Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica

František Machala; Marek Pomp

Homomorphisms of contexts and isomorphisms of concept lattices

Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica, Vol. 35 (1996), No. 1, 121--129

Persistent URL: http://dml.cz/dmlcz/120339

Acta Univ. Palacki. Olomuc., Fac. rer. nat.,
Mathematica **35** (1996) 121–129

Homomorphisms of Contexts and Isomorphisms of Concept Lattices *

FRANTIŠEK MACHALA [1], MAREK POMP [2]

[1] *Department of Algebra and Geometry, Faculty of Sciences,
Palacký University, Tomkova 40, 779 00 Olomouc, Czech Republic
e-mail: machala@risc.upol.cz*
[2] *Department of Mathematics, Faculty of Science, University of Ostrava,
Bráfova 6, 700 00 Ostrava, Czech Republic*

(Received September 7, 1995)

Abstract

Homomorphisms of contexts induce maps on corresponding concept lattices. We are studying a relationship between these homomorphisms and maps.

Key words: Incidence structures, concept lattice, homomorphism of incidence structures.

1991 Mathematics Subject Classification: 06B05, 08A35

Definition 1 Let G and M be a nonempty sets and $I \subseteq G \times M$. Then the triple $\mathcal{J} = (G, M, I)$ is called *a context*.

Definition 2 Let $A \subseteq G$, $B \subseteq M$ be a nonempty sets. Then we denote:

$$A^{\uparrow} = \{m \in M;\ gIm\ \forall g \in A\}, \qquad B^{\downarrow} = \{g \in G;\ gIm\ \forall m \in B\},$$
$$\emptyset^{\uparrow} = M, \qquad \emptyset^{\downarrow} = G.$$

Denotation 1 Let $A \subseteq G$, $B \subseteq M$. We denote $A^{\uparrow\downarrow} := (A^{\uparrow})^{\downarrow}$ and $B^{\downarrow\uparrow} := (B^{\downarrow})^{\uparrow}$, respectively. And moreover, for $g \in G$, $m \in M$, we denote $g^{\uparrow} := \{g\}^{\uparrow}$ and $m^{\downarrow} := \{m\}^{\downarrow}$.

*Supported by grant No. 201/95/1631 of The Grant Agency of Czech Republic.

Remark 1 Let $A, C \subseteq G$ and $B, D \subseteq M$. Then $A \subseteq C$ implies $C^{\uparrow} \subseteq A^{\uparrow}$ and $B \subseteq D$ implies $D^{\downarrow} \subseteq B^{\downarrow}$. Moreover, $A^{\uparrow\downarrow\uparrow} = A^{\uparrow}$, $B^{\downarrow\uparrow\downarrow} = B^{\downarrow}$. And finally $\bigcap_{i\in J} A_i^{\uparrow} = (\bigcup_{i\in J} A_i)^{\uparrow}$ for $A_i \subseteq G$, $i \in J$ and, similarly, $\bigcap_{i\in J} B_i^{\downarrow} = (\bigcup_{i\in J} B_i)^{\downarrow}$ holds for $B_i \subseteq M$ $i \in J$. (See [3]).

Lemma 1 *Let $\mathcal{J} = (G, M, I)$ be a context, and let us consider the set Q defined by*

$$Q = \{A \subseteq G;\ A = A^{\uparrow\downarrow}\}.$$

Then the (partially) ordered set $(Q, \subseteq)$ is a complete lattice. The operations $\wedge$, $\vee$ on this lattice are defined as follows:

$$\bigwedge_{i\in J} A_i = \bigcap_{i\in J} A_i\,, \qquad \bigvee_{i\in J} A_i = (\bigcap_{i\in J} A_i^{\uparrow})^{\downarrow}$$

for $A_i \in Q$. (See proof in [3]).

Definition 3 The complete lattice $(Q, \wedge, \vee)$ from the previous lemma is called *a concept lattice.* We denote it $K(\mathcal{J})$. The maximal element or the minimal element in $K(\mathcal{J})$ are denoted by 1, or 0 respectively.

Remark 2 If $\mathcal{G}_{\mathcal{J}} = \{g^{\uparrow\downarrow};\ g \in G\}$ and $\mathcal{M}_{\mathcal{J}} = \{m^{\downarrow};\ m \in M\}$, then $\mathcal{G}_{\mathcal{J}}, \mathcal{M}_{\mathcal{J}} \subseteq Q$. In what follows we denote $\mathcal{U}_{\mathcal{J}} = \mathcal{G}_{\mathcal{J}} \cup \{0\}$ and $\mathcal{V}_{\mathcal{J}} = \mathcal{M}_{\mathcal{J}} \cup \{1\}$.

Definition 4 A context $\mathcal{J} = (G, M, I)$ is called *faithful,* if

(1) $$g^{\uparrow\downarrow} = h^{\uparrow\downarrow} \Longrightarrow g = h,$$

(2) $$m^{\downarrow} = n^{\downarrow} \Longrightarrow m = n$$

for every $g, h \in G$ and every $m, n \in M$, respectively.

Example 1 Let $(L, \wedge, \vee)$ be a complete lattice and $\leq$ be the ordering of L defined by the operations $\wedge$, $\vee$. Then $\mathcal{J}_L = (L, L, \leq)$ is a context. Let $U(A)$ or $L(A)$ be the upper bound and the lower bound of a set $A \subseteq L$, respectively. Then $A^{\uparrow} = U(A)$ and $A^{\downarrow} = L(A)$. Hence $A^{\uparrow\downarrow} = LU(A) = L(x)$, where $x = \vee A$ for every $A \subseteq L$ and, particularly, $x^{\uparrow\downarrow} = L(x) = x^{\downarrow}$ for $x \in L$, with regard to $x = \vee x$. If $K(\mathcal{J}_L) = (Q, \subseteq)$ is a corresponding concept lattice, then $A \in Q$ if and only if $A = L(x)$ for a some element $x \in L$. The elements of lattice $K(\mathcal{J}_L)$ are lower bounds of elements of lattice L and $\mathcal{G}_{\mathcal{J}_L} = \mathcal{M}_{\mathcal{J}_L} = Q$. Evidently $x^{\uparrow\downarrow} = y^{\uparrow\downarrow} = L(x) = L(y)$ which implies that $x = y$ for $x, y \in L$. And similarly, $x^{\downarrow} = y^{\downarrow}$ implies $x = y$. Then the context $\mathcal{J}_L$ is faithful.

