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Numerical Solution of Some Iterative Differential Equation *

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Abstract

Some simple algorithm based on linear and quadratic splines is discussed for the numerical solution of the state-dependent iterative-differential equation (1). In the case of initial condition $x(t_0) = t_0$, where we do not need some starting interval, there are some new results known in the theory concerning the existence and uniqueness of the solution. The aim of this contribution is to describe some simple algorithm for the numerical solution of such problem and to show some computational results for various values of the parameters t_0, b.

Key words: Numerical solution of iterative-differential equation, applications of splines.

1991 Mathematics Subject Classification: 41A15, 65D05, 65L05, 34K15

1 Introduction

The mathematical models of state dependent processes lead to various types of functional differential equations with retarded argument or neutral type equations (see [2], [3]). We shall deal in this contribution with the numerical solution of the special state dependent equation (iterative-differential equation)

$$x'(t) = x(x(t)) - b \cdot x(t) \tag{1}$$

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which was studied in details (from the point of view of global properties) in [6] and for some special case in [7], [8]. The equation (1) has trivial solution $x(t) \equiv 0$ and a "singular solution" $x(t) = b(t+1)$ on the whole real axis, which depends on the parameter b from the equation only. When we search for the solution determined by some initial value $x(t_0) = x_0 \neq t_0$, then we need some starting interval for the solution to be prescribed. The spline-based algorithms for such cases have been investigated in [1]. Let us consider now therefore the case of the special initial value problem for the equation (1) with

$$x(t_0) = t_0 \quad \text{which results (through collocation) in} \quad x'(t_0) = t_0(1-b). \tag{2}$$

It is now possible to discuss the conditions for (local) existence of some increasing or decreasing solutions (see [6], [8]) of the initial value problem (1)–(2). To obtain the increasing solution of the equation with retarded argument, we suppose

$$0 \leq b < 1, \ 0 < t_0 < 1/(1-b) \qquad \text{or} \qquad t_0 < 0, \ b > 1.$$

The existence and uniqueness of the global solution in that case is proved in [6]. The decreasing solution we can obtain (see [8]) e.g. in case

$$b < 1, \ t_0 < 0 \qquad \text{or} \qquad t_0 > 0, \ b > 1.$$

The aim of this contribution is to develop and demonstrate on some examples the simple algorithm for the numerical solution of this special problem considered. We have used the experience from the algorithms for the neutral type FDE's developed in [1]. Some properties of the iterative term $x(x(t))$ in simple functional equations are described e.g. in [5]; some constructive approach to it is followed in [4], where the properties of the iterate function are shown and used to the numerical solution of the equation $x(x(t)) = f(t)$.

2 The algorithm for increasing solution

The algorithm described in the following for the numerical solution of the problem

$$x'(t) = x(x(t)) - b \cdot x(t), \quad x(t_0) = t_0, \quad 0 < t_0 < 1/(1-b), \quad 0 \leq b < 1 \tag{3}$$

is based on the use of some proper combination of linear splines for advancing the solution forwards (or backwards) and quadratic splines for computing the value of the delayed iterate term $x(x(t))$ from known values of $x(t), x'(t)$. The three stages of the computation from given initial values t_0, b on the equidistant knotset $t = \{t_j, j = -m(1)n\}$ (with the step $h > 0$ and t_0 generally inside the knotset) to the output values $\{t_j, x_j\}$ are the following:

1. Compute the value $x_1 = x(t_1)$ using two steps with the steplength $h/2$ and the formulas

$$x_0' = t_0(1-b), \quad x_{1/2} = x_0 + hx_0'/2, \quad q = (x_{1/2} - x_0)/(h/2)$$

$$v = (1-q^2)x_0 + q^2 x_{1/2} + \tfrac{1}{2}hq(1-q)x_0' \qquad \textit{(quadratic interpolation)}$$
$$w = v - bx_{1/2} \qquad \textit{(collocation to (1) for } x_{1/2}'\textit{)}$$
$$x_1 = x_{1/2} + \tfrac{1}{2}hw \tag{4}$$

(the starting interval needed for the computation of the delayed values in further steps—computed with higher accuracy).

