. □

Lemma 4

$$\frac{d^m([1,2,\ldots,n])}{dt^m} = [m+1, m+2, \ldots, n] + \sum_{c_1=1}^{m} \lambda_{c_1}[c_1, m+1, m+2, \ldots, n]$$
$$+ \sum_{\substack{c_1,c_2=1\\ c_1<c_2}}^{m} \lambda_{c_1}\lambda_{c_2}[c_1, c_2, m+1, m+2, \ldots, n] + \cdots$$
$$+ \sum_{\substack{c_1,c_2,\ldots,c_p=1\\ c_1<c_2<\cdots<c_p}}^{m} (\prod_{j=1}^{p} \lambda_{c_j})[c_1, c_2, \ldots, c_p, m+1, m+2, \ldots, n] + \cdots$$
$$+ (\prod_{j=1}^{m} \lambda_j)[1, 2, \ldots, n], \quad m = 1, \ldots, n-1. \tag{18}$$

Proof We will proceed by the mathematical induction method. Since for $m = 1$ the assertion follows from Lemma 3 (see (17)), we want to show its validity for $m = k$, provided it is true for $m = k-1$ $((n-2) \geq (k-1) \geq 1)$.

Hence,

$$\frac{d^k([1,2,\ldots,n])}{dt^k} = \frac{d}{dt}\left(\frac{d^{k-1}([1,2,\ldots,n])}{dt^{k-1}}\right) = ([k+1, k+2, \ldots, n])$$
$$+ \left(\lambda_k[k, k+1 \ldots, n] + \sum_{c_1=1}^{k-1} \lambda_{c_1}[c_1, k+1, k+2 \ldots, n]\right)$$

$$+\sum_{c_1=1}^{k-1}\lambda_{c_1}\lambda_k[c_1,k,k+1,\dots,n]+\cdots$$
$$+\sum_{\substack{c_1,c_2,\dots,c_{p-1}=1\\ c_1<c_2<\dots<c_{p-1}}}^{k-1}(\prod_{j=1}^{p-1}\lambda_{c_j})[c_1,c_2,\dots,c_{p-1},k+1,k+2,\dots,n]$$
$$+\left(\sum_{\substack{c_1,c_2,\dots,c_{p-1}=1\\ c_1<c_2<\dots<c_{p-1}}}^{k-1}(\prod_{j=1}^{p-1}\lambda_{c_j})\lambda_k[c_1,c_2,\dots,c_{p-1},k,k+1,\dots,n]\right.$$
$$\left.+\sum_{\substack{c_1,c_2,\dots,c_p=1\\ c_1<c_2<\dots<c_p}}^{k-1}(\prod_{j=1}^{p}\lambda_{c_j})[c_1,c_2,\dots,c_p,k+1,\dots,n]\right)$$
$$+\sum_{\substack{c_1,c_2,\dots,c_p=1\\ c_1<c_2<\dots<c_p}}^{k-1}(\prod_{j=1}^{p}\lambda_{c_j})\lambda_k[c_1,c_2,\dots,c_p,k,\dots,n]+\cdots$$
$$+\left((\prod_{j=1}^{k-1}\lambda_{c_j})\lambda_k[1,2,\dots,n]\right)=[k+1,k+2,\dots,n]$$
$$+\sum_{c_1=1}^{k}\lambda_{c_1}[c_1,k+1,k+2,\dots,n]+\cdots$$
$$+\sum_{\substack{c_1,c_2,\dots,c_p=1\\ c_1<c_2<\dots<c_p}}^{k}(\prod_{j=1}^{p}\lambda_{c_j})[c_1,c_2,\dots,c_p,k+1,k+2,\dots,n]+\cdots$$
$$+(\prod_{j=1}^{k}\lambda_j)[1,2,\dots,n]\,. \qquad \square$$

Lemma 5 *If the characteristic polynomial (6) has only real nonzero roots, then the derivatives of φ in (7) satisfy the estimates*

$$\sup_{t\in\mathbb{R}}\left|\frac{d^m\varphi(t)}{dt^m}\right|\le\frac{2^mP}{|a_n|}\prod_{j=1}^{m}|\alpha_j|\,,\quad m=1,\dots,n-1. \tag{19}$$

