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Jiří V. Horák

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On Solvability of One Special Problem of Coupled Thermoelasticity

Part I. — Classical boundary conditions and steady sources

JIŘÍ V. HORÁK

*Department of Mathematical Analysis, Faculty of Science, Palacký University,
Tomkova 40, Hejčín, 779 00 Olomouc, Czech Republic
E-mail: jhorak@risc.upol.cz*

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Abstract

The existence and uniqueness of a weak solution of special problem arising in linear theory of coupled thermoelasticity is proved by using Rothe method of discretization in time. First, model problem is derived from 3D theory of coupled thermoelasticity and then a-priory estimations of Rothe vector functions and their time derivatives for the case of classical boundary conditions and steady sources are shown. Approximative properties of the Rothe vector functions and their convergence to the weak solution as well as continuous dependence of the solution on the given data are also proved.

Key words: coupled thermoelasticity, Rothe method of discretization in time, bending of thermoelastic beam (plate), weak solution of BVP for coupled fields, abstract vector functions.

MS Classification: 35D05

1 Introduction

1.1 Brief review of the result

This paper presents an investigation of the solvability of one special problem within framework of the *linearized theory of coupled thermoelasticity* (*LTCTh*), see [25] for more details. From the mechanical point of view, the problem represents evolution process of bending of the thermoelastic beam or infinite plate strip under steady load and heat source. We also restrict ourselves here only to classical boundary conditions for both unknown abstract functions u (vertical displacements) and ϑ (temperature).

Firstly, at the beginning of our paper, we briefly mention the method of derivation of govern equations of our *model problem (MP)*: by dimensional reduction we derive two linear differential coupled equations of *MP* from the 3D *equation of motion* (equilibrium condition) coupled with the 3D *energy equation* (heat condition equation). Then we formulate *model problem* as a mixed (initial-boundary value) problem. And, by the method of factorization, there is also shown that for special type of boundary conditions the problem is not coupled one and can be solved sequentally.

Next we introduce weak formulation of the *MP* for continuous case and then its semidiscrete "instant" formulation by using *Rothe method (RM)* of discretization in time (see [26], [27], for general description of this method; brief mention about *RM* and mainly another approach can be also find in [8]). As a consequence of well known Lax-Millgram Theorem we proof (for any given partition of time interval) existence and uniqueness of finite system of instant semidiscrete solutions. This system is later used for construction of *abstract Rothe vector functions (RvF)*.

All necessary a-priory estimations of the *RvF* are derived through a-priory estimations of the instant semidiscrete solutions.

Finally, by using some technical means, there is also shown that a weak limit element of sequence of *RvF* is the unique weak solution of original *MP*. As a consequence of the uniqueness of the weak solution we can derive *strong convergence* of the *whole sequence* of *RvF* to the *weak solution* of *MP*.

Different problems of thermoelasticity and coupled thermoelasticity (*LTCTh*) are derived and studied in various publications—see [1], [4], [5], [16], [18], [19], [23], for example. 3D and 2D theory of *CTh* was mathematically treated in [2], [21], [22], [28] and so on, while beam and plate theory was studied in [3], [29] but only for uncoupled cases.

The meaningfull advantage of Rothe method we use herein for dealing with theoretic questions is that this is also a constructive method and can be therefore used directly for numerical solution (in combination with FEM, for example) of the problem.

1.2 Origin of the Problem

Govern equations of linearized theory of coupled thermoelasticity can be derived from the first and second law of thermodynamics and under special assumptions on behaviour of material (*constitutive relations, properties of materials*) and on course of evolution of thermodynamic process (see [11] or [15]). Detailed derivation is introduced in [7] or in czech in [24], for example, thus we briefly recall here first only starting system of field equations for nonlinear theory and then resulting linearized equations:

- The first thermodynamical law takes, in an admissible thermodynamic process, following form

$$\theta \cdot \dot{\eta} = -\operatorname{div} \mathbf{q} + r \ , \tag{1}$$

- equation of motion

$$\operatorname{div} \mathbf{S} + \mathbf{b} = \rho \, \ddot{\mathbf{U}} \ , \tag{2}$$