2. To compute the appoximate value of the solution in the knots t_{i+1}, $i = 1(1)n-1$, we have
   – to find the index $j \leq i$ such that $x_i \in (t_j, t_{j+1})$;
   – with $q = (x_i - t_j)/h$ to compute the approximation

$$v = (1-q^2)x_j + q^2 x_{j+1} + hq(1-q)x_j'; \quad \textit{(quadratic interpolation)}$$
$$x_i' = v - bx_i \quad \textit{(collocation to FDE's)};$$
$$x_{i+1} = x_i + hx_i' \quad \textit{(Euler's method)}. \tag{5}$$

3. In case that $t_{-m} < t_0$ we have to return to the starting interval and to continue the computation backwards for $i = 0(-1)-m$ in the following way:
   – compute $x_{i-1} = x_i - hx_i'$; *(Euler's method)*
   – find index $j \geq i$ such that $x_{i-1} \in (t_{j-1}, t_j)$;
   – with $q = (x_{i-1} - t_{j-1})/h$ compute *(with quadratic spline interpolation)*

$$v = (1-q)^2 x_{j-1} + q(2-q)x_j - hq(q-1)x_j' \tag{6}$$
$$x_{i-1}' = v - bx_{i-1} \qquad \textit{(collocation to FDE (1))}.$$

The algorithm described above was implemented in MATLAB's M-file fdesp21.m worked out by the author. This M-file was used for computations in the examples demonstrated in Section 4.

3 Convergence of the method

The numerical solution $\{x_i\}$ of the problem (3) is computed in the forward resp. backward stages as described in Section 2, formulas (5), (6). The resulting formulas for computation x_{i+1} (forwards), resp. x_{i-1} (backwards) we can write—with the variable index j determined during computation and neglecting the terms with $O(h^2)$—as

$$x_{i+1} = (1-hb)x_i + hq^2 x_{j+1} + h(1-q^2)x_j; \quad j \leq i \tag{7}$$
$$x_{i-1} = (1+hb)x_i - hq(2-q)x_{j+1} - h(1-q)^2 x_j; \quad j \geq i. \tag{8}$$

We can consider such a method as linear multistep method (LMM) with variable scheme and use some features from the standard LMM technique for the proof of the convergence of our method.

Let us denote $e_i = x(t_i) - x_i$ the global error in the knot t_i. For its propagation we obtain the following recursions in forward and backward stages

$$e_{i+1} - (1 - hb)e_i - hq^2 e_{j+1} - h(1 - q^2)e_j = 0; \tag{9}$$

$$e_{i-1} - (1 + hb)e_i - h(1 - q)^2 e_j + hq(2 - q)e_{j+1} = 0. \tag{10}$$

The roots of the characteristic polynomial of the forward recursion lie inside the unit disc for $h \to 0$ ($|q| \leq 1$, $0 \leq b < 1$). The principal root (the root x_p with maximal modulus) of the backward recursion may go slightly outside the unit disc, but still its modulus is $|x_p| = 1 + O(h)$ as follows from the continuous dependence of the roots on the coefficients. All its powers remain then bounded. In both cases we have then $|e_n| \to 0$ with $|e_0| \to 0$, $h \to 0$ for all n—e.g. the method is convergent on finite intervals.

4 Examples of increasing solutions

We can see the results of computations for the value $t_0 = 0.2$, with the parameter values $b = 0.1, 0.3, 0.5, 0.7$ on the knotset $t = -10 : 0.1 : 10$ plotted on Fig. 1a. In all cases we have obtained monotone increasing solutions. We can mention different character of solution corresponding to $b = 0.1$ (quickly growing for positive values t). The solutions corresponding to the values $b = 0.5$, $b = 0.7$ demonstrate the change of the shape for $t \to -\infty$. The results of computations with the parameter $b = 0.7$ and different starting points $t_0 = 0.1, 0.2, 0.3, 0.5, 0.6$ on the knotset $t = -20 : 0.1 : 5$ we can see on Fig. 1b. We can mention here the strong dependency of the solution on the relativelly small changes in the initial value t_0, similar shape properties for the values $b = 0.1, 0.2, 0.3$ and change in the shape for the values $b = 0.5, 0.6$.

