Jiří Rachůnek

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A Weakly Associative Generalization of the Variety of Representable Lattice Ordered Groups

JIŘÍ RACHŮNEK

*Department of Algebra and Geometry, Faculty of Science,
Palacký University, Tomkova 40, 779 00 Olomouc, Czech Republic
e-mail: rachunek@risc.upol.cz*

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Abstract

A semi-ordered group is a group endowed with a reflexive and antisymmetric binary relation compatible with the group addition. Circular totally semi-ordered groups (circular *to*-groups) are very close to linearly ordered groups. In the paper it is proved that the class of all subdirect sums of circular *to*-groups is a variety of weakly associative lattice groups (*wal*-groups). Further, an atom in the lattice of varieties of *wal*-groups is described.

Key words: Weakly associative lattice group, representable lattice ordered group, almost o-group.

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A *weakly associative lattice (wa-lattice)* is an algebra $A = (A, \vee, \wedge)$ with two binary operations satisfying the identities

(I)	$x \vee x = x\,;$	$x \wedge x = x\,;$
(C)	$x \vee y = y \vee x\,;$	$x \wedge y = y \wedge x\,;$
(Abs)	$x \vee (x \wedge y) = x\,;$	$x \wedge (x \vee y) = x\,;$
(WA)	$((x \wedge z) \vee (y \wedge z)) \vee z = z\,;$	$((x \vee z) \wedge (y \vee z)) \wedge z = z\,.$

The *wa*-lattices have been introduced by E. Fried in [3] and [4], and by H. L. Skala in [12] and [13]. It is obvious that the notion of a *wa*-lattice generalizes that of a lattice because the identities of associativity of the operations $\vee$

and $\wedge$ are special cases of the identities (WA) of weak associativity. Nevertheless, similarly as for lattices, the properties of $\vee$ and $\wedge$ make possible to define also for *wa*-lattices a binary relation $\leq$ on A as follows:

$$\forall x, y \in A;\, x \leq y \quad \Longleftrightarrow_{df} \quad x \wedge y = x\,.$$

The relation $\leq$ is reflexive and antisymmetric (i.e. $\leq$ is so-called *semi-order*) and each subset $\{a, b\} \subseteq A$ has the join $\sup\{a, b\} = a \vee b$ and the meet $\inf\{a, b\} = a \wedge b$ in A. It holds that $(A, \vee, \wedge)$ is a *wa*-lattice. Therefore we can equivalently view any *wa*-lattice as a set with a binary relation $\leq$. From this point of view, tournaments are special cases of *wa*-lattices.

Recall that a *tournament* is a set $T \neq \emptyset$ with a reflexive and antisymmetric binary relation $\leq$ satisfying

$$\forall x, y \in T;\; x \leq y \text{ or } y \leq x\,.$$

If $(G, +, 0, -(\cdot))$ is a group and $(G, \vee, \wedge)$ is a *wa*-lattice then the system $G = (G, +, 0, -(\cdot), \vee, \wedge)$ is called a *weakly associative lattice group* (*wal-group*) if G satisfies the following mutually equivalent identities and quasi-identity:

$$(\mathrm{D}_\vee) \quad x + (y \vee z) + v = (x + y + v) \vee (x + z + v)\,,$$
$$(\mathrm{D}_\wedge) \quad x + (y \wedge z) + v = (x + y + v) \wedge (x + z + v)\,,$$
$$(\mathrm{M}) \quad y \leq z \Longrightarrow x + y + u \leq x + z + u\,.$$

(See [7] and [8]. In [13] a *wal*-group is called a *trellis-group*.) If G is a *wal*-group then $G^+ = \{x \in G;\, 0 \leq x\}$ is called the *positive cone* of G and its elements are *positive*.

In contrast to lattice ordered groups (*l*-groups) that are torsion free, there are many finite *wal*-groups.

It is obvious that the class $\mathcal{G}_{wal}$ of all *wal*-groups is a variety of algebras of type $\langle +, 0, -(\cdot), \vee, \wedge\rangle$ of signature $\langle 2, 0, 1, 2, 2\rangle$. Some properties of the variety $\mathcal{G}_{wal}$ and the lattice of subvarieties of $\mathcal{G}_{wal}$ have been investigated in [11] and [10].

If for a *wal*-group G the *wa*-lattice $(G, \leq)$ is a tournament, then G is called a *totally semiordered group* (a *to-group*). A tournament $(T, \leq)$ is said to be *circular* (see e.g. [2]) if

(a) there exist $a, b, c \in T$ such that $a < b < c < a$,
(b) whenever $x, y, z \in T$ satisfy $x < y < z < x$ then there exists no $w \in T$ such that $w < \{x, y, z\}$ or $w > \{x, y, z\}$.