Definition 5 Let $\mathcal{U}$ and $\mathcal{V}$ be subsets of a complete lattice L. Then we call them *supremal* and *infimal dense sets* in L if there exists subsets $N \subseteq \mathcal{U}$ and $P \subseteq \mathcal{V}$ such that $x = \vee N$ and $x = \wedge P$ for every $x \in L$, respectively.

Theorem 1 *Let L be a complete lattice and $\leq$ a corresponding ordering of L. Let $\mathcal{U}$ or $\mathcal{V}$ be a supremal and an infimal dense subset of L, respectively. Then the context $\mathcal{J} = (\mathcal{U}, \mathcal{V}, \leq)$ is faithful.*

Proof 1. First we proof that $x = \vee(L(x) \cap \mathcal{U})$ for arbitrary $x \in L$. Evidently, $\vee L(x) = x$ and $L(x) \cap \mathcal{U} \subseteq L(x)$. Then $\vee(L(x) \cap \mathcal{U}) \leq \vee L(x) = x$. There exists $N \subseteq \mathcal{U}$ such that $x = \vee N$. Since $N \subseteq L(x)$, then $N \subseteq L(x) \cap \mathcal{U}$. Subsequently, $x = \vee N \leq \vee(L(x) \cap \mathcal{U})$ and $x = \vee(L(x) \cap \mathcal{U})$. Similarly $x = \wedge(U(x) \cap \mathcal{V})$.

2. Let $\mathcal{J} = (\mathcal{U}, \mathcal{V}, \leq)$ and $\mathcal{J}_L = (L, L, \leq)$ be contexts. In order to distinguish between these two contexts, we will write the arrows at the context $\mathcal{J}_L$ on the right hand side and at $\mathcal{J}$ on the left hand side. Now, ${}^{\uparrow}g = \{x \in \mathcal{V};\ g \leq x\} = U(g) \cap \mathcal{V}$ and ${}^{\uparrow\downarrow}g = \{x \in \mathcal{U};\ x \leq y\ \forall y \in U(g) \cap \mathcal{V}\}$ for every $g \in \mathcal{U}$. According to 1, $g = \wedge(U(g) \cap \mathcal{V})$ which implies ${}^{\uparrow\downarrow}g = L(g) \cap \mathcal{U} = g^{\uparrow\downarrow} \cap \mathcal{U}$.
Similarly ${}^{\downarrow}m = L(m) \cap \mathcal{U} = m^{\downarrow} \cap \mathcal{U}$ holds for $m \in \mathcal{V}$.

3. Let ${}^{\uparrow\downarrow}g = {}^{\uparrow\downarrow}h$ for $g, h \in \mathcal{U}$. Then $L(g) \cap \mathcal{U} = L(h) \cap \mathcal{U}$ and $g = h$ according to 1. Similarly ${}^{\downarrow}m = {}^{\downarrow}n$ implies $m = n$ holds. □

Definition 6 Let $\mathcal{J} = (G, M, I)$ be a context, and let $G_1 \subseteq G$, $M_1 \subseteq M$ be nonempty subsets and $I_1 \subseteq G_1 \times M_1$.

1. If $I_1 \subseteq I$, then the context $\mathcal{J}_1 = (G_1, M_1, I_1)$ is said to be *an embedded context* into $\mathcal{J}$.
2. If $I_1 = I \cap (G_1 \times M_1)$, then the context $\mathcal{J}_1 = (G_1, M_1, I_1)$ is said to be *a subcontext* of the context $\mathcal{J} = (G, M, I)$.

Remark 3 Let L be a complete lattice. Then the context $\mathcal{J} = (\mathcal{U}, \mathcal{V}, \leq)$ from Theorem 1 is the subcontext of the context $\mathcal{J}_L$.

Definition 7 Let $\mathcal{J} = (G, M, I)$, $\mathcal{J}_1 = (G_1, M_1, I_1)$ be contexts. Then a map $\varphi\colon G \cup M \to G_1 \cup M_1$ satisfying the conditions

$$\varphi(G) \subseteq G_1, \quad \varphi(M) \subseteq M_1, \tag{1}$$

$$gIm \Longrightarrow \varphi(g)\, I_1\, \varphi(m) \tag{2}$$

is said to be *a homomorphism* of the context $\mathcal{J}$ into the context $\mathcal{J}_1$.

Definition 8 Let φ be a homomorphism of a context $\mathcal{J} = (G, M, I)$ into a context $\mathcal{J}_1 = (G_1, M_1, I_1)$. We define the incidence relation $I_\varphi \subseteq \varphi(G) \times \varphi(M)$ on the context $\varphi(\mathcal{J}) = (\varphi(G), \varphi(M), I_\varphi)$ by

$$\varphi(g) I_\varphi \varphi(m) \Longleftrightarrow \begin{array}{l} \exists h \in G \quad \varphi(g) = \varphi(h), \\ \exists n \in M \quad \varphi(m) = \varphi(n), \end{array} hIn.$$

Remark 4 In what follows, we will consider homomorphisms φ satisfying one of the following conditions:

$$\varphi(g) I_1 \varphi(m) \Longrightarrow gIm, \tag{H1}$$

$$\varphi(g) I_1 \varphi(m) \Longrightarrow \begin{array}{ll} (a) \ \exists n \in M, & \varphi(n) = \varphi(m),\ gIn, \\ (b) \ \exists h \in G, & \varphi(g) = \varphi(h),\ \ hIm, \end{array} \tag{H2}$$

$$\varphi(g) I_1 \varphi(m) \Longrightarrow \begin{array}{l} \exists h \in G \quad \varphi(g) = \varphi(h), \\ \exists n \in M \quad \varphi(m) = \varphi(n), \end{array} hIn. \tag{H3}$$

Remark 5 Evidently, (H1) $\Longrightarrow$ (H2) $\Longrightarrow$ (H3).