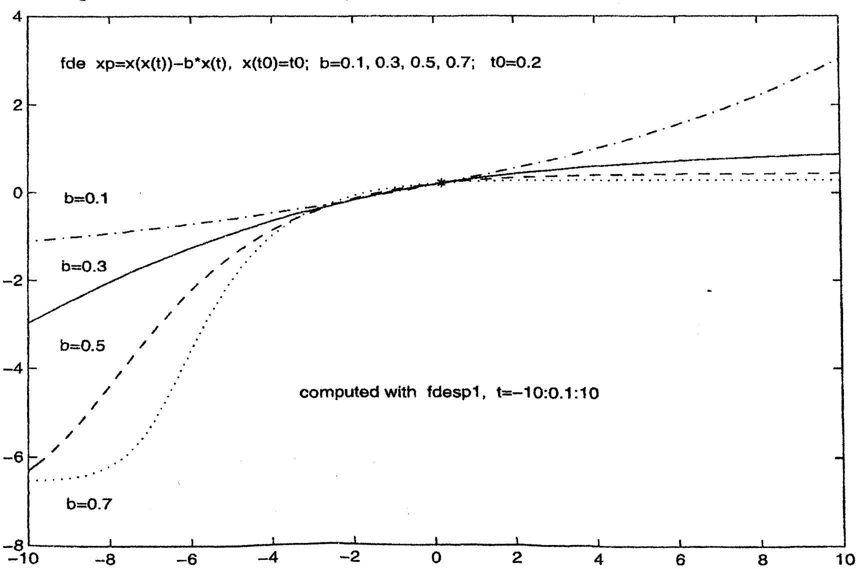

Fig. 1a

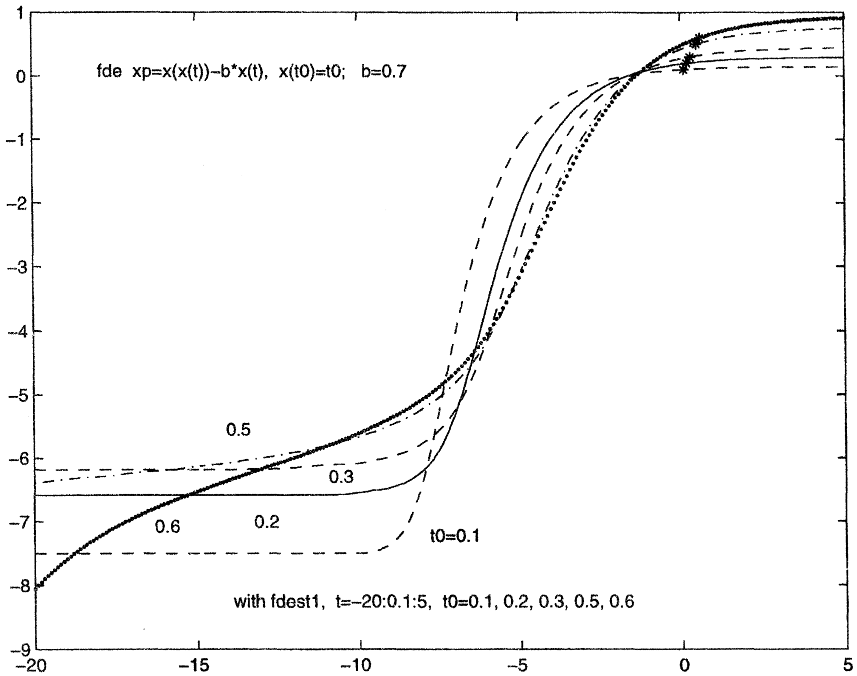

Fig. 1b

We can demonstrate some conclusions concerning the global behaviour of the solutions of the equation (1) mentioned in [6] on the following example with yet more prolongated left part of the knotset, taken here as $t = -30 : 0.1 : 5$.

We can follow the results of computations for the value $b = 0.5$ and initial values $t_0 = 0.1, 0.2, 0.5, 0.7, 0.9$ on Fig. 2a. We can see here, that the solutions corresponding to $b = 0.1, 0.2$ are approaching some limits for $t \to -\infty$ and the solutions corresponding to $b = 0.7, 0.9$ behave nearly linearly for $t \in [-30, 0]$.

The change of the parameter b to the value $b = 0.2$ and the results of computations for initial conditions $t_0 = 0.1, 0.2, 0.3, 0.4, 0.5$ on the knotset $t = -20 : 0.1 : 5$ are given on Fig. 2b. The shapes of the solutions corresponding to $t_0 = 0.3, 0.4, 0.5$ are similar, but we can see significant changes in the shape of the solutions for $t_0 = 0.1, 0.2$.

