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The Degrees of Regularity in Varieties

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Abstract

Congruence regular varieties are characterized by a Mal'cev condition containing m-ary terms. We prove that this number m is the degree of regularity, i.e. the number of elements which generate the congruence class of every principal congruence.

Key words: Mal'cev condition, regularity, 0-regularity, regularity with respect to $g_1, \ldots, g_n$.

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Recall from [1] that if $g_1(x), \ldots, g_n(x)$ are unary terms, we say that an algebra A is *regular with respect to* $g_1, \ldots, g_n$ if for any $a \in A$, $\Theta = \Phi$ for $\Theta, \Phi \in \operatorname{Con} A$ whenever $[g_i(a)]_\Theta = [g_i(a)]_\Phi$ for $i = 1, \ldots, n$. A variety $\mathcal{V}$ is *regular with respect to* $g_1, \ldots, g_n$ if all its members have this property.

Let us remark that if $g_i(x) = x$ for $i = 1, \ldots, n$ then it gives the common concept of *regularity*. If $g_i(x) = 0$ for $i = 1, \ldots, n$ (where 0 is a nullary term) then we obtain the concept of *0-regularity* alias *weak regularity*. Moreover, the concept of regularity with respect to $g_1, \ldots, g_n$ coincides with that of subregularity introduced by J. Duda, [4], see [1] for some details. The following statement was proven in [1]:

Proposition 1 *The following conditions on a variety $\mathcal{V}$ with unary terms*

$$g_1(x), \ldots, g_n(x)$$

are equivalent:

(1) $\mathcal{V}$ is regular with respect to $g_1, \ldots, g_n$;

(2) for some positive integer m, there exist ternary terms $p_1, \ldots, p_m$ and a function $r \mapsto i_r$ from $\{1, \ldots, m\}$ to $\{1, \ldots, n\}$ such that $\mathcal{V}$ satisfies

$$[p_1(x,y,z) = g_{i_1}(z) \& \cdots \& p_m(x,y,z) = g_{i_m}(z)] \Rightarrow x = y;$$

(3) for some positive integers m, k there exist ternary terms $p_1, \ldots, p_m$, $(m+3)$-ary terms $t_1, \ldots, t_k$ and a function $r \mapsto i_r$ from $\{1, \ldots, m\}$ to $\{1, \ldots, n\}$ such that for $j = 1 \ldots, k-1$ and $r = 1, \ldots, m$, $\mathcal{V}$ satisfies $p_r(x,x,z) = g_{i_r}(z)$ and

$$(*)\begin{cases} x = t_1(x,y,z,g_{i_1}(z), \ldots, g_{i_m}(z)) \\ t_j(x,y,z,p_1(x,y,z), \ldots, p_m(x,y,z)) = t_{j+1}(x,y,z,g_{i_1}(z), \ldots, g_{i_m}(z)) \\ y = t_k(x,y,z,p_1(x,y,z), \ldots, p_m(x,y,z)). \end{cases}$$

Moreover, if one of the foregoing equivalent conditions holds for $(*)$ then k is the smallest integer for which $\mathcal{V}$ is $(k+1)$-permutable.

Hence, the Proposition characterizes the degree of permutability by the number of terms t_i in $(*)$. On the other hand, it was not clear what is the dependence of the integer m in (ii) or (iii). We are going to introduce a degree of regularity which relates this m.

At first, we will solve the simplest case for $k = 1$ and $m = 1$, i.e. for permutable varieties.

Definition 1 Let $g_1(x), \ldots, g_n(x)$ be unary terms. An algebra A has *transferable congruences with respect to $g_1, \ldots, g_n$* if for any $a, b, x \in A$ there exist $c_1, \ldots, c_n \in A$ such that $\Theta(a,b) = \Theta(g_i(x), c_i)$ holds for each $i \in \{1, \ldots, n\}$. A variety $\mathcal{V}$ has *transferable congruences with respect to $g_1, \ldots, g_n$* if each $A \in \mathcal{V}$ has this property.