Definition 9 A homomorphism φ satisfying (H1) is called *an I-homomorphism.* If φ is an I-homomorphism, $\varphi(G) = G_1$, $\varphi(M) = M_1$, and φ induces a bijective maps of sets G, G_1 and M, M_1, then φ is an isomorphism. If φ is an isomorphism of the context $\mathcal{J}$ onto the context $\mathcal{J}_1$, then $\mathcal{J}$, $\mathcal{J}_1$ are called isomorphic contexts and we denote them $\mathcal{J} \simeq \mathcal{J}_1$.

Theorem 2 *Let $\mathcal{J} = (G, M, I)$, $\mathcal{J}_1 = (G_1, M_1, I_1)$, $\mathcal{J}_2 = (G_2, M_2, I_2)$ be contexts and $\varphi : \mathcal{J} \to \mathcal{J}_1$ and $\alpha : \mathcal{J}_1 \to \mathcal{J}_2$ respectively be surjective homomorphisms. Then $\xi = \alpha\varphi$ is a surjective homomorphism of the context $\mathcal{J}$ onto $\mathcal{J}_2$ and following conditions are fulfilled.*

1. *ξ is an I-homomorphism if and only if α and φ are I-homomorphisms.*
2. *(a) If α and φ satisfy (H3), then ξ satisfies (H3).*
   (b) If ξ satisfies (H3), then α satisfies (H3). In addition let α be a bijective map, then φ satisfies (H3).
3. *Condition 2 is valid for (H2), too.*

Proof 1. Immediately, the map ξ is a surjective homomorphism. Let α, φ be an I-homomorphisms and let us suppose $\xi(g)I_2\xi(m)$. Then $\alpha\varphi(g)\, I_2\, \alpha\varphi(m)$. With regard to the fact that α is an I-homomorphism we obtain $\varphi(g)I_1\varphi(m)$, and moreover, gIm because φ is an I-homomorphism, too.

Let ξ be an I-homomorphism and $\varphi(g)I_1\varphi(m)$. Then $\xi(g)I_2\xi(m)$ and consequently gIm, hence φ is an I-homomorphism. Let be $\alpha(g_1)I_2\alpha(m_1)$ for $g_1 \in G_1$, $m_1 \in M_1$. Because φ is a map onto $\mathcal{J}_1$, there exist $g \in G$, $m \in M$ such that $\varphi(g) = g_1$, $\varphi(m) = m_1$. Hence $\xi(g)I_2\xi(m)$ and thus gIm and $\varphi(g)I_1\varphi(m)$ which imply $g_1I_1m_1$ and α is an I-homomorphism.

2. (a) Let us assume that α, φ satisfy (H3). Let $\xi(g)I_2\xi(m)$.
Then $\alpha(\varphi(g))I_2\alpha(\varphi(m))$ and there are $g_1 \in G_1$, $m_1 \in M_1$ such that $\alpha(g_1) = \alpha(\varphi(g))$, $\alpha(m_1) = \alpha(\varphi(m))$ and $g_1I_1m_1$. Certainly there exist $h' \in G$, $n' \in M$ such that $\varphi(h') = g_1$, $\varphi(n') = m_1$, then $\varphi(h')I_1\varphi(n')$. Because φ satisfies (H3), there are $h \in G$, $n \in M$ such that $\varphi(h) = \varphi(h')$, $\varphi(n) = \varphi(n')$ and hIn. Evidently $\xi(h) = \alpha(\varphi(h)) = \alpha(\varphi(h')) = \alpha(g_1) = \alpha(\varphi(g)) = \xi(g)$. Similarly $\xi(n) = \xi(m)$.

(b) Let ξ satisfy (H3). Let us assume $\alpha(g_1)I_2\alpha(m_1)$ for $g_1 \in G_1$, $m_1 \in M_1$. There are $g \in G$, $m \in M$ such that $\varphi(g) = g_1$, $\varphi(m) = m_1$. We obtain $\xi(g)I_2\xi(m)$. Then $h \in G$, $n \in M$ exist such that $\alpha(\varphi(h)) = \alpha(\varphi(g))$, $\alpha(\varphi(n)) = \alpha(\varphi(m))$ and hIn. Subsequently, $\varphi(h)I_1\varphi(n)$ and $\alpha(\varphi(h)) = \alpha(\varphi(g)) = \alpha(g_1)$, $\alpha(\varphi(n)) = \alpha(m_1)$ and α satisfies (H3).

Let us assume, that α is a bijective map onto $\mathcal{J}_2$. Let $\varphi(g)I_1\varphi(m)$. Then $\alpha(\varphi(g))I_2\alpha(\varphi(m))$ and $\xi(g)I_2\xi(m)$. Then $h \in G$, $m \in M$ exist such that $\alpha(\varphi(g)) = \alpha(\varphi(h))$, $\alpha(\varphi(m)) = \alpha(\varphi(n))$ and hIn. Therefore $\varphi(g) = \varphi(h)$ and $\varphi(m) = \varphi(n)$. Hence φ satisfies (H3).