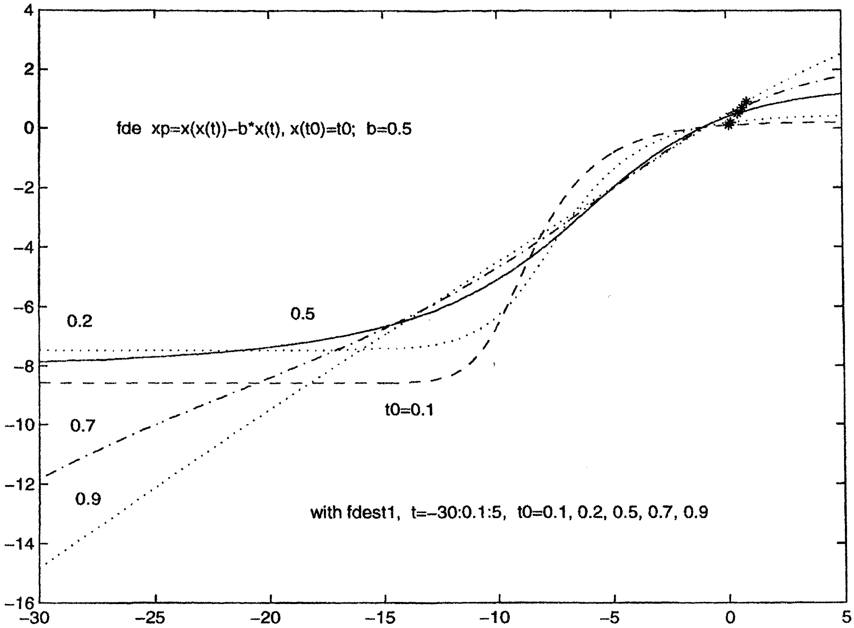
fde xp=x(x(t))-b*x(t), x(t0)=t0; b=0.5
0.2
0.5
0.7
0.9
t0=0.1
with fdest1, t=-30:0.1:5, t0=0.1, 0.2, 0.5, 0.7, 0.9

Fig. 2a

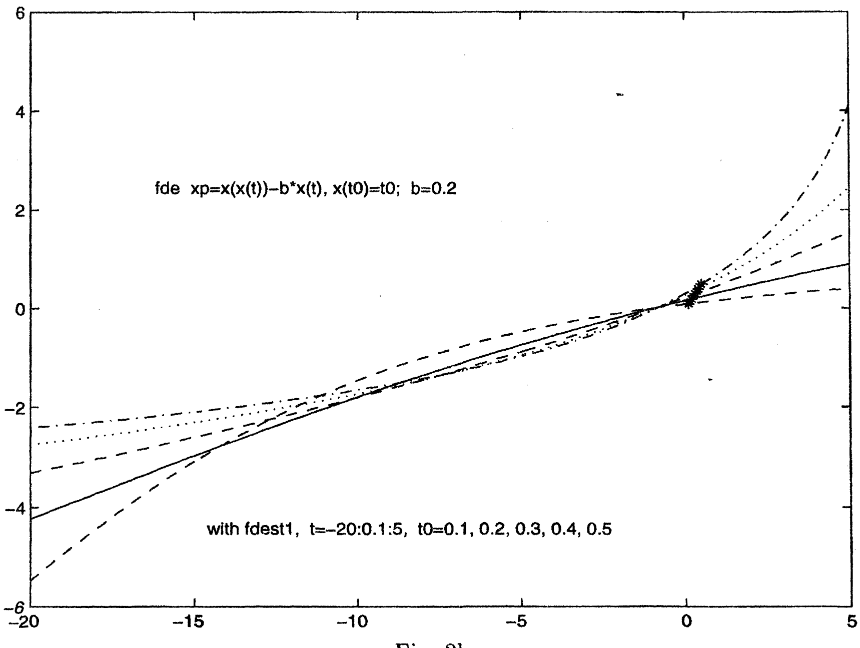
fde xp=x(x(t))-b*x(t), x(t0)=t0; b=0.2
with fdest1, t=-20:0.1:5, t0=0.1, 0.2, 0.3, 0.4, 0.5

Fig. 2b

Some more details about the solutions corresponding to the value $b = 0.2$ and $t_0 = 0.4, 0.5$ are given on Fig. 3a, where the results of computing on the knotset $t = -30 : 0.1 : 5$ are plotted. We can see the convex shape of the solutions.

The detailed results of computations for $t_0 \to 0$ on the interval $[-2, 0.5]$ and with $b = 0$ are plotted on Fig. 3b. We can see here the nearly linear shape of solutions corresponding to the values $t_0 = 0.1, 0.05, 0.01, 0.001$ and their convergence to the trivial solution.