Theorem 1 *The following conditions are equivalent for a variety $\mathcal{V}$ with unary terms $g_1(x), \ldots, g_n(x)$:*

(1) $\mathcal{V}$ has transferable congruences with respect to $g_1, \ldots, g_n$;

(2) for each $i \in \{1, \ldots, n\}$ there exists an integer k and a ternary term p_i and 5-ary terms $t_1, \ldots, t_k$ such that $p_i(x,x,z) = g_i(z)$ and
$x = t_1(x,y,z,g_i(z),p_i(x,y,z))$,
$t_j(x,y,z,p_i(x,y,z),g_i(z)) = t_{j+1}(x,y,z,g_i(z),p_i(x,y,z))$ for $j = 1, \ldots, k-1$,
$y = t_k(x,y,z,p_i(x,y,z),g_i(z))$;

(3) for each $i \in \{1, \ldots, n\}$ there exists a ternary term p_i such that

$$p_i(x,x,z) = g_i(z) \quad \text{iff} \quad x = y.$$

Proof (1) $\Rightarrow$ (2): Put $A = F_{\mathcal{V}}(x,y,z)$. By (1), for each $i \in \{1,\ldots,n\}$ there exists $c_i \in A$ with $\Theta(x,y) = \Theta(g_i(z), c_i)$. Hence, $c_i = p_i(x,y,z)$ for some 3-ary term p_i and, immediately, $p_i(x,x,z) = g_i(z)$. Since $\langle x,y\rangle \in \Theta(g_i(z), p_i(x,y,z))$, there exist 5-ary terms $t_1,\ldots,t_k$ satisfying (2).

(2) $\Rightarrow$ (1): Let $A \in \mathcal{V}$ and $a,b,x \in A$. By (2) we have

$$\langle a,b\rangle \in \Theta(g_i(x), p_i(a,b,x)).$$

Further, $\langle g_i(x), p_i(a,b,x)\rangle = \langle p_i(a,a,x), p_i(a,b,x)\rangle \in \Theta(a,b)$, i.e. $\Theta(a,b) = \Theta(g_i(x), p_i(a,b,x))$ proving (1).

(1) $\Rightarrow$ (3) is implicitely contained in (1) $\Rightarrow$ (2) since for those p_i we have $p_i(x,y,z) = g_i(z)$ iff $x = y$.

(3) $\Rightarrow$ (1): Let $A \in \mathcal{V}$ and $x,y,z \in A$. Put $c_i = p_i(x,y,z)$. Then $\langle g_i(z), c_i\rangle = \langle p_i(x,x,z), p_i(x,y,z)\rangle \in \Theta(x,y)$. Denote by $\Theta = \Theta(g_i(z), p_i(x,y,z))$. Then in A/Θ we have $[g_i(z)]_\Theta = [p_i(x,y,z)]_\Theta = p_i([x]_\Theta, [y]_\Theta, [z]_\Theta)$. However, $A/\Theta \in \mathcal{V}$, thus also A/Θ satisfies (3), i.e. we obtain $[x]_\Theta = [y]_\Theta$ giving

$$\langle x,y\rangle \in \Theta = \Theta(g_i(z), c_i).$$

Altogether, $\Theta(x,y) = \Theta(g_i(z), c_i)$ proving (1). □

By *(iii)* of the Proposition, we conclude

Corollary 1 *If a variety $\mathcal{V}$ has transferable congruences with respect to $g_1,\ldots,g_n$ then $\mathcal{V}$ is regular with respect to $g_1,\ldots,g_n$.*

Now, we can characterize the simplest case:

Theorem 2 *For a variety $\mathcal{V}$, the following are equivalent:*

(1) $\mathcal{V}$ is permutable and has transferable congruences with respect to $g_1,\ldots,g_n$;

(2) for each $i \in \{1,\ldots,n\}$ there exists a 3-ary term p_i and a 4-ary term t_i such that $p_i(x,x,z) = g_i(z)$ and $x = t_i(x,y,z,g_i(z))$, $y = t_i(x,y,z,p_i(x,y,z))$.

Proof (1) $\Rightarrow$ (2): Consider again $F_{\mathcal{V}}(x,y,z)$ and $\Theta = \Theta(x,y)$. For each $i \in \{1,\ldots,n\}$ there exists $c_i \in F_{\mathcal{V}}(x,y,z)$ with $\Theta(x,y) = \Theta(g_i(z), c_i)$. Hence $c_i = p_i(x,y,z)$ for some 3-ary term $p_i(x,y,z)$ and $p_i(x,x,z) = g_i(z)$. Moreover, the permutability implies

$$\Theta(g_i(z), p_i(x,y,z)) = R(g_i(z), p_i(x,y,z))$$

whence $\langle x,y\rangle \in R(g_i(z), p_i(x,y,z))$. It is a routine way to prove (2).

(2) $\Rightarrow$ (1): for permutability, put $m(x,y,z) = t_i(x,z,y,p_i(y,z,y))$. Then $m(x,y,z)$ is a Mal'cev term, i.e. $\mathcal{V}$ is permutable.

