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Optimal Control of a System Governed by Petrowsky Type Equation with an Infinite Number of Variables

WIESŁAW KOTARSKI

Institute of Mathematics, Silesian University,
Bankowa 14, 40-007 Katowice, Poland
e-mail: Kotarski@gate.math.us.edu.pl

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Abstract

We derive necessary and sufficient conditions of optimality for a system described by an evolution equation of Petrowsky type with an infinite number of variables. The time of control is assumed to be fixed. Constraints on controls and states are imposed. The performance index is more general than quadratic one and has an integral form. To obtain optimality conditions we use the well-known Dubovitskii–Milyutin method.

Key words: Petrowsky type equation, infinite number of variables, optimal control, Dubovitskii–Milyutin method.

1991 Mathematics Subject Classification: 49K20, 93C20, 35R15

1 Introduction

In [4], [5], [6] optimal control problems for systems described by operators with an infinite number of variables have been considered. That operators are similar to the stationary Schrödinger operator. The interests in the study of that class of operators is stimulated by problems in quantum field theory [2], [4]. In [4], [5], [6] to obtain optimality conditions arguments of [12] have been applied.

In our paper making use of the Dubovitskii–Milyutin method [3], [7], [13] similarly as in [9], [10] we derive necessary and sufficient conditions of optimality for a system described by Petrowsky type equation with an infinite number of variables. That problem with quadratic performance index and contraints imposed only on controls has been earlier considered by Gali and El-Saify in [6].

2 Some functional spaces ([4], [5], [6], [9])

Let $(P_k(t))_{k=1}^{\infty}$ be a fixed sequence of weights such that $0 < P_k(t) \in C^{\infty}(R^1)$, $\int_{R^1} P_k(t)dt = 1$. With respect to this sequence on $R^{\infty} = R^1 \times R^1 \times \ldots$ with the boundary Γ (Γ is meant as the boundary of the support of the measure $d\varrho(x)$ defined below) it can be introduced the measure $d\varrho(x)$ in the following way

$$d\varrho(x) = (P_1(x_1)dx_1) \otimes (P_2(x_2)dx_2) \otimes \ldots, \quad (R^{\infty} \ni x = (x_k)_{k=1}^{\infty}, \; x_k \in R^1).$$

The examples of the construction of the measure $d\varrho(x)$ are given in [1]. On R^{∞} one can construct the space $L_2(R^{\infty}) := L_2(R^{\infty}, d\varrho(x))$ with the norm

$$||u||_{L_2(R^{\infty})} := \left(\int_{R^{\infty}} |u|^2 d\varrho(x) \right)^{1/2} < +\infty.$$

The space $L_2(R^{\infty})$ is a Hilbert one with the scalar product

$$(u, v)_{L_2(R^{\infty})} = \int_{R^{\infty}} u(x)v(x)d\varrho(x).$$

For functions which are $l = 1, 2, \ldots$ times continuosly differentiable up to the boundary Γ of R^{∞} and which vanish on Γ it can be introduced the scalar product

$$(u, v)_{H^l(R^{\infty})} = \sum_{|\alpha| \leq l} (D^{\alpha}u, D^{\alpha}v)_{L_2(R^{\infty})}$$

where D^{α} is defined by

$$D^{\alpha} = \frac{\partial^{|\alpha|}}{(\partial x_1)^{\alpha_1}(\partial x_2)^{\alpha_2}\ldots}, \qquad |\alpha| = \sum_{i=1}^{\infty} \alpha_i.$$

Understanding the differentation in the distributional sense, after the standard procedure of completion one can obtain Sobolev spaces $H^l(R^{\infty})$ ($l = 1, 2, \ldots$). The space $H^0(R^{\infty})$ is equivalent to $L_2(R^{\infty})$.

To the spaces $H^l(R^{\infty})$ ($l = 1, 2, \ldots$) one can construct their duals $H^{-l}(R^{\infty})$. The duality between the spaces $H^l(R^{\infty})$ and $H^{-l}(R^{\infty})$ is induced by the scalar product of the space $L_2(R^{\infty})$.