3. Proof is similar to the previous one.

Theorem 3 *Let $\varphi : \mathcal{J} = (G, M, I) \to \mathcal{J}_1 = (G_1, M_1, I_1)$ be a homomorphism. Then the context $\varphi(\mathcal{J})$ is embedded into $\mathcal{J}_1$ and φ is a homomorphism onto $\varphi(\mathcal{J})$. The context $\varphi(\mathcal{J})$ is a subcontext of $\mathcal{J}_1$ if and only if φ satisfies (H3).*

Proof It is evident, that φ is a homomorphism of $\mathcal{J}$ onto $\varphi(\mathcal{J})$. And moreover, $\varphi(g)\ I_\varphi\ \varphi(m)$ implies that there exist $h \in G$, $n \in M$ such that $\varphi(g) = \varphi(h)$, $\varphi(m) = \varphi(n)$ and hIn. This implies that $\varphi(g) I_1 \varphi(m)$, and hence $I_\varphi \subseteq I_1$. If φ satisfies (H3), then the converse implications hold and $\varphi(\mathcal{J})$ is a subcontext of $\mathcal{J}_1$. Let $\varphi(\mathcal{J})$ be a subcontext of $\mathcal{J}_1$. Then $\varphi(g) I_1 \varphi(m)$ implies $\varphi(g) I_\varphi \varphi(m)$ and, according to the definition of the relation I_φ, φ satisfies (H3). □

Remark 6 The following lemmas are proved in [1] and [2].

Lemma 2 *Let $\mathcal{J} = (G, M, I)$ be a context. Then the map α defined by*

$$\alpha:\ g \mapsto g^{\uparrow\downarrow} \quad \forall g \in G,$$
$$m \mapsto m^{\downarrow}\ \forall m \in M$$

is an I-homomorphism of the context $\mathcal{J}$ into the context $\mathcal{J}_{K(\mathcal{J})}$, and the sets $\mathcal{U}_\mathcal{J}$ and $\mathcal{V}_\mathcal{J}$ are supremal or infimal dense in $K(\mathcal{J})$, respectively.

Remark 7 The map α from Lemma 2 satisfies (H3) and, therefore, the context $\alpha(\mathcal{J}) = (\mathcal{G}_\mathcal{J}, \mathcal{M}_\mathcal{J}, \subseteq)$ is a subcontext of $\mathcal{J}_{K(\mathcal{J})}$. The context $\varphi(\mathcal{J}) = (\mathcal{U}_\mathcal{J}, \mathcal{V}_\mathcal{J}, \subseteq)$ is faithful according to Theorem 1.

Lemma 3 *Let $\mathcal{J} = (G, M, I)$ be a context, L be a complete lattice and φ be an I-homomorphism of the context $\mathcal{J}$ into $\mathcal{J}_L$ such that $\mathcal{U} = \varphi(G) \cup \{0\}$ and $\mathcal{V} = \varphi(M) \cup \{1\}$ are dense sets in L. There exists an isomorphism ψ of (complete) lattices $K(\mathcal{J})$ and L, which induces bijective maps of sets $\mathcal{G}_\mathcal{J}$, $\varphi(G)$ respectively $\mathcal{M}_\mathcal{J}$, $\varphi(M)$ such that*

$$\psi(g^{\uparrow\downarrow}) = \varphi(g) \quad \forall g \in G, \tag{1}$$

$$\psi(m^{\downarrow}) = \varphi(m) \quad \forall m \in M. \tag{2}$$

Remark 8 Let α, φ be maps according to Lemma 2 and Lemma 3. Then the contexts $\alpha(\mathcal{J}) = (\mathcal{G}_\mathcal{J}, \mathcal{M}_\mathcal{J}, \leq)$ and $\varphi(\mathcal{J}) = (\varphi(G), \varphi(M), \leq)$ are subcontexts of the contexts $\mathcal{J}_{K(\mathcal{J})}$ and $\mathcal{J}_L$. The isomorphism $\psi : K(\mathcal{J}) \to L$, described in Lemma 3, induces a map $\bar{\psi} : \alpha(\mathcal{J}) \to \varphi(\mathcal{J})$. With regard to α, φ are I-homomorphisms, equivalences $\alpha(g) \leq \alpha(m)$ iff $g^{\uparrow\downarrow} \subseteq m^{\downarrow}$ iff gIm iff $\varphi(g) \leq \varphi(m)$ hold for all $g \in G$, $m \in M$. Then $x \leq y$ iff $\bar{\psi}(x) \leq \bar{\psi}(y)$ for $x, y \in \alpha(\mathcal{J})$ and $\bar{\psi}$ is an isomorphism of contexts $\alpha(\mathcal{J})$, $\varphi(\mathcal{J})$.

Remark 9 Let L be a complete lattice and $\mathcal{J}_L = (L, L, \leq)$ the corresponding context. The identity map $\varphi : L \to L$ satisfies the conditions from Lemma 3. The map ξ defined by $\xi(L(x)) = x\ \forall x \in L$, is an isomorphism of the lattices $K(\mathcal{J}_L)$ and L.

Remark 10 Let $\varphi\colon \mathcal{J} \to \mathcal{J}_1$ be a homomorphism onto the context $\mathcal{J}_1$. Then φ induce the map ξ of the concept lattices $K(\mathcal{J}) = (Q, \wedge, \vee)$, $K(\mathcal{J}_1) = (Q_1, \wedge, \vee)$, $\xi: K(\mathcal{J}) \to K(\mathcal{J}_1)$ such that $\xi(A) = (\varphi(A))^{\uparrow\downarrow}$ for every $A \in Q$. With regard to $A \subseteq C$ implies $\varphi(A) \subseteq \varphi(C)$ implies $(\varphi(A))^{\uparrow\downarrow} \subseteq (\varphi(C))^{\uparrow\downarrow}$ for $A, C \in Q$, ξ is a homomorphism of the context $\mathcal{J}_{K(\mathcal{J})} = (Q, Q, \subseteq)$ into the context $\mathcal{J}_{K(\mathcal{J}_1)} = (Q_1, Q_1, \subseteq)$. But ξ need not to be a map onto $K(\mathcal{J}_1)$ and ξ need not to be a homomorphism of the lattice $K(\mathcal{J})$ into the lattice $K(\mathcal{J}_1)$.

