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On f-planar Mappings of Affine-connection Spaces with Infinite Dimension *

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Abstract

This paper is concerned with certain fundamental equations of f-planar mappings of affine-connection spaces with infinite dimension, which is constructed on a base of Banachian space. For spaces with finite dimension these equations are known very well.

Key words: Geodesic mappings, f-planar mapping, affine connection space, infinite dimension.

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The geodesic and f-planar mappings were earlier studied for affine-connected and Riemannian spaces with finite dimension [1], [4], [6], [7]. For infinite dimensional spaces geodesic mappings were first studied by V. E. Fomin [2], [3]. This paper deals with geodesic and f-planar mappings on affine connected spaces with infinite dimensions.

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1 Introduction

Let M be a manifold which is constructed on a base of Banachian space E in [2], [3]. For each point $x \in M$ the tangent bundle T_x is a Banachian space E. In the local study we will suppose that M is identical with the open regions $U \subset E$.

There are defined two affine connections ∇ and $\overline{\nabla}$ in U which on M define structures of affine connected spaces without torsion A and $\overline{A}$. The connections ∇ and $\overline{\nabla}$ are in each point $x \in U$ defined as the smooth bilinear operators is

$$\nabla_X Y \in E, \qquad \overline{\nabla}_X Y \in E,$$

where $X, Y \in E$.

A *geodesic curve* γ: $x = x(t)$, $t \in [a, b]$, of an affine-connection space A is a curve whose tangent vector λ, being parallel displaced, is recurrent along it, i.e., the conditions

$$\nabla_\lambda \lambda = \rho(t)\, \lambda, \tag{1}$$

where ρ is a function of the parameter t, are fulfilled.

The diffeomorphism of A onto $\overline{A}$ is a *geodesic mapping* if it transforms any geodesic of the space A into a geodesic of $\overline{A}$.

V. E. Fomin [3] proved the following

Theorem 1 *The necessary and sufficient conditions for the geodesic mappings of A onto $\overline{A}$ are the following:*

$$T(X, Y) = \psi(X) \cdot Y + \psi(Y) \cdot X, \tag{2}$$

where

$$T(X, Y) \equiv \overline{\nabla}_X Y - \nabla_X Y \tag{3}$$

is the deformation tensor *of the affine connection A and $\overline{A}$, $X, Y \in T_U$ are vector fields, ψ is a linear form.*

Now, we will show a shorter and more simple proof of this statement, which uses the smooth affine connections ∇ and $\overline{\nabla}$.

Proof Let us admit that A is geodesically mapped onto $\overline{A}$. Then any geodesic curve $\gamma \subset U$ in space A, determined by (1), is mapped into the geodesic curve $\overline{\gamma}$ in $\overline{A}$ which is determined by

$$\overline{\nabla}_\lambda \lambda = \overline{\rho}(t)\, \lambda \tag{4}$$

where $\overline{\rho}$ is a function of the argument t.

If we substract (4) from (1) in each point $x \in U$ using the notation (3) we obtain

$$T(\lambda, \lambda) = a\, \lambda,$$

where $a \equiv \overline{\rho}(t) - \rho(t)$.

These formulae are satisfied in each point $x \in U$ and for any vector $\lambda \in E$. Obviously, a is a function of the arguments x and λ. We can suppose that it is a continuous function of both its arguments.

For any non-zero vector λ and real number $\alpha \neq 0$ we write $T(\alpha \cdot \lambda, \alpha \cdot \lambda)$. Since T is a bilinear form we have the rule

$$a(\alpha \cdot \lambda) = \alpha \cdot a(\lambda). \tag{6}$$

Since a is a continuous function, the condition (6) is true for each $\lambda \in E$ and each $\alpha \in R$.

Since A and $\overline{A}$ are torsion-free, the deformation tensor T is a symmetric bilinear form. The calculation $T(X+Y, X+Y)$ gives for all $X, Y \in T_x$ the formula

$$T(X,Y) = (a(X+Y) - a(X))\,X - (a(X+Y) - a(Y)Y)\,. \tag{7}$$

We now express $T(X, \alpha \cdot Y)$. It is clear from (6) and (7) that we get

$$2\alpha T(X,Y) = (a(X+\alpha Y) - a(X))\,X - (a(X+\alpha Y) - \alpha \cdot a(Y)) \cdot \alpha Y.$$

Let us suppose X, Y linearly independent and $\alpha \neq 0$, then multiplying (7) by a number α and substracting the result from the last formula, we get a system of equations

$$\alpha \cdot (a(X+Y) - a(X)) = a(X+\alpha Y) - a(X),$$
$$a(X+Y) - a(Y) = a(X+\alpha Y) - \alpha \cdot a(Y).$$

Substracting these two equations when $\alpha \neq 1$, we obtain

$$a(X+Y) = a(X) + a(Y). \tag{8}$$

This equality is satisfied for all $X, Y \in E$, of course. Therefore, $a(X)$ is a linear form. Moreover, if we denote $\psi(X) = \frac{1}{2}a(X)$, then (7) implies (2).

Conversely, if (2) is satisfied then the mapping is geodesic, which is trivial.

2 f-planar mappings

Let us consider the space A of affine connection without torsion, for which the affine connection ∇ and the vector field $f \neq 0$ are defined.

The curve ℓ: $x = x(t)$ is said to be *f-planar* (in the terminology of K. Yano [8] it is *subplanar* and in that of S. Bácsó and L. Verhócki [1] it is *E_1-planar*) if the tangent vector λ, being translated along it, lies in the plane area formed by the tangent vector λ and the vector f, i.e.

$$\nabla_\lambda \lambda = \rho_1(t) \cdot \lambda + \rho_2(t) \cdot f, \tag{9}$$

where ρ_1 and ρ_2 are functions of the argument t. If $\rho_2(t) \equiv 0$, the f-planar curve is geodesic.