Next one can define the following space

$$H_0^l(R^{\infty}) := \{u : u \in H^l(R^{\infty}), \; D^{\alpha}u = 0 \quad \text{on } \Gamma, \; |\alpha| \leq l - 1, \; l > 1\}$$

and its dual $H_0^{-l}(R^{\infty})$.

For spaces mentioned above we have the following chains

$$\begin{aligned} H^l(R^\infty) &\subseteq L_2(R^\infty) = H^0(R^\infty) \subseteq H^{-l}(R^\infty) \\ H_0^l(R^\infty) &\subseteq L_2(R^\infty) \subseteq H_0^{-l}(R^\infty) \end{aligned}$$

and

$$||u||_{H_0^l(R^\infty)} \geq ||u||_{L_2(R^\infty)} \geq ||u||_{H_0^{-l}(R^\infty)}.$$

$L_2(0,T;\, H_0^l(R^\infty))$ denotes the space of measurable functions $(0,T) \to H_0^l(R^\infty)$: $t \mapsto f(t)$, where $T < +\infty$, such that

$$||f||_{L_2(0,T;\, H_0^l(R^\infty))} := \left(\int_0^T ||f(t)||^2_{H_0^l(R^\infty)} dt \right)^{1/2} < +\infty.$$

The space $L_2(0,T;\, H_0^l(R^\infty))$ is a Hilbert one with the scalar product

$$(f,g)_{L_2(0,T;\, H_0^l(R^\infty))} = \int_0^T (f(t),g(t))_{H_0^l(R^\infty)} dt.$$

Analogously it can be defined the spaces

$$L_2(0,T;\, L_2(R^\infty)) \quad \text{and} \quad L_2(0,T;\, H_0^{-l}(R^\infty)).$$

For them we have the chain

$$L_2(0,T;\, H_0^l(R^\infty)) \subseteq L_2(0,T;\, L_2(R^\infty)) \subseteq L_2(0,T;\, H_0^{-l}(R^\infty)).$$

3 Petrowsky type equation with an infinite number of variables

For the operator $A(t)$ in the form [6]

$$(A(t)\Phi)(x) = \sum_{|\alpha| \leq l} \sum_{k=1}^{\infty} (-1)^{|\alpha|} \frac{1}{\sqrt{P_k(x_k,t)}} \frac{\partial^{2\alpha}}{\partial x_k^{2\alpha}} \left(\sqrt{P_k(x_k,t)}\Phi(x) \right) + q(x,t)\Phi(x)$$

where $q(x,t)$ for all $t \in (0,T)$ is a real-valued function in x that is bounded and measurable on R^∞, such that $q(x,t) \geq c > 0$, c a constant, the bilinear form $\pi(t;\Phi,\Psi) := (A(t)\Phi,\Psi)_{L_2(R^\infty)}$ is coercive on $H_0^l(R^\infty)$.

The operator $A(t)$ is a bounded self-adjoint elliptic operator of $2l^{\text{th}}$ order with an infinite number of variables mapping $H_0^l(R^\infty)$ onto $H_0^{-l}(R^\infty)$.

Now let us consider the following evolution equation of Petrowsky type

$$A(t)y + \frac{\partial^2 y}{\partial t^2} = f \qquad x \in R^\infty,\ t \in (0,T) \tag{3.1}$$

$$y(x,0) = y_1(x) \qquad x \in R^\infty \tag{3.2}$$

$$\frac{\partial y}{\partial t}(x,0) = y_2(x) \qquad x \in R^\infty \tag{3.3}$$

$$y(x,t) = 0 \qquad x \in \Gamma,\ t \in (0,T) \tag{3.4}$$

where $f \in L_2(0,T;\ H_0^{-l}(R^\infty)),\ y_1 \in H_0^l(R^\infty),\ y_2 \in L_2(R^\infty)$.