A *to*-group G is called *circular* if the tournament $(G, \leq)$ is circular. The circular *to*-groups have been introduced and studied in [9].

In this paper we will deal with circular *to*-groups and linearly ordered groups (*o*-groups) (and classes of *wal*-groups obtained from them) and discuss a question concerning atoms in the lattice of varieties of *wal*-groups.

For necessary results concerning *l*-groups and *o*-groups see e.g. [1], [5], [6].

Definition A *to*-group G is called an *almost o-group* (an *ao-group*) if G is either an *o*-group or a circular *to*-group.

Proposition 1 *Let G be a to-group. Then G is an ao-group if and only if G^+ is a linearly ordered set.*

Proof a) Let G be a circular *to*-group, $a, b, c \in G^+ \setminus \{0\}$, $a < b < c$, but $a > c$. Then $a < b < c < a$ and $0 < \{a, b, c\}$, a contradiction. Hence $a < c$, therefore the restriction of $<$ on G^+ is transitive.

b) Let G^+ be linearly ordered set and let G not be a linearly ordered group. Then there exist $a, b, c, d \in G$ such that $a < b < c < a$ and, for example, $d < \{a, b, c\}$. Hence $-d+a < -d+b < -d+c < -d+a$, and $0 < \{-d+a, -d+b, -d+c\}$. Thus G^+ is not a linearly ordered set, a contradiction. Similarly for $d > \{a, b, c\}$. □

Now we will recall some notions and results concerning *wal*-groups and their subgroups. Subalgebras of *wal*-groups are called *wal-subgroups.* That means if G is a *wal*-group and $\emptyset \neq H \subseteq G$ then H is a *wal*-subgroup of G if H is both subgroup and *wa*-sublattice of G. A normal convex *wal*-subgroup H of a *wal*-group G is called a *wal-ideal* of G if it satisfies the following mutually equivalent conditions:

(a) $\forall a, b \in H,\ x, y \in G;\ (x \leq a,\ y \leq b \Rightarrow \exists c \in H;\ x \vee y \leq c)$;
(b) $\forall a, b, c \in H,\ x, y \in G;\ x \leq a,\ y \leq b \Rightarrow (x \vee y) \vee c \in H$.

By [7] and [8], the *wal*-ideals of *wal*-groups coincide with the kernels of homomorphisms of *wal*-groups.

If H is a *wal*-ideal of G, we can define the structure of a *wa*-lattice on G/H by

$$x + H \leq y + H \quad \Longleftrightarrow_{df} \quad \exists a \in H;\ x + a \leq y\,,$$

and with this relation G/H is a *wal*-group.

A *wal*-ideal H of G is called *straightening* if it satisfies the following mutually equivalent conditions (see [8]):

(a) $x, y \in G,\ 0 \leq x \wedge y \in H \implies x \in H$ or $y \in H$,
(b) $x, y \in G,\ x \wedge y = 0 \implies x \in H$ or $y \in H$,
(c) G/H is a *to*-group.

A *wal*-group G is called *representable* if it is isomorphic to a subdirect sum of *to*-groups. It is obvious (see also [8]) that a *wal*-group is representable if and only if the intersection of all its straightening *wal*-ideals is equal to $\{0\}$. Let us denote by $\mathcal{R}_{wal}$ the class of all representable *wal*-groups. By [11], Proposition 7, $\mathcal{R}_{wal}$ is a variety of *wal*-groups.

Now we will deal with a class of representable *wal*-groups which is close to the class $\mathcal{R}_l$ of representable *l*-groups.

Definition A *wal*-ideal H of a *wal*-group G is called an *ao-straightening wal-ideal* of G if G/H is an *ao*-group.

(Obviously, every *ao*-straightening *wal*-ideal is also straightening.)

Definition A *wal*-group G is called *ao-representable* if it is isomorphic to a subdirect sum of *ao*-groups.

Let us denote by $\mathcal{RA}o$ the class of all *ao*-representable *wal*-groups and $\mathcal{VA}o$ the variety of *wal*-groups generated by all *ao*-groups. We have:

Lemma 2 *If G is a wal-group, then $G \in \mathcal{RA}o$ if and only if the intersection of all its ao-straightening wal-ideals is equal to $\{0\}$.*

Theorem 3 *The class $\mathcal{RA}o$ is a variety of wal-groups.*

Proof We will use Birkhoff's characterization of varieties as classes of algebras of a given type closed with respect to products, subalgebras and homomorphic images. Let us put $\mathcal{U} = \mathcal{RA}o$.

a) It is obvious that the product (the cardinal sum) of *wal*-groups from $\mathcal{U}$ belongs also to $\mathcal{U}$.

b) Let $G \in \mathcal{U}$ be a subdirect sum of *ao*-groups G_i $(i \in I)$ and H be a *wal*-subgroup of G. Let us consider any *ao*-straightening *wal*-ideal S_j of G and denote $H_j = H \cap S_j$. By [11], proof of Proposition 7, H_j is a straightening *wal*-ideal of H.