Theorem 4 *Let $\mathcal{J} = (G, M, I)$ be a context, L be a complete lattice, and φ be an I-homomorphism of the context $\mathcal{J}$ into $\mathcal{J}_L$. If there exist an isomorphism ψ of the complete lattices $K(\mathcal{J})$, L and if ψ induces a bijective map of the sets $\mathcal{G}_{\mathcal{J}}$, $\varphi(G)$ and $\mathcal{M}_{\mathcal{J}}$, $\varphi(M)$, respectively, such that $\psi(g^{\uparrow\downarrow}) = \varphi(g)$, for every $g \in G$, $\psi(m^{\downarrow}) = \varphi(m)$, for every $m \in M$, then the sets $\mathcal{U} = \varphi(G) \cup \{0\}$, $\mathcal{V} = \varphi(M) \cup \{1\}$ are dense in the lattice L.*

Proof Immediately from the assumption of our theorem we have $0 = \vee 0$, $1 = \wedge 1$ and $0 \in \mathcal{U}$, $1 \in \mathcal{V}$. If $x \in L$, $x \notin \{0, 1\}$, then there exists an element $A \in K(\mathcal{J})$, A is not the minimal or maximal element in $K(\mathcal{J})$, such that $\psi(A) = x$. According to Lemma 1, $A \subseteq G$ and $A = A^{\uparrow\downarrow}$. Furthermore, $A^{\uparrow\downarrow} = (\bigcup_{a\in A} \{a\})^{\uparrow\downarrow} = ((\bigcup_{a\in A} \{a\})^{\uparrow})^{\downarrow} = (\bigcap_{a\in A} a^{\uparrow})^{\downarrow} = (\bigcap_{a\in A} a^{\uparrow\downarrow\uparrow})^{\downarrow} = (\bigcap_{a\in A} (a^{\uparrow\downarrow})^{\uparrow})^{\downarrow} = \bigvee_{a\in A} a^{\uparrow\downarrow}$. We obtain $x = \psi(A) = \psi(\bigvee_{a\in A} a^{\uparrow\downarrow}) = \bigvee_{a\in A} \psi(a^{\uparrow\downarrow}) = \bigvee_{a\in A} \varphi(a) = \vee\varphi(A)$. Since $A \subseteq G$, then $\varphi(A) \subseteq \varphi(G)$. The set U is supremal dense in L.

Subsequently $A^{\uparrow\downarrow} = (A^{\uparrow})^{\downarrow} = B^{\downarrow} = (\bigcup_{b\in B} \{b\})^{\downarrow} = \bigcap_{b\in B} b^{\downarrow} = \bigwedge_{b\in B} b^{\downarrow}$. Then $x = \psi(A) = \psi(B^{\downarrow}) = \psi(\bigwedge_{b\in B} b^{\downarrow}) = \bigwedge_{b\in B} \psi(b^{\downarrow}) = \bigwedge_{b\in B} \varphi(b) = \wedge\varphi(B)$. Since $B \subseteq M$, then $\varphi(B) \subseteq \varphi(M)$. The set $\mathcal{V}$ is infimal dense in L. □

Remark 11 Let L be a complete lattice and G, M be nonempty subsets of L such that $\mathcal{J} = (G, M, \leq)$ is a subcontext of the context $\mathcal{J}_L = (L, L, \leq)$. According to Lemma 3 and Theorem 4 and since the map $\varphi : g \to g\ \forall g \in G$, $m \to m\ \forall m \in M$ is an I-homomorphism of the context $\mathcal{J}$ into the context $\mathcal{J}_L$ we obtain that the following conditions are equivalent.

1. The sets $\mathcal{U} = G \cup \{0\}$, $\mathcal{V} = M \cup \{1\}$ are dense in L.
2. There exists an isomorphism ψ of lattices $K(\mathcal{J})$, L which induces bijective maps of the sets $\mathcal{G}_{\mathcal{J}}$, G or $\mathcal{M}_{\mathcal{J}}$, M, respectively and $\psi(g^{\uparrow\downarrow}) = g\ \forall g \in G$, $\psi(m^{\downarrow}) = m\ \forall m \in M$.

Denotation 2 1. Let $\mathcal{A}_1 = \{\bar{a};\ a \in A\}$, $\mathcal{A}_2 = \{\tilde{a};\ a \in A\}$ be decompositions of the set A. If $\tilde{a} \subseteq \bar{a}$, for any $a \in A$, then the decomposition $\mathcal{A}_1$ is so-called covering of the decomposition $\mathcal{A}_2$ and we denote it $\mathcal{A}_2 \leq \mathcal{A}_1$.

2. Let $\mathcal{J} = (G, M, I)$ be a context and $\mathcal{G} = \{\bar{g};\ g \in G\}$, $\mathcal{M} = \{\bar{m};\ m \in M\}$ be decompositions of the sets G, M. Let us denote the corresponding decomposition of the set $G \times M$ by $\mathcal{R} = (\mathcal{G}, \mathcal{M})$. We have the new context $\mathcal{J}_{\mathcal{R}} = (\mathcal{G}, \mathcal{M}, I_{\mathcal{R}})$, where $\bar{g} I_{\mathcal{R}} \bar{m}$ iff $\exists h \in \bar{g}$, $n \in \bar{m}$ and hIn. We define the map $\varphi_{\mathcal{R}}\colon G \cup M \to \mathcal{G} \cup \mathcal{M}$ by $\varphi_{\mathcal{R}}(g) = \bar{g}\ \forall g \in G$, $\varphi_{\mathcal{R}}(m) = \bar{m}\ \forall m \in M$.