New features of the solutions corresponding to the input values $\{t_0 = 0.6,\ b = 0.4\}$, $\{t_0 = 0.5,\ b = 0.5\}$, $\{t_0 = 1,\ b = 0.8\}$, $\{t_0 = 1.5,\ b = 0.9\}$ and the knotset $t = -30 : 0.1 : 20$ we can see on Fig. 4a. Mention please the shapes of last two solutions in the interval $[-20, 0]$ and then some smoothing towards the limiting value.

The solutions corresponding to $b = 0.4$ are quickly growing with growing values of t_0. The results of computations on the knotset $t = 0 : 0.01 : 3$ for the initial values $t_0 = 0.7, 0.9, 1.2$ are plotted on Fig. 4b. We can compare this solution here with the solution corresponding to the parameters $b = 0.8$, $t_0 = 1.2$.

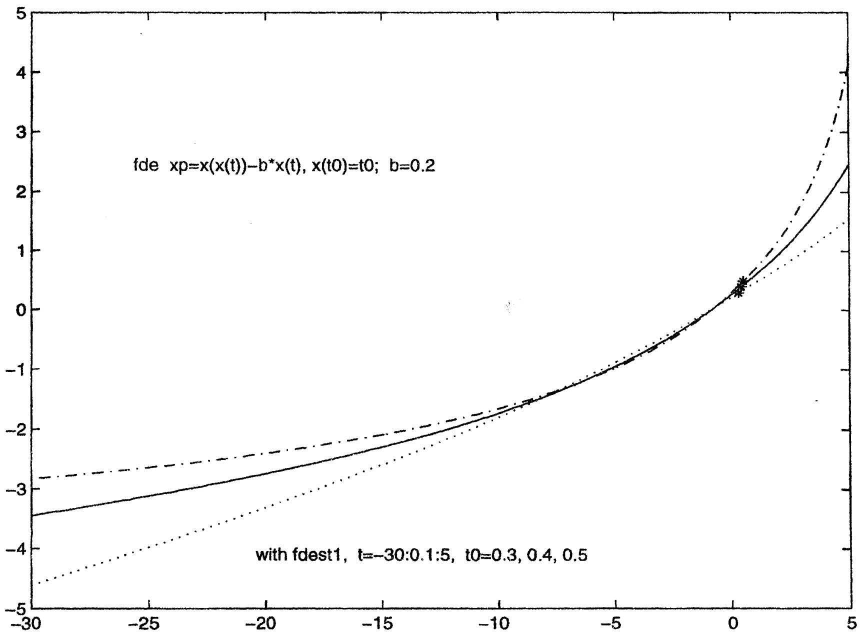

Fig. 3a

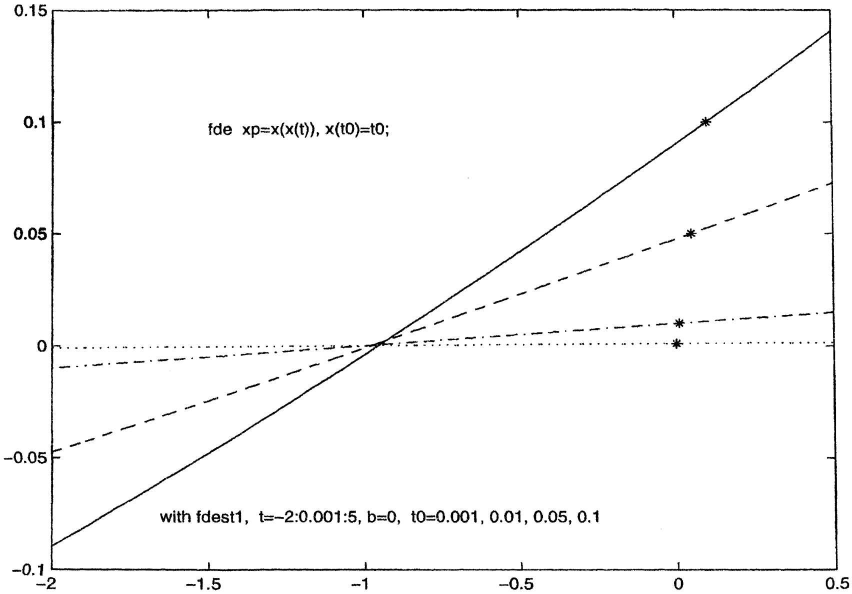
fde xp=x(x(t)), x(t0)=t0;
with fdest1, t=-2:0.001:5, b=0, t0=0.001, 0.01, 0.05, 0.1
0.15
0.1
0.05
0
-0.05
-0.1
-2
-1.5
-1
-0.5
0
0.5