Proof At first, we can estimate each term in (18) (see (16))

$$\left|\left(\prod_{j=1}^{p}\alpha_{c_j}\right)[c_1,c_2,\dots,c_p,m+1,m+2,\dots,n]\right|\le$$
$$\le\left|\left(\prod_{j=1}^{p}\alpha_{c_j}\right)\right|\frac{P}{|(\prod_{j=1}^{p}\alpha_{c_j})(\prod_{j=m+1}^{n}\alpha_j)|}=\frac{P}{|\prod_{j=m+1}^{n}\alpha_j|}=\frac{P}{|a_n|}\left|\prod_{j=1}^{m}\alpha_j\right|.$$

Now we can sum these 2^m estimates to establish the desired formula. □

Remark 1 Independence of the estimate (19), under the permutation of the roots, is evident and so we have

$$\sup_{t\in\mathbb{R}}\left|\frac{d^m\varphi(t)}{dt^m}\right| \le \frac{2^m P}{|a_n|}\prod_{j=1}^{m}|\alpha_{i_j}|\,, \quad m=1,\dots,n-1. \tag{20}$$

Lemma 6 *Let all roots of the characteristic polynomial (6) be negative and denote $a_0 := 1$. Then*

$$\sup_{t\in\mathbb{R}}\left|\frac{d^m\varphi(t)}{dt^m}\right| \le \frac{2^m a_m P}{\binom{n}{m}a_n}\,, \quad m=0,\dots,n-1. \tag{21}$$

Proof Consequence 1 ensures the validity of (21) for $m=0$. Fix $m\in\{1,\dots,n-1\}$. Then the inequality in (20) can be written in the form

$$\sup_{t\in\mathbb{R}}\left|\frac{d^m\varphi(t)}{dt^m}\right| \le \frac{2^m P}{a_n}(-1)^m\prod_{j=1}^{m}\alpha_{i_j}. \tag{22}$$

We have $\binom{n}{m}$ choices of the roots $\alpha_{i_1}\alpha_{i_2}\dots\alpha_{i_m}$ for n roots of (6). Sum the $\binom{n}{m}$ inequalities of the type (22) and divide by $\binom{n}{m}$. Now, to verify the lemma, it is sufficient to apply the Vieta formula

$$\sum_{\substack{c_1,\dots,c_m=1\\ c_1<\dots<c_m}}^{n}(-1)^m\prod_{j=1}^{m}\alpha_{c_j}=a_m\,. \qquad \square$$

Lemma 7 *Consider the sequence of the "shifted polynomials"*

$$\lambda^{n-p}+\sum_{j=1}^{n-p}a_j\lambda^{n-p-j}\,, \quad p=0,1,\dots,n-1$$

and assume that each of them has only real nonzero roots. Then the estimate

$$\sup_{t\in\mathbb{R}}\left|\frac{d^m\varphi(t)}{dt^m}\right| \le \frac{2^m P}{|a_{n-m}|}\,, \quad m=0,\dots,n-1 \tag{23}$$

holds for φ in (7).

Proof As we could see, the derivatives of the solution φ of (4), up to the $(n-1)$-th order, are entirely bounded. Substituting $y=x'$, we get the equation

$$y^{(n-1)}+\sum_{j=1}^{n-1}a_j y^{(n-1-j)}=p(t)-a_n\varphi(t) \tag{24}$$

with exactly one entirely bounded solution φ'. Applying (14), we obtain

$$\sup_{t\in\mathbb{R}}\left|\frac{d\varphi(t)}{dt}\right| \le \frac{\sup_{t\in\mathbb{R}}|p(t)-a_n\varphi(t)|}{|a_{n-1}|} \le \frac{P+|a_n|\frac{P}{|a_n|}}{|a_{n-1}|}=\frac{2P}{|a_{n-1}|}\,.$$

Writing $z = y'$, we have the equation

$$z^{(n-2)} + \sum_{j=1}^{n-2} a_j y^{(n-2-j)} = p(t) - a_n\varphi(t) - a_{n-1}\varphi'(t)$$

with exactly one entirely bounded solution φ''. Applying (14) again, we get

$$\sup_{t\in\mathbb{R}} |p(t) - a_n\varphi(t) - a_{n-1}\varphi'(t)| \le P + \frac{|a_n|P}{|a_n|} + \frac{2|a_{n-1}|P}{|a_{n-1}|} = 4P\,,$$

and consequently

$$\sup_{t\in\mathbb{R}} \left|\frac{d^2\varphi(t)}{dt^2}\right| \le \frac{4P}{|a_{n-2}|}\,,$$

by the same reason as above. Proceeding by the same way, till the substitution $w = \varphi^{(n-1)}$, leading to

$$\sup_{t\in\mathbb{R}} \left|\frac{d^{n-1}\varphi(t)}{dt^{n-1}}\right| \le \frac{2^{n-1}P}{|a_1|}\,,$$

we get successively all the estimates in (23), which proves the theorem. □

Remark 2 It can be proved for $m \neq 0$ that the estimate in (23) is better than the one in (21) (see [5]).