- set of *reduced constitutive equations* that are subjected to the thermodynamics restriction (due to equivalence to local dissipation inequality and principle of material frame-indifference (see again [7] or [24], for example)),

where θ denotes actual absolute temperature, η is entropy, $\mathbf{q}$ vector of heat flux, r inner heat source, $\mathbf{U} = \mathbf{U}(x, y, z, t) = \{u_1, u_2, u_3\}$ displacement vector, $\mathbf{S}$ stress tensor, $\mathbf{b}$ vector of body forces (per unit mass), ρ mass density.

After application of all necessary axioms, assumptions and procedure of linearization we obtain resulting set of relations of linear theory of coupled thermoelasticity (see [7], [24], for example) which reads as follows:

$$\mathbf{g} = \nabla \theta \ , \tag{3}$$

$$\mathbf{q} = \mathbf{K} \ \mathbf{g} \ , \tag{4}$$

$$\mathbf{E} = \frac{1}{2} \left(\nabla \mathbf{U} + \nabla \mathbf{U}^T \right) \ , \tag{5}$$

$$\mathbf{S} = \mathbf{C}[\mathbf{E}] + (\theta - \theta_0)\mathbf{M} \ , \tag{6}$$

$$\operatorname{div} \mathbf{S} + \mathbf{b} = \rho \, \ddot{\mathbf{U}} \ , \tag{7}$$

$$-\operatorname{div} \mathbf{q} + \theta_0 \mathbf{M} \cdot \dot{\mathbf{E}} + r = c\dot{\vartheta} \ , \tag{8}$$

where θ_0 denotes reference absolute temperature ($\theta_0 = 290K$, for example), $\mathbf{g}$ is thermal gradient, $\mathbf{K}$ tensor of thermal conductivity, $\mathbf{E}$ infinitesimal strain tensor, $\mathbf{C}[.]$ tensor of elasticity, $\mathbf{M}$ tensor of the relation between $\mathbf{S}$ and temperature θ and c is specific heat.

Equations (7) are well known Cauchy equations of motion, (8) is energy equation and (6) are constitutive relations (Duhamel–Neumann law).

After introducing of assumption of zero inertia forces ($\ddot{\mathbf{U}} = \mathbf{0}$), notation $\vartheta = \theta - \theta_0$ and substitution of (5) into (6), (6) into (7), (3) into (4), (4) and (5) into (8) we obtain resulting system of equations for linear and *quasistatic theory of coupled thermoelasticity* in the following form

$$\operatorname{div} \mathbf{C}[\nabla \mathbf{U}] + \operatorname{div} (\vartheta \mathbf{M}) + \mathbf{b} = \mathbf{0}, \tag{9}$$

$$\operatorname{div} (\mathbf{K} \nabla \vartheta) + \theta_0 \mathbf{M} \cdot \nabla \dot{\mathbf{U}} + r = c\dot{\vartheta} \ . \tag{10}$$

Remark 1 From the previous text we can easily see the origin of both thermoelastic coupling terms: the term $\theta_0 \mathbf{M} \cdot \nabla \dot{\mathbf{U}}$ comes from the (1) after expansion of entropy and linearization while "standard" coupling term $\operatorname{div} (\vartheta \mathbf{M})$ comes from Duhamel–Neumann law (6) (see also [7] for more details).

Remark 2 The set of four linear equations (9) and (10) creates starting point for our next study of quasistatic problem of *LTCTh*.

2 Derivation of the Model Problem Equations

2.1 Assumptions and Dimensional Reduction

First, for the sake of simplicity, we assume: *technical theory of beam (plate) bending* (see [3], [10] or [29]); *prismatic cross section* of the beam with constant height H and unit width $B = 1$; and *elastic* and *thermal isotropic homogeneous material*. Next, because the domain of our MP has special geometrical shape, it is a "thin" body, we can essentially reduce dimensions of the problem. The assumption of body shape is used directly for dimensional reduction of the heat equation while technical theory of bending is used for dimensional reduction of the equilibrium equation. Thus dimensional reduction of the problem can be realized as follows. We start with heat equation.