3. Let $\varphi\colon \mathcal{J} \to \mathcal{J}_1$ be a homomorphism. We denote $\bar{g} = \{h \in G;\ \varphi(h) = \varphi(g)\}$, $\mathcal{G}_\varphi = \{\bar{g};\ g \in G\}$, $\bar{m} = \{n \in M;\ \varphi(n) = \varphi(m)\}$, $\mathcal{M}_\varphi = \{\bar{m};\ m \in M\}$, and $\mathcal{R}_\varphi = (\mathcal{G}_\varphi, \mathcal{M}_\varphi)$

4. Let $\mathcal{J} = (G, M, I)$ be a context. We denote $\vec{g} = \{h \in G;\ h^\uparrow = g^\uparrow\}$, $\vec{m} = \{n \in M;\ n^\downarrow = m^\downarrow\}$, and $\vec{G} = \{\vec{g};\ g \in G\}$, $\vec{M} = \{\vec{m};\ m \in M\}$, $\mathcal{R}_\mathcal{J} = (\vec{G}, \vec{M})$, and $F(\mathcal{J}) = \mathcal{J}_{\mathcal{R}_\mathcal{J}}$.

Remark 12 The map $\varphi_\mathcal{R}$, according to 2, is a homomorphism of the context $\mathcal{J}$ onto the context $\mathcal{J}_\mathcal{R}$. The context $F(\mathcal{J})$, according to 4, is faithful. If the context $\mathcal{J}$ is faithful, then $F(\mathcal{J}) = \mathcal{J}$.

Theorem 5 *Let $\varphi\colon \mathcal{J} = (G, M, I) \to \mathcal{J}_1 = (G_1, M_1, I_1)$ be a homomorphism onto $\mathcal{J}_1$ and let us consider a map ξ_φ of the context $\mathcal{J}$ into the lattice $K(\mathcal{J}_1)$ such that*

$$g \mapsto (\varphi(g))^{\uparrow\downarrow} \quad \forall g \in G, \tag{1}$$

$$m \mapsto (\varphi(m))^{\downarrow} \quad \forall m \in M. \tag{2}$$

Then the following statements hold.

1. *ξ_φ is a homomorphism of the context $\mathcal{J}$ into the context $\mathcal{J}_{K(\mathcal{J}_1)}$ and the sets $\xi_\varphi(G) \cup \{0\}$ and $\xi_\varphi(M) \cup \{1\}$ are dense in $K(\mathcal{J}_1)$.*
2. *The decomposition $\mathcal{R}_{\xi_\varphi}$ is covering of the decomposition $\mathcal{R}_\varphi$, and $\mathcal{R}_{\xi_\varphi} = \mathcal{R}_\varphi$ if and only if the context $\mathcal{J}_1$ is faithful.*
3. *ξ_φ is an I-homomorphism if and only if φ is an I-homomorphism.*
4. *If φ satisfies (H3), then ξ_φ satisfies (H3). If ξ_φ satisfies (H3) and if the context $\mathcal{J}_1$ is faithful, then φ satisfies (H3).*
5. *Condition 4 is valid for (H2), too.*
6. *If φ satisfies (H3), then $\mathcal{J}_{\mathcal{R}_{\xi_\varphi}} \simeq F(\mathcal{J}_1)$. If φ is an I-homomorphism, then $\mathcal{J}_{\mathcal{R}_{\xi_\varphi}} = F(\mathcal{J})$.*

Proof 1. The map $\alpha_1 : g_1 \mapsto g_1^{\uparrow\downarrow}\ \forall g_1 \in G_1$, $m_1 \mapsto m_1^{\downarrow}\ \forall m_1 \in M_1$ is an I-homomorphism of the context $\mathcal{J}_1$ into the context $\mathcal{J}_{K(\mathcal{J}_1)}$ according to Lemma 2. Moreover, $\xi_\varphi(g) = (\varphi(g))^{\uparrow\downarrow} = \alpha_1(\varphi(g))\ \ \forall g \in G$ and $\xi_\varphi(m) = (\varphi(m))^{\downarrow} = \alpha_1(\varphi(m))\ \ \forall m \in M$, hence $\xi_\varphi = \alpha_1\varphi$. With regard to φ is a map onto the context $\mathcal{J}_1$ and α_1 is a map onto the context $(\mathcal{G}_{\mathcal{J}_1}, \mathcal{M}_{\mathcal{J}_1}, \leq)$, $\xi_\varphi(G) = \mathcal{G}_{\mathcal{J}_1}$, $\xi_\varphi(M) = \mathcal{M}_{\mathcal{J}_1}$ and ξ_φ is a homomorphism onto the context $\xi_\varphi(\mathcal{J}) = (\xi_\varphi(G), \xi_\varphi(M), \leq)$ according to Theorem 3. According to Lemma 2, the sets $\xi_\varphi(G) \cup \{0\}$, $\xi_\varphi(M) \cup \{0\}$ are dense in $K(\mathcal{J}_1)$.

2. If $\varphi(g) = \varphi(h)$ implies $(\varphi(g))^{\uparrow\downarrow} = (\varphi(h))^{\uparrow\downarrow}$ for $g, h \in G$ and $\varphi(m) = \varphi(n)$ implies $(\varphi(m))^{\downarrow} = (\varphi(n))^{\downarrow}$ for $m,\ n \in M$, then $\mathcal{R}_\varphi \leq \mathcal{R}_{\xi_\varphi}$. Furthermore, the equality $\mathcal{R}_\varphi = \mathcal{R}_{\xi_\varphi}$ holds if and only if $(\varphi(g))^{\uparrow\downarrow} = (\varphi(h))^{\uparrow\downarrow}$, then $\varphi(g) = \varphi(h)$, $(\varphi(m))^{\downarrow} = (\varphi(n))^{\downarrow}$ then $\varphi(m) = \varphi(n)$. With regard to φ is a map onto $\mathcal{J}_1$, this equality holds if and only if the context $\mathcal{J}_1$ is faithful.

3. We have the I-homomorphism α_1 such that $\xi_\varphi = \alpha_1\varphi$. Theorem 2 yields, that ξ_φ is an I-homomorphism if and only if φ is an I-homomorphism.

