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Notice on the Chipman Generalization of the Matrix Inverse *

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Abstract

Approximative solutions of inconsistent linear matrix equations can be obtained by the Chipman inverse. This inverse of a given matrix must fulfil some necessary and sufficient conditions. The aim of the paper is to give a direct proof that a special class of matrices fulfils these conditions.

Key words: Inconsistent linear matrix equation, least squares minimum norm g-inverse of a matrix.

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Introduction

In the theory of the linear estimation several types of generalized inverse (g-inverse) of matrices have been used. The Chipman g-inverse is an important representant of them. It can be generalized and then not only one however a whole class of such matrices exists. This class is characterized by necessary and sufficient conditions. The aim of the paper is to prove that a special class of matrices fulfils these conditions.

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1 Definition and auxiliary statements

Let A be an $m \times n$ matrix, M an $m \times m$ positively definite (p.d.) matrix and N an $n \times n$ p.d. matrix.

Definition 1.1 The matrix $A^+_{M,N}$ is the Chipman g-inverse of A, if

$$AA^+_{M,N}A = A \quad \& \quad A^+_{M,N}AA^+_{M,N} = A^+_{M,N}$$

$$MAA^+_{M,N} = (MAA^+_{M,N})' \quad \& \quad NA^+_{M,N}A = (NA^+_{M,N}A)'$$

($'$ denotes a transposition).

Lemma 1.2 *Let A be any $m \times n$ matrix and M and N be fixed. Then there exists just one matrix $A^+_{M,N}$.*

Proof Cf. [1] and Note 6 in [2], p. 52. □

Let R^m denote the m-dimensional real linear space,

$$||Ax - y||_M = \sqrt{(Ax-y)'M(Ax-y)} \quad \text{and} \quad ||x||_N = \sqrt{x'Nx}\,.$$

Lemma 1.3 *Properties of any matrix $A^+_{M,N}$ are characterized by the following*

$$\forall\{y \in R^m\}\ \forall\{x \in R^n\} \quad \left(||AA^+_{M,N}y - y||_M \le ||Ax - y||_M\right) \ \&$$
$$\& \ \left(||A^+_{M,N}y||_N \le ||x||_N \Leftarrow ||Ax - y||_M = ||AA^+_{M,N}y - y||_M\right). \qquad (1)$$

Proof Cf. [2], pp. 53–54. □

Remark 1.4 The matrix $A^+_{M,N}$ is called M-least squares N-minimum norm g-inverse of the matrix A (in more detail cf. [2]).

2 A generalization

Let M and N need not be p.d., i.e. they can be positively semidefinite only.

Lemma 2.1 *Necessary and sufficient conditions for $A^+_{M,N}$ to satisfy (1) is*

$$MAA^+_{M,N}A = MA \quad \& \quad MAA^+_{M,N} = (MAA^+_{M,N})' \quad \&$$
$$\& \quad NA^+_{M,N}A = (NA^+_{M,N}A)' \quad \& \quad NA^+_{M,N}AA^+_{M,N} = NA^+_{M,N}. \qquad (2)$$

Proof Cf. [2], pp. 53. □

Let the class of all matrices $A^+_{M,N}$ satisfying (2) be denoted as $\mathcal{A}^+_{M,N}$.

Lemma 2.2 *The class of matrices*

$$G = (N + A'MA)^- A'M[A'MA(N + A'MA)^- A'M]^- A'M \qquad (3)$$

is included into $\mathcal{A}^+_{M,N}$.

Proof It is given in [2], pp. 53–54 and it is based on a minimization of the function $\Phi(x) = x'Nx$ under the condition $A'MAx = A'My$. □

The problem is to prove directly the validity of (2) for (3).

Let $\mathcal{M}(A) \perp \mathcal{M}(B)$ mean $A'B = 0$.