Denote by $Q = R^\infty \times (0,T)$ and $L_2(Q) := L_2(0,T;\ L_2(R^\infty))$. From [6] and the results of [12] we know that there is a unique solution

$$\left(y, \frac{\partial y}{\partial t}\right) \in L_2(0,T;\ H_0^l(R^\infty)) \times L_2(Q)$$

to the equations (3.1)–(3.4) and the mapping $(f, y_1, y_2) \mapsto \left(y, \frac{\partial y}{\partial t}\right)$:

$$L_2(0,T; H_0^{-l}(R^\infty)) \times H_0^l(R^\infty) \times L_2(R^\infty)) \to L_2(0,T;\ H_0^l(R^\infty)) \times L_2(Q)$$

is (norm,norm)-continuous. Moreover, the operator $A(t) + \frac{\partial^2}{\partial t^2}$ [6] is a linear bounded operator which maps $L_2(0,T; H_0^l(R^\infty))$ onto $L_2(0,T; H_0^{-l}(R^\infty))$.

4 Statement of optimal control problem Optimality conditions

We consider the following optimization problem

$$A(t)y + \frac{\partial^2 y}{\partial t^2} = u \qquad x \in R^\infty,\ t \in (0,T) \tag{4.1}$$

$$y(x,0) = y_1(x) \qquad x \in R^\infty \tag{4.2}$$

$$\frac{\partial y}{\partial t}(x,0) = y_2(x) \qquad x \in R^\infty \tag{4.3}$$

$$y(x,t) = 0 \qquad x \in \Gamma,\ t \in (0,T) \tag{4.4}$$

Let us denote by $Y = L_2(0,T;\ H_0^l(R^\infty)) \times L_2(Q)$ the space of states and by $U = L_2(Q)$ the space of controls.

The control time T is assumed to be fixed.

The performance functional is given by

$$I(y,u) = \int_Q F(x,t,y,u)d\varrho(x)dt \to \min \tag{4.5}$$

where $F : R^\infty \times [0,T] \times R^1 \times R^1 \mapsto R^1$ satisfies the following conditions:

A1) $F(x,t,y,u)$ is continuous with respect to (x,t,y,u),

A2) there exist $F_y(x,t,y,u), F_u(x,t,y,u)$ which are continuous with respect to (x,t,y,u),

A3) $F(x,t,y,u)$ is strictly convex with respect to the pair (y,u) i.e.

$$F(x,t,\lambda y_1+(1-\lambda)y_2,\lambda u_1+(1-\lambda)u_2) < \lambda F(x,t,y_1,u_1)+(1-\lambda)F(x,t,y_2,u_2),$$
$$\forall y_1,y_2,u_1,u_2 \in R^1,\ (y_1,u_1) \neq (y_2,u_2),\ \lambda \in (0,1).$$

We assume the following contraints on controls:

$$u \in U_{ad} \text{ is a closed, convex subset of the space } L_2(Q) \tag{4.6}$$

and on states:

$$\begin{array}{l} y \in U_{ad} \text{ is a closed, convex subset of } L_2(0,T;\ H_0^l(R^\infty)) \\ \qquad\text{with non-empty interior.} \end{array} \tag{4.7}$$

Also we assume the following condition: there exists $(\tilde{y},\tilde{u})$ such that $\tilde{y} \in \text{int}Y_{ad}$, $\tilde{u} \in U_{ad}$ and $(\tilde{y},\tilde{u})$ satisfies the equation (4.1)–(4.4) (the so-called Slater's condition).