Fig. 3b

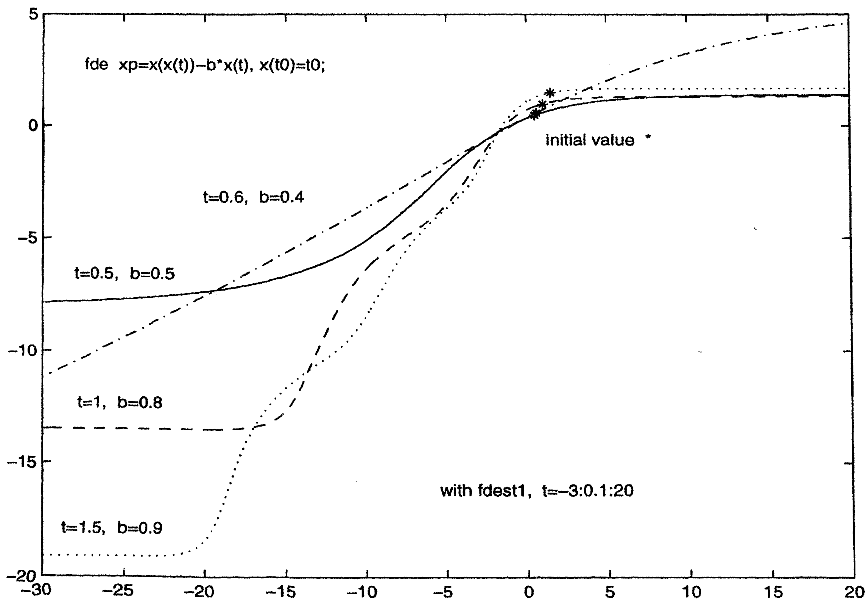
fde xp=x(x(t))-b*x(t), x(t0)=t0;
initial value *
t=0.6, b=0.4
t=0.5, b=0.5
t=1, b=0.8
t=1.5, b=0.9
with fdest1, t=-3:0.1:20
5
0
-5
-10
-15
-20
-30
-25
-20
-15
-10
-5
0
5
10
15
20

Fig. 4a

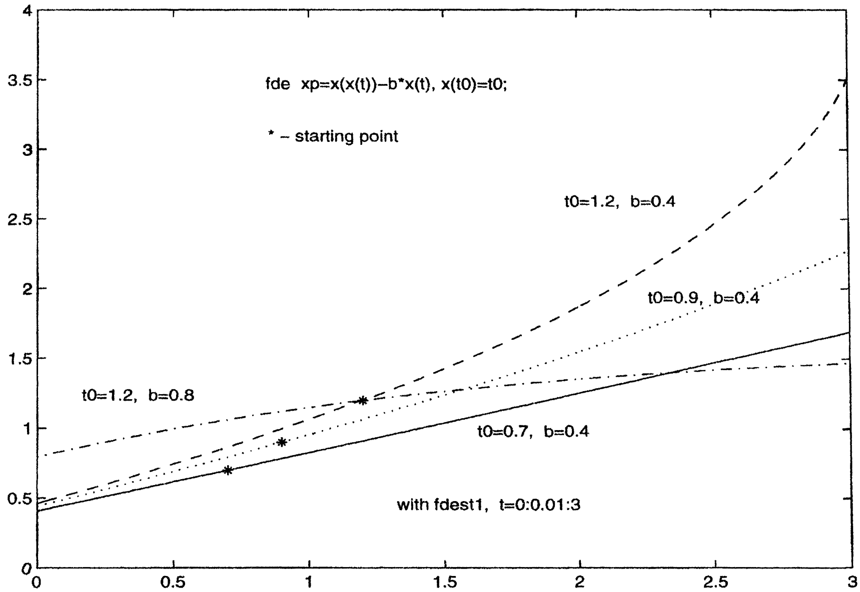

Fig. 4b

Remark The introduced algorithm does not work in cases of $t_0 < 0$ and for the decreasing solutions. The necessary modifications, changes and improvements will be the subject of some further research.

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