4. The map α_1 is an I-homomorphism, then (H3) is valid for it. It follows, according to Theorem 2, if φ satisfies (H3), then ξ_φ satisfies (H3), too. Let $\mathcal{J}_1$ be a faithful context. For $g_1, g_2 \in G_1$, $\alpha_1(g_1) = \alpha_1(g_2)$ implies $(g_1)^{\uparrow\downarrow} = (g_2)^{\uparrow\downarrow}$ implies $g_1 = g_2$. Similarly $\alpha_1(m_1) = \alpha_1(m_2)$ implies $(m_1)^\downarrow = (m_2)^\downarrow$ implies $m_1 = m_2$ for $m_1, m_2 \in M$. Hence α_1 is a bijective map onto the context $\xi_\varphi(\mathcal{J})$. According to Theorem 2, φ satisfies (H3).

5. Proof is similar to the previous one.

6. The map ξ_φ is a homomorphism of the context $\mathcal{J}$ onto the context $\xi_\varphi(\mathcal{J}) = (\mathcal{G}_{\mathcal{J}_1}, \mathcal{M}_{\mathcal{J}_1}, \leq)$. Let φ satisfies (H3). Subsequently, according to Condition 4 ξ_φ satisfies (H3) too, and $\xi_\varphi(\mathcal{J})$ is a subcontext of $\mathcal{J}_{K(\mathcal{J}_1)}$. The sets $\mathcal{G}_{\mathcal{J}_1} \cup \{0\}$, $\mathcal{M}_{\mathcal{J}_1} \cup \{1\}$ are dense in $K(\mathcal{J}_1)$. According to Remark 11, there exists the isomorphism ψ of the lattices $K(\xi_\varphi(\mathcal{J}))$, $K(\mathcal{J}_1)$, which induces bijective maps of sets $\mathcal{G}_{\xi_\varphi(\mathcal{J})}$, $\mathcal{G}_{\mathcal{J}_1}$, and $\mathcal{M}_{\xi_\varphi(\mathcal{J})}$, $\mathcal{M}_{\mathcal{J}_1}$. According to Lemma 2 from [1], $F(\xi_\varphi(\mathcal{J})) \simeq F(\mathcal{J}_1)$. According to Theorem 1, the context $\xi_\varphi(\mathcal{J})$ is faithful and according to Remark 12, $F(\xi_\varphi(\mathcal{J})) = \xi_\varphi(\mathcal{J})$. Hence $F(\mathcal{J}_1) \simeq \xi_\varphi(\mathcal{J})$. According to Theorem 2 from [2], ξ_φ induces an isomorphism of the contexts $\mathcal{J}_{\mathcal{R}_{\xi_\varphi}}$, $\xi_\varphi(\mathcal{J})$, and then $\mathcal{J}_{\mathcal{R}_{\xi_\varphi}} \simeq F(\mathcal{J}_1)$.

Let φ is an I-homomorphism. According to Theorem 16 from [2], $F(\mathcal{J}) \simeq F(\mathcal{J}_1)$ and then $\mathcal{J}_{\mathcal{R}_{\xi_\varphi}} \simeq F(\mathcal{J})$. Then $(\varphi(g))^\uparrow = (\varphi(h))^\uparrow$ iff $g^\uparrow = h^\uparrow$, $(\varphi(m))^\downarrow = (\varphi(n))^\downarrow$ iff $m^\downarrow = n^\downarrow$. Let us denote $\bar{g}$, $\bar{h}, \dots$ respectively $\bar{m}$, $\bar{n}, \dots$ elements of the decomposition $\mathcal{R}_{\xi_\varphi}$ and $\vec{g}$, $\vec{h}, \dots$ respectively $\vec{m}$, $\vec{n}, \dots$ elements of the decomposition $\mathcal{R}_{\mathcal{J}}$. For $g \in G$, $h \in \bar{g}$ iff $\xi_\varphi(h) = \xi_\varphi(g)$ iff $h^\uparrow = g^\uparrow$ iff $h \in \vec{g}$, then $\bar{g} = \vec{g}$. Similarly $\vec{m} = \bar{m}$ for every $m \in M$. □

Theorem 6 *Let φ be a homomorphism of a context $\mathcal{J} = (G, M, I)$ onto $\mathcal{J}_1 = (G_1, M_1, I_1)$. The following conditions are equivalent.*

1. *φ is an I-homomorphism.*
2. *φ satisfies (H3) and there exists an isomorphism ψ of lattices $K(\mathcal{J})$, $K(\mathcal{J}_1)$ which induces a bijective map of the sets $\mathcal{G}_{\mathcal{J}}$, $\mathcal{G}_{\mathcal{J}_1}$ respectively $\mathcal{M}_{\mathcal{J}}$, $\mathcal{M}_{\mathcal{J}_1}$, such that $\psi(g^{\uparrow\downarrow}) = (\varphi(g))^{\uparrow\downarrow}$ $\forall g \in G$, $\psi(m^\downarrow) = (\varphi(m))^\downarrow$ $\forall m \in M$.*

Proof (1) $\Longrightarrow$ (2). The homomorphism φ satisfies (H1), and this implies that φ satisfies (H2). According to Condition 3 from Theorem 5, the map ξ_φ is an I-homomorphism of the context $\mathcal{J}$ into the context $\mathcal{J}_{K(\mathcal{J}_1)}$. With regard to the sets $\xi_\varphi(G) \cup \{0\}$, $\xi_\varphi(M) \cup \{1\}$ are dense in $K(\mathcal{J}_1)$, then according to Lemma 3, there is an isomorphism ψ from Condition 2.

(2) $\Longrightarrow$ (1). According to Theorem 1 from [1], $F(\mathcal{J}) \simeq F(\mathcal{J}_1)$. With regard to φ satisfies (H3), we obtain Condition 2 from Theorem 16 from [2]. □

Remark 13 Theorem 17 from [2] introduces a lot of characterization of the I-homomorphism and Theorem 6 introduces other.