The necessary and sufficient optimality conditions to the problem (4.1)–(4.7) are formulated in the following theorem:

Theorem 1 *Under the assumptions mentioned above, there is a unique solution* (y^0,u^0) *to the problem* (4.1)–(4.7). *Moreover, there is the adjoint state p,*

$$\left(p, \frac{\partial p}{\partial t}\right) \in L_2(0,T;\ H_0^l(R^\infty)) \times L_2(Q),$$

which satisfies (in the weak sense) the adjoint equation given below and the necessary and sufficient conditions of optimality are characterized by the following system of partial differential equations and inequalities:
state equation

$$A(t)y^0 + \frac{\partial^2 y^0}{\partial t^2} = u^0 \qquad x \in R^\infty,\ t \in (0,T) \tag{4.8}$$

$$y^0(x,0) = y_1(x) \qquad x \in R^\infty \tag{4.9}$$

$$\frac{\partial y^0}{\partial t}(x,0) = y_2(x) \qquad x \in R^\infty \tag{4.10}$$

$$y^0(x,t) = 0 \qquad x \in \Gamma,\ t \in (0,T) \tag{4.11}$$

adjoint equation

$$A(t)p + \frac{\partial^2 p}{\partial t^2} = F_y \qquad x \in R^\infty,\ t \in (0,T) \tag{4.12}$$

$$p(x,T) = 0 \qquad x \in R^\infty \tag{4.13}$$

$$\frac{\partial p}{\partial t}(x,T) = 0 \qquad x \in R^\infty \tag{4.14}$$

$$p(x,t) = 0 \qquad x \in \Gamma,\ t \in (0,T) \tag{4.15}$$

maximum conditions

$$\int_Q (p+F_u)(u-u^0)d\varrho(x)dt \geq 0 \qquad \forall u \in U_{ad} \tag{4.16}$$

$$\int_Q (F_y+F_u\mathcal{F})(y-y^0)d\varrho(x)dt \geq 0 \qquad \forall y \in Y_{ad} \tag{4.17}$$

where F_y, F_u *are Fréchet derivatives of* F *with respect to* y, u, *respectively at the point* (y^0, u^0), $\mathcal{F} : \left\{y \in L_2(0,T;\ H_0^1(R^\infty));\ A(t)y + \frac{\partial^2 y}{\partial t^2} \in L_2(Q)\right\} \to U$ *is the operator related to the equation* (4.1)–(4.4) *with zero-initial conditions.*

Proof We apply the generalized Dubovitskii–Milyutin theorem (Theorem 4.1 [13]).

Denote by Q_1, Q_2, Q_3 the sets in the space $E := Y \times U$ with elements $z = ((y, \frac{\partial y}{\partial t}), u)$.

$$Q_1 := \left\{ z \in E;\ \begin{array}{ll} A(t)y + \dfrac{\partial^2 y}{\partial t^2} = u & x \in R^\infty,\ t \in (0,T) \\ y(x,0) = y_1(x) & x \in R^\infty \\ \dfrac{\partial y}{\partial t}(x,0) = y_2(x) & x \in R^\infty \\ y(x,t) = 0 & x \in \Gamma,\ t \in (0,T) \end{array} \right\}$$

$$Q_2 := \left\{ z \in E :\quad \left(y, \frac{\partial y}{\partial t}\right) \in Y,\ u \in U_{ad} \right\}$$

$$Q_3 := \left\{ z \in E;\quad y \in Y_{ad},\ \frac{\partial y}{\partial t} \in L^2(Q),\ u \in U \right\}$$

Thus the optimization problem may be formulated in the form

$$I(y,u) \to \min \quad \text{subject to} \quad (y,u) \in Q_1 \cap Q_2 \cap Q_3$$

We aproximate the sets Q_1 and Q_2 by the regular tangent cones (RTC), the set Q_3 by the regular admissible cone (RAC) and the performance index by the regular improvement cone (RFC) [7], [13].