Lemma 2.3 *Let E be an idempotent matrix, i.e. $E^2 = E$. Then $\mathcal{M}(E) \perp \mathcal{M}(I-E) \Rightarrow E = E'$.*

Proof $\mathcal{M}(E) \perp \mathcal{M}(I-E) \Leftrightarrow E'(I-E) = 0 \Leftrightarrow E' = E'E$, i.e. E' and thus E is symmetric. □

Lemma 2.4 *Let A and B be $m \times n$ and $n \times r$, respectively, matrices with the property $\mathcal{M}(A') = \mathcal{M}(B)$. Then $B(AB)^- A$ is idempotent and symmetric.*

Proof $\mathcal{M}(A') = \mathcal{M}(B) \Rightarrow \mathcal{M}(AA') = \mathcal{M}(AB)$ and $\mathcal{M}(B') = \mathcal{M}(B'B) = \mathcal{M}(B'A')$. It implies $B(AB)^- AB(AB)^- A = B(AB)^- A$, thus $B(AB)^- A$ is idempotent. The expression $B(AB)^- A$ is invariant with respect to a choice of the g-inverse $(AB)^-$, i.e. $B(AB)^- A = B[(B'A')^-]'A$. Obviously

$$\mathcal{M}[B(AB)^- A] = \mathcal{M}(B) = \mathcal{M}(A') = \mathcal{M}[A'(B'A')^- B'] \perp \mathcal{K}er(A).$$

Since $B(AB)^- A$ is idempotent and $\mathcal{K}er(A) = \mathcal{M}[I - B(AB)^- A]$, the matrix $B(AB)^- A$ is symmetric with respect to Lemma 2.3. □

Theorem 2.5 *Any matrix G from (3) satisfies (2).*

Proof (i) Since M is p.s.d. there exists a matrix J of the full rank in columns such that $M = JJ'$. Thus

$$MAGA = JJ'A(N + A'MA)^- A'M[A'JJ'A(N + A'MA)^- A'M]^- A'MA.$$

Let $U = J'A(N + A'MA)A'M, V = A'J$. Obviously $\mathcal{M}(U) = \mathcal{M}(V')$. Thus $MAGA = JU(VU)^- VJ'A$ and $U(VU)^- V$ is the Euclidean projection matrix on $\mathcal{M}(J'A)$, with respect to Lemma 2.4. Thus $JU(VU)^- VJ'A = JJ'A = MA$.

(ii) $MAG = MA(N + A'MA)^- A'M[A'MA(N + A'MA)^- A'M]^- A'M$ and

$$\begin{aligned} G'A'MAG &= MA\{[A'MA(N + A'MA)^- A'M]^-\}'MA(N + A'MA)^- \\ &\quad \times A'MA(N + A'MA)^- A'M[A'MA(N + A'MA)^- A'M]^- A'M \\ &= MA\{[A'MA(N + A'MA)^- A'M]^-\}'MA(N + A'MA)^- A'M = G'A'M. \end{aligned}$$

Thus $G'A'M = G'A'MAG$ (symmetric) $= MAG$.

(iii) Since $(N + A'MA)^+$ is symmetric (this matrix can be used instead of $(N + A'MA)^-$) and p.s.d. there exists a matrix K of the full rank in columns such that $(N + A'MA)^+ = KK'$. Thus

$$\begin{aligned} NGA &= NKK'A'M[A'MA(N + A'MA)^- A'M]^- A'MA \\ &= NKK'A'M[A'MA(N + A'MA)^- A'M]^- A'MA(N + A'MA)^+(N + A'MA). \end{aligned}$$

Let $U = K'A'M$ and $V = A'MAK$. The matrix

$$K'A'M(A'MAKK'A'M)^{-}A'MAK = U(VU)^{-}V$$

is a projection matrix in Euclidean norm on $\mathcal{M}(K'A'M)$ with respect to Lemma 2.4. Thus

$$\begin{aligned} NGA &= NGA'MAKK'(N + A'MA) \\ &= NKP_{K'A'M}K'N + NKP_{K'A'M}K'A'MA \\ &= NKP_{K'A'M}K'N + NKK'A'MA \\ &= NKP_{K'A'M}K'N + A'MA - A'MAKK'A'MA, \end{aligned}$$

what is a symmetric matrix.

(iv)

$$\begin{aligned} NGAG &= N(N + A'MA)^{-}A'M[A'MA(N + A'MA)^{-}A'M]^{-}A'MA \\ &\quad \times (N + A'MA)^{-}A'M[A'MA(N + A'MA)^{-}A'M]^{-}A'M \\ &= N(N + A'MA)^{-}A'M[A'MA(N + A'MA)^{-}A'M]^{-}A'M = NG. \end{aligned}$$ □

References

[1] Chipman, J. S.: *On least squares with insufficient observations.* J. Amer. Statist. Assoc. **59** (1964), 1078–1111.

[2] Rao, C. R., Mitra, K. S.: *Generalized Inverse of Matrices and Its Application.* J. Wiley, New York 1971.