Let us consider two spaces A and $\overline{A}$ in which there are defined vector fields f and $\overline{f}$, respectively.

The diffeomorphism $A \to \overline{A}$ is called *f-planar* (see [5], *E_1-planar*, see [1]) if, under this mapping, any f-planar curve of space A is mapped into an $\overline{f}$-planar curve of space $\overline{A}$.

Similarly as in the example mentioned we can define in an open region $U \subset E$ the affine connections ∇ and $\overline{\nabla}$, which determine spaces A and $\overline{A}$ and also vector fields f and $\overline{f}$.

Theorem 2 *The mapping of A onto $\overline{A}$ is f-planar if and only if the conditions*

$$T(X,Y) = \psi(X)\,Y + \psi(Y)\,X + \sigma(X,Y)\cdot f, \tag{10}$$

$$\overline{f} = \alpha \cdot f, \tag{11}$$

where $\psi(X)$ is a linear form and $\sigma(X,Y)$ a bilinear symmetric form, α a function and $T(X,Y)$ the deformation tensor (3), are satisfied.

Proof Let (A, f) be f-planar and let it be mapped onto $(\overline{A}, \overline{f})$. Then, any f-planar curve $\gamma \subset A$ is mapped onto an $\overline{f}$-planar curve $\overline{\gamma} \subset \overline{A}$.

a) Since geodesic curves are f-planar, by an f-planar mapping, any geodesic curve $\gamma \subset A$ will be mapped onto an $\overline{f}$-planar curve $\overline{\gamma} \subset \overline{A}$. Let γ be defined by the equation (1) and $\overline{\gamma}$

$$\overline{\nabla}_\lambda \lambda = \overline{\rho}_1(t)\cdot\lambda + \overline{\rho}_2(t)\cdot\overline{f}. \tag{12}$$

If we substract (1) from (12) in each point $x \in A$, then with the notation (3) we get

$$T(\lambda,\lambda) = a\cdot\lambda + b\cdot\overline{f}, \tag{13}$$

where $a \equiv \overline{\rho}_1(t) - \rho(t)$, $b \equiv \overline{\rho}_2(t)$.

These conditions are fulfilled in any point $x \in A$ and for any vector $\lambda \in E$. Therefore, we can consider a and b to be functions of x and λ, moreover, they are continuous with respect to these arguments.

For any vector λ, we write $T(\alpha\lambda, \alpha\lambda)$. With regard to the bilinearity of T, $\alpha \neq 0$ and non-collinearity of λ and $\overline{f}$, we get

$$a(\alpha\cdot\lambda) = \alpha\cdot a(\lambda) \qquad \text{and} \qquad b(\alpha\cdot\lambda) = \alpha^2\cdot b(\lambda). \tag{14}$$

Since a and b are continuous, the power of the set of the vectors $\alpha\cdot\lambda \parallel \overline{f}$ is zero in E and this condition is fulfilled for any vectors $X, Y \in E$. This means that $a(X)$ is a linear mapping $E \to R$. Note

$$B(X,Y) = T(X,Y) - (\psi(X)Y + \psi(Y)X) \tag{16}$$

where $\psi(X) \equiv \frac{1}{2}a(X)$. The operator $B(X,Y)$ is bilinear and symmetric. Then, (13) can be written as

$$B(\lambda,\lambda) = b(\lambda)\cdot\overline{f}. \tag{17}$$

The operator $B(X,Y)$ is divided into parts $\overline{f}$ and its complement $\overline{f}^{\perp}$, then

$$B(X,Y) = \sigma^*(X,Y) \cdot \overline{f} + \overline{f}^{\perp}(X,Y)$$

where σ^* is symmetrical bilinear form. Then, (17) can be written as

$$\sigma(\lambda,\lambda) \cdot \overline{f} + \overline{f}^{\perp}(\lambda,\lambda) = b(\lambda) \cdot \overline{f}.$$

Hence

$$b(\lambda) = \sigma^*(\lambda,\lambda) \qquad \text{and} \qquad \overline{f}^{\perp}(\lambda,\lambda) = 0, \text{ i.e. } \overline{f}^{\perp}(X,Y) = 0.$$

Therefore, from (16) we get

$$T(X,Y) = \psi(X) \cdot Y + \psi(Y) \cdot X + \sigma^*(X,Y) \cdot \overline{f} \tag{18}$$

where ψ is a linear form and σ^* is a bilinear form.

b) Further we will define a special type of f-planar curves in a space A, which are characterized by the equations

$$\nabla_\lambda \lambda = f. \tag{19}$$

According to formulae (3) and (17) these curves are mapped by an f-planar mapping into curves in $\overline{A}$, for which

$$\overline{\nabla}_\lambda \lambda = f + 2\,\psi(\lambda) \cdot \lambda + \sigma(\lambda,\lambda) \cdot \overline{f} \tag{20}$$

holds. On the other hand, this curve will be $\overline{f}$-planar if and only if the vector f is expressed by help of

$$f = a_1(\lambda) \cdot \lambda + a_2(\lambda) \cdot \overline{f}. \tag{21}$$

The formula (21) is satisfied in any point $x \in A$ and for an arbitrary vector $\lambda \in T_x$. Similarly as in the example mentioned above we prove that the validity of (11) follows from (21). On the other hand (10) and (11) are also sufficient conditions for the mapping to be f-planar.

Remarks. For a space A_n with finite dimension n ($n \geq 3$) the statement (2) was proved by S. Bácsó and L. Verhóczki [1]. The proof stated above can applied for A_n ($n \geq 3$) and it is much easier.

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