The cone tangent to the set Q_1 at z^0 has the form

$$\mathrm{RTC}\,(Q_1, z^0) = \{\overline{z} \in E;\ P'(z^0)\overline{z} = 0\} =$$

$$\left\{ \overline{z} \in E;\ \begin{array}{ll} A(t)y + \dfrac{\partial^2 \overline{y}}{\partial t^2} = \overline{u} & x \in R^\infty,\ t \in (0,T) \\ \overline{y}(x,0) = y_1(x) & x \in R^\infty \\ \dfrac{\partial \overline{y}}{\partial t}(x,0) = y_2(x) & x \in R^\infty \\ \overline{y}(x,t) = 0 & x \in \Gamma,\ t \in (0,T) \end{array} \right\}$$

where $P'(z^0)\overline{z}$ is the Fréchet differential of the operator

$$P\left(y, \frac{\partial y}{\partial t}, u\right) := \left(A(t)y + \frac{\partial^2 y}{\partial t^2} - u,\ y(x,0) - y_1(x), \frac{\partial y}{\partial t}(x,0) - y_2(x)\right)$$

mapping from the space $\mathcal{V} := L_2(0,T;\ H_0^l(R^\infty)) \times L_2(Q) \times L_2(Q)$ into the space $\Omega := L_2(0,T;\ H_0^{-l}(R^\infty)) \times H_0^l(R^\infty) \times L_2(R^\infty)$.

Knowing that there exists a unique solution to the equation (4.1)–(4.4) for every u, y_1 and y_2 it is easy to prove that $P'(z^0)$ is the mapping from the space $\mathcal{V}$ onto Ω, as it is needed in the Lusternik theorem (Theorem 9.1 [7]).

The tangent cone RTC(Q_2, z^0) to the set Q_2 at z^0 has the form $Y\times$ RTC (U_{ad}, u^0), where RTC (U_{ad}, u^0) is the tangent cone to the set U_{ad} at the point u^0.

Following [14] it is easy to show that

$$\mathrm{RTC}\,(Q_1 \cap Q_2, z^0) = \mathrm{RTC}\,(Q_1, z^0) \cap \mathrm{RTC}\,(Q_2, z^0).$$

We only need to show the inclusion "$\supset$", because always we have "$\subset$" [11]. It can be easily checked that in the neighbourhood V_0 of the point

$$\left(\left(y^0, \frac{\partial y^0}{\partial t}\right), u^0\right)$$

the operator P satisfies the assumptions of the implicit function theorem [8]. Consequently the set Q_1 can be represented in the neighbourhood V_0 in the form

$$\left\{\left(\left(y, \frac{\partial y}{\partial t}\right), u\right) \in E;\ \left(y, \frac{\partial y}{\partial t}\right) = \varphi(u)\right\} \tag{4.18}$$

where $\varphi : L_2(Q) \mapsto L_2(0,T;\ H_0^l(R^\infty)) \times L_2(Q)$ is an operator of the class C^1 satisfying the condition $P(\varphi(u), u) = 0$ for u such that $(\varphi(u), u) \in V_0$.

From this we have

$$\mathrm{RTC}(Q_1, z^0) = \left\{\overline{z} \in E;\ \left(\overline{y}, \frac{\partial \overline{y}}{\partial t}\right) = \varphi_u(u^0)\overline{u}\right\}. \tag{4.19}$$

Let $((\overline{y}, \frac{\partial \overline{y}}{\partial t}), \overline{u})$ be any element of the set RTC $(Q_1, z^0) \cap$ RTC(Q_2, z^0).

From the definition of the tangent cone [7] we have that there exists an operator $r_u^2 : R^1 \mapsto U$ such that $\frac{r_u^2(\varepsilon)}{\varepsilon} \to 0$ as $\varepsilon \to 0^+$ and

$$\left(\left(y^0, \frac{\partial y^0}{\partial t}\right), u^0\right) + \varepsilon\left(\left(\overline{y}, \frac{\partial \overline{y}}{\partial t}\right), \overline{u}\right) + (r_y^2, r_u^2) \in Q_2 \tag{4.20}$$

for a sufficiently small ε and with any $r_y^2(\varepsilon)$.