Prove transferability: let $A \in \mathcal{V}$ and $a,b,x \in A$. Then $\langle g_i(x), p_i(a,b,x)\rangle = \langle p_i(a,a,x), p(a,b,x)\rangle \in \Theta(a,b)$, $\langle a,b\rangle = \langle t_i(a,b,x,g_i(x)), t_i(a,b,x,p_i(a,b,x))\rangle \in \Theta(g_i(x), p_i(a,b,x))$ thus $\Theta(a,b) = \Theta(g_i(x), p_i(a,b,x))$. □

Remark 1 By Theorem 2, if regularity is replaced by transferability in a permutable variety, then $m = 1$ in the Proposition. Hence, this condition has an influence on this number. We can generalize the concept of transferability to obtain a full characterization of this m. Theorem 2 is a generalization of the result of [2], [3] for regular and permutable varieties.

Definition 2 An algebra A is said to have *m-transferable congruences with respect to* $g_1, \dots, g_n$ if for any a, b, x of A there exist $c_1, \dots, c_m \in A$ such that

$$\Theta(a,b) = \Theta(g_{i_1}(x), c_1) \vee \cdots \vee \Theta(g_{i_m}(x), c_m)$$

for any subset $\{i_1, \dots, i_m\} \subseteq \{1, \dots, n\}$. A variety $\mathcal{V}$ has *m-transferable congruences w.r.t.* $g_1, \dots, g_n$ if each $A \in \mathcal{V}$ has this property.

Theorem 3 *A variety $\mathcal{V}$ has m-transferable congruence with respect to $g_1, \dots, g_n$ if and only if $\mathcal{V}$ satisfies* (ii) *of the Proposition.*

Proof Consider $F_{\mathcal{V}}(x,y,z)$ of $\mathcal{V}$. By the definition, there exist $c_1, \dots, c_m \in F_{\mathcal{V}}(x,y,z)$ with

$$\Theta(x,y) = \Theta(g_{i_1}(z), c_1) \vee \cdots \vee \Theta(g_{i_m}(z), c_m)$$

for any $\{i_1, \dots, i_m\} \subseteq \{1, \dots, n\}$. Hence, $c_j = p_j(x,y,z)$ $(i = 1, \dots, m)$ and

$$[p_1(x,y,z) = g_{i_1}(z) \& \cdots \& p_m(x,y,z) = g_{i_m}(z)] \quad \text{iff} \quad x = y\,.$$

The converse implication can be shown similarly as in the proof of Theorem 1.
□

Corollary 2 *A variety $\mathcal{V}$ is regular with respect to $g_1, \dots, g_n$ if and only if $\mathcal{V}$ has m-transferable congruences w.r.t. $g_1, \dots, g_n$ for some integer $m \geq 1$.*

Combining the approach developed in [1] with the foregoing results, we can easily prove:

Theorem 4 *If a variety $\mathcal{V}$ satisfies $(*)$ of the Proposition for some integers m, k, then k is the smallest integer for which $\mathcal{V}$ is $(k+1)$-permutable and $\mathcal{V}$ has m-transferable congruences with respect to $g_1, \dots, g_n$.*

Let us remark that if $g_i(z) = \cdots = g_n(z) = z$ then $\mathcal{V}$ has *m-transferable congruences*, i.e. $\forall A \in \mathcal{V}$ and for each $a, b, d \in A$ there exist $c_1, \dots, c_m \in A$ with

$$\Theta(a,b) = \Theta(d, c_1, \dots, c_m)\,.$$

If $g_1(z) = \cdots = g_n(z) = 0$ then $\mathcal{V}$ has *m-transferable congruences at* 0, i.e. for each $A \in \mathcal{V}$, any $a, b \in A$ there are $c_1, \dots, c_m \in A$ with $\Theta(a,b) = \Theta(0, c_1, \dots, c_m)$.

Hence, a variety $\mathcal{V}$ is regular (or 0-regular) if and only if $\mathcal{V}$ has m-transferable congruences (at 0, respectively) for some integer $m \geq 1$.

References

[1] Barbour, G. D., Raftery, J. G.: *On the degrees of permutability of subregular varieties.* preprint 1994.

[2] Chajda I.: *Regularity and permutability of congruences*, Algebra Univ. **11** (1980), 159–162.

[3] Duda J.: *Mal'cev condition for regular and weakly regular subalgebras of the square.* Acta Sci. Math. (Szeged) **46** (1983), 29–34.

[4] Duda J.: *Mal'cev conditions for subalgebras algebras.* Acta Sci. Math. (Szeged) **51** (1987), 329–334.