4 Main results

Using Proposition 1, Lemma 6 and Consequence 2, we are ready to establish

Theorem 2 *Let all roots of the characteristic polynomial (6) be negative. Put* $a_0 := 1$ *and assume the existence of real nonnegative numbers* $C_0, \dots, C_{n-1}$ *such that the inequalities*

$$\operatorname*{ess\,sup}_{\substack{t\in\mathbb{R},\ |x_j|\le C_j\\ j=0,\dots,n-1}} |f(t, x_0, \dots, x_{n-1})| \le \binom{n}{m} \frac{a_n}{2^m a_m} C_m\,, \quad m = 0, \dots, n-1 \qquad (25)$$

hold for f. *Then the equation (1) admits an entirely bounded weak solution.*

If, additionally, f *is T-periodic in* t, *then there exists at least one weak T-periodic solution of the equation (1).*

At last, under the assumption (25), the equation (1) admits a weak T-antiperiodic solution provided the equality

$$f(t + T, -x_0, -x_1, \dots, -x_{n-1}) = -f(t, x_0, x_1, \dots, x_{n-1})$$

holds for every $(t, x_0, \dots, x_{n-1}) \in \mathbb{R}^{n+1}$.

Proof Define (the evidently nonempty, closed, convex and bounded subset of the Fréchet space $C^{(n-1)}(\mathbb{R},\mathbb{R})$):

$$S := \{q \in C^{(n-1)}(\mathbb{R},\mathbb{R}) : |q^m(t)| \le C_m,\ t \in \mathbb{R},\ m = 0,\dots,n-1\}.$$

Choose any $q \in S$ and put

$$p(t) := f(t, q(t), \dots, q^{(n-1)}(t)), \quad t \in \mathbb{R}.$$

Using the inequalities (25), we obtain from (21) the uniqueness and solvability for the problem (3). Since (25) immediately ensures also the validity of the condition (ii) in Proposition 1, we proved the existence of an entirely bounded weak solution of the equation (1).

Defining

$$S_1 := S \cap \{q \in C(\mathbb{R},\mathbb{R}) : q(t) = q(t+T),\ t \in \mathbb{R}\}$$

and using, additionally, Consequence 2 we get by the same way the existence of a T-periodic weak solution of (1).

To prove the third assertion we put

$$S_2 := S \cap \{q \in C(\mathbb{R},\mathbb{R}) : q(t) = -q(t+T),\ t \in \mathbb{R}\}$$

and repeat the previous procedure. □

Theorem 3 *Consider the sequence of the "shifted polynomials"*

$$\lambda^{n-p} + \sum_{j=1}^{n-p} a_j \lambda^{n-p-j}, \quad p = 0, 1, \dots, n-1$$

and assume that each of them has only real nonzero roots. Moreover, let there exist real nonnegative numbers $D_0, \dots, D_{n-1}$ such that the function f satisfies the following inequalities

$$\operatorname*{ess\,sup}_{\substack{t\in\mathbb{R},\ |x_j|\le D_j \\ j=0,\dots,n-1}} |f(t, x_0, \dots, x_{n-1})| \le \frac{|a_{n-m}|}{2^m} D_m, \quad m = 0, \dots, n-1. \tag{26}$$

Then the same assertions as in Theorem 1 holds for the equation (1).

Proof Using the estimates (23) instead of (21), we can repeat the method of the previous proof. □

Remark 3 The periodicity assertions in the last two theorems complete several former results due to the other authors like e.g. [1], [2], [8], [9], [10], [11].

Remark 4 Clearly, a theorem analogous to Theorem 2 can be given (see Consequence 1 and Lemma 5). In this way, (25) can be replaced by

$$\operatorname*{ess\,sup}_{\substack{t\in\mathbb{R},\ |x_j|\le C_j \\ j=0,\dots,n-1}} |f(t, x_0, \dots, x_{n-1})| \le \frac{|a_n|}{2^m \prod_{j=1}^m |\alpha_j|} C_m, \quad m = 0, \dots, n-1,$$

where $\prod_{j=1}^0 |\alpha_j| := 1$.

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