From (4.18) follows that for sufficiently small ε, we have

$$\varphi(u^0 + \varepsilon\overline{u} + r_u^2(\varepsilon)) = \varphi(u^0) + \varepsilon\varphi_u(u^0)\overline{u} + r_y^1(\varepsilon)$$

for some $r_y^1(\varepsilon)$ such that $\frac{r_y^1(\varepsilon)}{\varepsilon} \to 0$ as $\varepsilon \to 0^+$.

Taking into account (4.18) and (4.19), we get

$$\left(\left(y^0, \frac{\partial y^0}{\partial t}\right), u^0\right) + \varepsilon\left(\left(\overline{y}, \frac{\partial \overline{y}}{\partial t}\right), \overline{u}\right) + (r_y^1(\varepsilon), r_u^2(\varepsilon)) \in Q_1. \tag{4.21}$$

If in (4.20) we have $r_y^2(\varepsilon) = r_y^1(\varepsilon)$, then it follows from (4.20) and (4.21) that $\left(\left(\overline{y}, \frac{\partial \overline{y}}{\partial t}\right), \overline{u}\right)$ is an element of the cone tangent to the set $Q_1 \cap Q_2$ at z^0.

It finishes the proof of the inclusion "$\supset$".

From [11] it is known that tangent cones are closed.

Further applying Theorem 3.3 [13] we can prove that the adjoint cones $[\mathrm{RTC}\, Q_1, z^0)]^*$ and $[\mathrm{RTC}\,(Q_2, z^0)]^*$ are of the same sense [13].

The admissible cone $\mathrm{RAC}\,(Q_3, z^0)$ to the set Q_3 at z^0 is $\mathrm{RAC}\,(Y_{ad}, y^0) \times L_2(Q) \times U$, where $\mathrm{RAC}\,(Y_{ad}, y^0)$ is the admissible cone to the set Y_{ad} at y^0.

Using Theorem 7.5 [7] we find the regular improvement cone

$$\mathrm{RFC}\,(I, z^0) = \{\overline{z} \in E;\ I'(z^0)\overline{z} < 0\}$$

where $I'(z^0)\overline{z}$ is the Fréchet differential of the performance functional.

By the assumptions $(A1)$ $(A2)$ this differential exists (compare with the example 7.2 [7]) and can be written as

$$\int_Q (F_y \overline{y} + F_u \overline{u}) d\varrho(x) dt.$$

If $\mathrm{RFC}\,(I, z^0) \neq \emptyset$, then the adjoint cone to it consists of the elements of the form (Theorem 10.2 [7]):

$$f_4(\overline{z}) = -\lambda_0 \int_Q (F_y \overline{y} + F_u \overline{u}) d\varrho(x) dt$$

where $\lambda_0 \geq 0$.

Since $\mathrm{RTC}\,(Q_1, z^0)$ is a subspace of E, then the functionals belonging to $[\mathrm{RTC}\,(Q_1, z^0)]^*$ are (Theorem 10.1 [7]):

$$f_1(\overline{z}) = 0 \qquad \forall \overline{z} \in \mathrm{RTC}\,(Q_1, z^0).$$

The functionals $f_2(\overline{z}) \in [\mathrm{RTC}\,(Q_2, z^0)]^*$ can be expressed as follows

$$f_2(\overline{z}) = f_2^1\left(\overline{y}, \frac{\partial \overline{y}}{\partial t}\right) + f_2^2(\overline{u})$$

where $f_2^1\left(\overline{y}, \frac{\partial \overline{y}}{\partial t}\right) = 0\ \forall \left(\overline{y}, \frac{\partial \overline{y}}{\partial t}\right) \in Y$ (Theorem 10.1 [7]), $f_2^2(\overline{u})$ is the support functional to the set U_{ad} at the point u^0 (Theorem 10.5 [7]).