Let $(S_j;\, j \in J)$ be the system of all *ao*-straightening *wal*-ideals of G. Then

$$\bigcap_{j\in J} H_j = \bigcap_{j\in J}(H \cap S_j) \subseteq \bigcap_{j\in J} S_j = \{0\},$$

hence by Lemma 2, $H \in \mathcal{U}$.

c) Let f be a *wal*-homomorphism of a *wal*-group G onto a *wal*-group G'. For any *wal*-ideal H of G put $H' = f(H)$. If H is a straightening *wal*-ideal of G then, by [11], proof of Proposition 7, H' is a straightening *wal*-ideal of G'. Let now H be an *ao*-straightening *wal*-ideal of G. Let us consider $a'+H'$, $b'+H'$, $c'+H' \in (G'/H')^+$ such that $a'+H' \leq b'+H'$, $b'+H' \leq c'+H'$. Let $a, b, c \in G$ be such that $a' = f(a), b' = f(b), c' = f(c)$, and $a+H$, $b+H$, $c+H \in (G/H)^+$. Since G/H is a *to*-group, $a+H$ and $b+H$ are comparable. If $a+H \geq b+H$ then $a'+H' \geq b'+H'$, hence $a'+H' = b'+H'$, and thus $a'+H' \leq c'+H'$. Similarly for $b+H \geq c+H$. Therefore we can suppose that $a+H \leq b+H$ and $b+H \leq c+H$. Since G/H is an *ao*-group, we have, by Proposition 1, $a+H \leq c+H$, and hence also $a'+H' \leq c'+H'$. Therefore, by Proposition 1, H' is an *ao*-straightening *wal*-ideal of G'.

Let now $G \in \mathcal{U}$ and let $(H_i;\, i \in I)$ be the system of all *ao*-straightening *wal*-ideals of G. If there exists $j \in I$ such that $H'_j = f(H_j) = \{0'\}$, then $\{0'\}$ is an *ao*-straightening *wal*-ideal of G', and thus G' is an *ao*-group. Let $H'_i = f(H_i) \neq \{0'\}$ for each $i \in I$. Because f induces a bijection (which respects set inclusions) between the set of *wal*-ideals of G which are not contained in $\mathcal{K}$er f and the set of all *wal*-ideals of G', and because the *wa*-lattices G/H_i and G'/H'_i are isomorphic, f also induces a bijection between the set of *ao*-straightening *wal*-ideals of G and the set of *ao*-straighteninig *wal*-ideals of G'.

If $H' = \bigcap_{i\in I} H'_i \neq \{0'\}$, then $H = f^{-1}(H')$ is a *wal*-ideal which is contained in all *ao*-straightening *wal*-ideals of G, hence $H = \{0\}$, a contradiction. Thus $H' = \{0'\}$, and therefore by Lemma 2, G' is an *ao*-group. □

Corollary 4 *The class $\mathcal{RA}o$ and the variety $\mathcal{VA}o$ coincide.*

Let us consider the following identities:

$$(\mathrm{A}^+)\ \begin{cases} (x \vee 0) \vee ((y \vee 0) \vee (z \vee 0)) = ((x \vee 0) \vee (y \vee 0)) \vee (z \vee 0), \\ (x \vee 0) \wedge ((y \vee 0) \wedge (z \vee 0)) = ((x \vee 0) \wedge (y \vee 0)) \wedge (z \vee 0). \end{cases}$$

It is obvious that any $G \in \mathcal{RA}o$ satisfies both identities (A^+) because G^+ is a lattice.

At the same time, for any *to*-group G which is not an *ao*-group, the tournament G^+ is not a linearly ordered set, hence such G does not satisfy (A^+).

Let us consider the variety $\mathcal{R}_l$ of all representable l-groups, i.e. the variety of l-groups generated by all linearly ordered groups. By the preceding we have:

Theorem 5 $\mathcal{R}_l \subset \mathcal{RA}o \subset \mathcal{R}_{wal}$.