Theorem 7 *Let φ be a homomorphism of a context $\mathcal{J} = (G, M, I)$ onto a context $\mathcal{J}_1 = (G_1, M_1, I_1)$. We denote by $\bar{g}, \bar{h}, \ldots \in \mathcal{G}_\varphi$, $\bar{m}, \bar{n}, \ldots \in \mathcal{M}_\varphi$ the elements of the context $\mathcal{J}_{\mathcal{R}_\varphi} = (\mathcal{G}_\varphi, \mathcal{M}_\varphi, I_{\mathcal{R}_\varphi})$. Then following conditions are equivalent.*

1. *φ satisfies (H3).*
2. *There is an isomorphism ψ of lattices $K(\mathcal{J}_{\mathcal{R}_\varphi})$, $K(\mathcal{J}_1)$ which induces a bijective map of sets $\mathcal{G}_{\mathcal{J}_{\mathcal{R}_\varphi}}$, $\mathcal{G}_{\mathcal{J}_1}$, or $\mathcal{M}_{\mathcal{J}_{\mathcal{R}_\varphi}}$, $\mathcal{M}_{\mathcal{J}_1}$, respectively, such that $\psi(\bar{g}^{\uparrow\downarrow}) = (\varphi(g))^{\uparrow\downarrow}$ $\forall g \in G$, $\psi(\bar{m}^{\downarrow}) = (\varphi(m))^{\downarrow}$ $\forall m \in M$.*

Proof (1) $\Longrightarrow$ (2). According to Theorem 2 from [2], with regard to φ satisfies (H3), the map $\bar{\varphi}\colon \bar{g} \mapsto \varphi(g)$ $\forall \bar{g} \in \mathcal{G}_\varphi$, $\bar{m} \mapsto \varphi(m)$ $\forall m \in \mathcal{M}_\varphi$ is an isomorphism of contexts $\mathcal{J}_{\mathcal{R}_\varphi}$, $\mathcal{J}_1$. Let $\xi_{\bar{\varphi}} : \bar{g} \mapsto (\bar{\varphi}(\bar{g}))^{\uparrow\downarrow}$ $\forall \bar{g} \in \mathcal{G}_\varphi$, $\bar{m} \mapsto (\bar{\varphi}(\bar{m}))^{\downarrow}$ $\forall \bar{m} \in \mathcal{M}_\varphi$ be a map of the context $\mathcal{J}_{\mathcal{R}_\varphi}$ into $\mathcal{J}_{K(\mathcal{J}_1)}$. With regard to $\bar{\varphi}$ is an isomorphism and according to Theorem 5, Condition 3, $\xi_{\bar{\varphi}}$ is an I-homomorphism. Because $\xi_{\bar{\varphi}}(\mathcal{G}_\varphi) = \mathcal{G}_{\mathcal{J}_1}$, $\xi_{\bar{\varphi}}(\mathcal{M}_\varphi) = \mathcal{M}_{\mathcal{J}_1}$, the sets $\xi_{\bar{\varphi}}(\mathcal{G}_\varphi) \cup \{0\}$, $\xi_{\bar{\varphi}}(\mathcal{M}_\varphi) \cup \{1\}$ are dense in $K(\mathcal{J}_1)$ and we obtain our Condition 2 from Lemma 3.

(2) $\Longrightarrow$ (1) According to Lemma 1, the map $\bar{g} \mapsto \bar{g}^{\uparrow\downarrow}$ $\forall \bar{g} \in \mathcal{G}_\varphi$, $\bar{m} \mapsto \bar{m}^{\downarrow}$ $\forall \bar{m} \in \mathcal{M}_\varphi$ is an I-homomorphism of the context $\mathcal{J}_{\mathcal{R}_\varphi}$ into the context $\mathcal{J}_{K(\mathcal{J}_{\mathcal{R}_\varphi})}$. Then $\bar{g} I_{\mathcal{R}_\varphi} \bar{m}$ iff $\bar{g}^{\uparrow\downarrow} \subseteq \bar{m}^{\downarrow}$. Similarly the map $\varphi(g) \mapsto (\varphi(g))^{\uparrow\downarrow}$ $\forall g \in G$, $\varphi(m) \mapsto (\varphi(m))^{\downarrow}$ $\forall m \in M$ is an I-homomorphism of $\mathcal{J}_1$ into $\mathcal{J}_{K(\mathcal{J}_1)}$ and then $\varphi(g) I_1 \varphi(m)$ iff $(\varphi(g))^{\uparrow\downarrow} \subseteq (\varphi(m))^{\downarrow}$. With regard to ψ in an isomorphism of $K(\mathcal{J}_{\mathcal{R}_\varphi})$ onto $K(\mathcal{J}_1)$, we obtain $\bar{g}^{\uparrow\downarrow} \subseteq \bar{m}^{\downarrow}$ iff $(\varphi(g))^{\uparrow\downarrow} \subseteq (\varphi(m))^{\downarrow}$. Then $\varphi(g) I_1 \varphi(m)$ iff $\bar{g} I_{\mathcal{R}_\varphi} \bar{m}$. From the definition of relation $I_{\mathcal{R}_\varphi}$ we obtain $\varphi(g) I_1 \varphi(m)$ implies $\bar{g} I_{\mathcal{R}_\varphi} \bar{m}$ implies $\exists h \in G$, $n \in M$, $\bar{h} = \bar{g}$, $\bar{n} = \bar{m}$, hIn, which means that $\varphi(h) = \varphi(g)$, $\varphi(n) = \varphi(m)$, hIn and φ satisfies (H3). □

References

[1] Machala, F.: *Isomorphismen von Kontexten und Konzeptualverbänden.* Acta Univ. Palacki. Olomuc., Fac. rer. nat. **110**, Math. (1993), 123–139.

[2] Machala, F.: *Über Homomorphismen der Kontexte.* Acta Univ. Palacki. Olomuc., Fac. rer. nat. **114**, Math. (1994), 95–104.

[3] Wille, R.: *Restructuring lattice theory: an approach based on hierarchies of concepts.* I. Rival (ed.), Ordered sets, Reidel, Dordrecht–Boston, 1982, 445–470.