Similarly, the functionals $f_3(\overline{z}) \in [\mathrm{RAC}\,(Q_3, z^0)]^*$ can be expressed as follows

$$f_3(\overline{z}) = f_3^1(\overline{y}) + f_3^2\left(\frac{\partial \overline{y}}{\partial t}\right) + f_3^3(\overline{u})$$

where $f_3^1(\overline{y})$ is the support functional to the set Y_{ad} at the point y^0 (Theorem 10.5 [7]), $f_3^2\left(\frac{\partial \overline{y}}{\partial t}\right) = 0 \ \forall \frac{\partial \overline{y}}{\partial t} \in L_2(Q)$ and $f_3^3(\overline{u}) = 0 \ \forall \overline{u} \in U$ (Theorem 10.1 [7]).

Since all assumptions of the Dubovitskii–Milyutin theorem (Theorem 4.1 [13]) are satisfied and we know suitable cones we are now ready to write down the Euler–Lagrange equation in the following form

$$f_2^2(\overline{u}) + f_3^1(\overline{y}) = \lambda_0 \int\limits_Q (F_y\overline{y} + F_u\overline{u})d\varrho(x)dt = \frac{1}{2}\lambda_0 \int\limits_Q (F_y\overline{y} + F_u\overline{u})d\varrho(x)dt + \\ + \frac{1}{2}\lambda_0 \int\limits_Q (F_y\overline{y} + F_u\overline{u})d\varrho(x)dt \qquad \forall \overline{z} \in \mathrm{RTC}\,(Q_1, z^0). \tag{4.22}$$

We transform $\int_Q F_y\overline{y}d\varrho(x)dt$ introducing the adjoint variable p by the equation (4.12)–(4.15) and taking into account that $\left(\overline{y}, \frac{\partial \overline{y}}{\partial t}\right)$ is the solution of $P'(z^0)\overline{z} = 0$ for any fixed $\overline{u}$.

In turn, we get

$$\begin{aligned} \int\limits_Q F_y\overline{y}d\varrho(x)dt &= \int\limits_Q \left(A(t)p + \frac{\partial^2 p}{\partial t^2}\right)\overline{y}d\varrho(x)dt = \int\limits_Q pA(t)\overline{y}d\varrho(x)dt + \\ &+ \int\limits_{R^\infty} \frac{\partial p}{\partial t}\overline{y}\bigg|_0^T d\varrho(x) - \int\limits_{R^\infty} p\frac{\partial \overline{y}}{\partial t}\bigg|_0^T d\varrho(x) + \int\limits_Q p\frac{\partial^2 \overline{y}}{\partial t^2}d\varrho(x)dt = \\ &= \int\limits_Q p\overline{u}d\varrho(x)dt. \end{aligned}$$

Further $\int\limits_Q F_u\overline{u}d\varrho(x)dt$ can be replaced by $\int\limits_Q F_u\mathcal{F}\,\overline{y}d\varrho(x)dt$.

Taking the above into account from (4.22), we obtain

$$f_2^2(\overline{u}) + f_3^1(\overline{y}) = \frac{1}{2}\lambda_0 \int\limits_Q (p + F_u)\overline{u}d\varrho(x)dt + \frac{1}{2}\lambda_0 \int\limits_Q (F_y + F_u\mathcal{F})\overline{y}d\varrho(x)dt. \tag{4.23}$$

λ_0 in (4.23) cannot be equal to zero, because in this case all functionals in the Euler–Lagrange equation would be zero, which is impossible according to the Dubovitskii–Milyutin theorem.

Using the definition of the support functional [7] and dividing both sides of the obtained inequalities by $\frac{1}{2}\lambda_0$, we finally get (4.16), (4.17).

If $\mathrm{RFC}\,(I, z^0) = \emptyset$, then optimality conditions (4.8)–(4.17) are fulfilled with equalities in the maximum conditions (4.16), (4.17).

In order to prove sufficiency of the derived conditions of optimality we use the fact that the constraints are convex, the performance functional is continuous and convex and the Slater condition is satisfied (Theorem 15.2 [7]).

The uniqueness of the solution to the problem (4.1)–(4.7) follows from the strict convexity of the performance functional (4.5) (assumption ($A3$)).

This last remark completes the proof of Theorem 4.1.

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