2.1.1 Dimensional reduction of the heat equation

Due to assumptions on the shape of the domain we can assume only linear distribution of the temperature and heat source along the height of the cross section, thus we introduce

$$\vartheta(x,y,z,t) = \vartheta^{(0)}(x,t) + y\vartheta^{(1)}(x,t), \tag{11}$$

$$r(x,y,z,t) = r^{(0)}(x,t) + yr^{(1)}(x,t), \tag{12}$$

and then we define $u_1 = -y\frac{\partial u}{\partial x}$, where $u = u(x,t)$ is a new notation of the vertical displacement of the beam (in the direction of component u_2 of the displacement vector $\mathbf{U}$, where we put $\mathbf{U} = \{u_1(x,y,t), u_2(x,t), 0\}$. Then we have $\operatorname{div}\dot{\mathbf{U}} = \frac{\partial}{\partial x}\dot{u}_1 = -y\frac{\partial}{\partial t}(\frac{\partial^2 u}{\partial x^2})$ (coupling term) and after multiplying energy equatin (10) by simple test function $y^0 = 1$ and using integration over cross section of the beam we obtain (for homogeneous and isotropic material)

$$\int_{-\frac{H}{2}}^{\frac{H}{2}} \frac{\partial^2\vartheta(x,y,z,t)}{\partial x^2}\,dy + \int_{-\frac{H}{2}}^{\frac{H}{2}} r(x,y,z,t)\,dy + \int_{-\frac{H}{2}}^{\frac{H}{2}} \theta_0 \frac{E\alpha}{k}\frac{\partial^2}{\partial x^2}\left(\frac{\partial u(x,t)}{\partial t}\right) y\,dy$$

$$+\frac{\partial\vartheta}{\partial y}\left(\frac{H}{2}\right) - \frac{\partial\vartheta}{\partial y}\left(-\frac{H}{2}\right) = \int_{-\frac{H}{2}}^{\frac{H}{2}} a\frac{\partial\vartheta(x,y,z,t)}{\partial t}\,dy\,,$$

where $a = \frac{c}{k}$, k is coefficient of thermal conductivity of material, α is coefficient of thermal linear expansion of material and E is Young's modulus (suppose $E, k, \alpha > 0$).

Remark 3 For isotropic and homogeneous material we generally have (for 3D problems) $\mathbf{M} = m\mathbf{1}$, $\mathbf{K} = k\mathbf{1}$, $m = -3(\lambda + 2\mu)\alpha$ (λ, μ are well known Lame's coefficients and $\lambda > 0, \mu > 0$), while for 1D case we have $m = -E\alpha$ (for beam) and $m = -\frac{E\alpha}{1-2\nu}$ (for infinity plate strip) and ν is Poisson's coefficient.

Now we can apply given boundary conditions, originally prescribed on body surface: we suppose here free exchange of heat with surrounding environment

(Newton boundary condition, see [1] or [16], for example). Thus we introduce convection boundary condition

$$\frac{\partial \vartheta}{\partial y} + \frac{\alpha_h}{k}(\vartheta - \vartheta_h) = 0 \quad \text{on surface} \quad y = +\frac{H}{2},$$
$$-\frac{\partial \vartheta}{\partial y} + \frac{\alpha_d}{k}(\vartheta - \vartheta_d) = 0 \quad \text{on surface} \quad y = -\frac{H}{2},$$

where α_h, α_d are coefficients of thermal exchange on upper and down surface and ϑ_h, ϑ_d are temperatures of surrounding environment. Then the previous equation, obtained from (10), has the following form