Let us denote by **WAL** the class of all varieties of *wal*-groups (considered in the language $\mathcal{L} = (+, 0, -(\cdot), \vee, \wedge))$. It is clear that **WAL** ordered by inclusion is a complete lattice. By [11], Theorem 5, it holds that the lattice **WAL** is distributive and contains the lattice **L** of all varieties of l-groups (considered also in the language $\mathcal{L}$) as a complete $\wedge$-sublattice. Furthemore, in [11], pp. 238–239, it is shown that the variety $\mathcal{A}b_l$ of all abelian l-groups is an atom of **WAL**, but it is not the least non-trivial variety of **WAL** (in contrast to the lattice **L**).

Now we will describe another atom of the lattice **WAL**. Let us consider the group $\mathbb{Z}_3 = \{0, 1, 2\}$ with the addition mod 3. If we put $\mathbb{Z}_3^+ = \{0, 1\}$ then $\mathbb{Z}_3^+$ is the positive cone of the total semi-order on $\mathbb{Z}_3$ such that $0 < 1$, $1 < 2$, $2 < 0$. $\mathbb{Z}_3$ is then an *ao*-group. Let us denote $\mathcal{V}_3 = \mathcal{V}_{wal}(\mathbb{Z}_3)$ (i.e. the variety of *wal*-groups generated by $\mathbb{Z}_3$) and $\mathcal{T}_3$ the variety of *wal*-groups satisfying the identity

$$(\mathrm{T}_3) \qquad 3x = 0.$$

Obviously $\mathcal{V}_3 \subseteq \mathcal{RA}o$.

Theorem 6 *$\mathcal{V}_3$ is an atom of the lattice **WAL**.*

Proof Let $\{0\} \neq G \in \mathcal{V}_3$. Since $G \neq \{0\}$, there exists $0 < a \in G$. Obviously $\mathcal{V}_3 \subseteq \mathcal{T}_3$, thus $3a = 0$. Hence we have $0 < a$, $a < 2a$, $2a < 0$, that means the subgroup $[a] = \mathrm{grp}(a)$ is a *wal*-subgroup of G which is (as a *wal*-group) isomorphic to $\mathbb{Z}_3$. Thus $\mathbb{Z}_3 \in \mathcal{V}_{wal}(G)$, and therefore $\mathcal{V}_{wal}(G) = \mathcal{V}_3$. □

Now, let us consider the group $\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$ with the addition mod 5 and put $\mathbb{Z}_5^+ = \{0, 1, 2\}$. Then $\mathbb{Z}_5$ is an *ao*-group. Moreover, $\mathbb{Z}_5^+$ is, up to isomorphism, the unique positive cone of a *wa*-lattice semi-order of the group $\mathbb{Z}_5$.

Let us denote $\mathcal{V}_5 = \mathcal{V}_{wal}(\mathbb{Z}_5)$ and consider $\{0\} \neq G \in \mathcal{V}_5$. (It holds again that $\mathcal{V}_5 \subseteq \mathcal{RA}o$.) Let us choose any $0 < a \in G$. Then we have also $a < 2a$, $2a < 3a$, $3a < 4a$, $4a < 0$. If $0 < 2a$ then $[a] = \mathrm{grp}(a)$ is a *to*-subgroup of G and the *to*-groups $[a]$ and $\mathbb{Z}_5$ are isomorphic.

Let $0 > 2a$, Then again $[a]$ is a *to*-group with the positive cone $\{0, 3a, a\}$ of G which is isomorphic to $\mathbb{Z}_5$.

The preceding considerations imply:

Theorem 7 *If* $G \in \mathcal{V}_5$ *and* G *is a to-group then* $\mathcal{V}_{wal}(G) = \mathcal{V}_5$.

Question It remains as an open problem: Is $\mathcal{V}_5$ an atom of **WAL**?

Note Similarly as the varieties $\mathcal{V}_3$ and $\mathcal{V}_5$ have been introduced, one can also define the varieties $\mathcal{V}_n$ for arbitrary $n \geq 3$ odd. Then one can also asks, more generally, the question, whether $\mathcal{V}_p$ is an atom of **WAL** for any p prime. But note that for $p > 5$ the situation becomes more complicated. For instance, for $p = 7$, there are *wa*-lattice semi-orders on $\mathbb{Z}_7$ such that the corresponding *wal*-groups are not mutually isomorphic. For example, for $\mathbb{Z}_7^+ = \{0, 1, 2, 3\}$ we get an *ao*-group which generates $\mathcal{V}_7$, for $\mathbb{Z}_7^+ = \{0, 1, 2, 4\}$ we get a *to*-group which is not an *ao*-group ($1 < 2 < 4 < 1$ and $0 < \{1, 2, 4\}$), and for $\mathbb{Z}_7^+ = \{0, 1, 5\}$ we get a *wal*-group which is not a *to*-group.

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