$$\frac{\partial^2}{\partial x^2}\int_{-\frac{H}{2}}^{\frac{H}{2}} \vartheta(x,y,z,t)dy - \left[\frac{\alpha_h}{k}\left(\vartheta\left(\frac{H}{2}\right) - \vartheta_h\right) + \frac{\alpha_d}{k}\left(\vartheta\left(\frac{H}{2}\right) - \vartheta_d\right)\right]$$
$$+ \int_{-\frac{H}{2}}^{\frac{H}{2}} r(x,y,z,t)\, dy + \frac{\partial^2}{\partial x^2}\left(\int_{-\frac{H}{2}}^{\frac{H}{2}} \theta_0 \frac{E\alpha}{k}\left(\frac{\partial u(x,t)}{\partial t}\right) y\, dy\right) = \int_{-\frac{H}{2}}^{\frac{H}{2}} a\frac{\partial \vartheta(x,y,z,t)}{\partial t} dy.$$

Finally, by using expansion (11) and (12) we obtain first equation for components of the unknown couple of the functions $\{\vartheta^{(0)}, \vartheta^{(1)}\}$ (where we use notation $\vartheta^{(0)} = \vartheta^{(0)}(x,t), \vartheta^{(1)} = \vartheta^{(1)}(x,t)$) in the form

$$\frac{\partial^2 \vartheta^{(0)}}{\partial x^2} - \frac{\alpha_h + \alpha_d}{kH}(\vartheta^{(0)} - \overline{\vartheta}) + r^{(0)} + \frac{\alpha_h - \alpha_d}{2k}\vartheta^{(1)} = a\frac{\partial \vartheta^{(0)}}{\partial t}, \tag{13}$$

where the following notation has been used

$$\overline{\vartheta} = \frac{\alpha_h \vartheta_h + \alpha_d \vartheta_d}{\alpha_h + \alpha_d}.$$

Second required equation can be then derived again by multiplying of the simplified equation (10) (for homogeneous and isotropic material) by function $y^1 = y$ and after that by its integration over cross section of the beam. Thus we have

$$\int_{-\frac{H}{2}}^{\frac{H}{2}} \left\{\Delta\vartheta y + ry + \theta_0 \frac{E\alpha}{k}\, y^2 \frac{\partial^2}{\partial x^2}\left(\frac{\partial u}{\partial t}\right)\right\} dy = \int_{-\frac{H}{2}}^{\frac{H}{2}} a\frac{\partial \vartheta}{\partial t} y\, dy,$$

and for the next simplification we will use the following identity

$$y\frac{\partial^2 \vartheta}{\partial y^2} = \frac{\partial}{\partial y}\left(y\frac{\partial \vartheta}{\partial y}\right) - \frac{\partial \vartheta}{\partial y}.$$

After incorporating of this identity and boundary conditions on upper and down surface of the beam into former equation and also after its further simplification we obtain

$$\frac{\partial^2}{\partial x^2}\int_{-\frac{H}{2}}^{\frac{H}{2}} \vartheta y\, dy - \frac{H}{2}\left\{\frac{\alpha_h}{k}\left(\vartheta\left(\frac{H}{2}\right) - \vartheta_h\right) - \frac{\alpha_d}{k}\left(\vartheta\left(-\frac{H}{2}\right) - \vartheta_d\right)\right\} - \int_{-\frac{H}{2}}^{\frac{H}{2}} ry\, dy$$
$$-\left(\vartheta\left(\frac{H}{2}\right) - \vartheta\left(-\frac{H}{2}\right)\right) + \int_{-\frac{H}{2}}^{\frac{H}{2}} \theta_0 \frac{E\alpha}{k}\frac{\partial^2}{\partial x^2}\left(\frac{\partial u}{\partial t}\right) y^2 dy = \int_{-\frac{H}{2}}^{\frac{H}{2}} a\frac{\partial \vartheta}{\partial t} y\, dy.$$

Now we use the assumptions (11) and (12) to obtain second necessary equation. Thus we have obtained two resulting equations for couple $\{\vartheta^{(0)}, \vartheta^{(1)}\}$, first one is the equation (13) and second one is the following equation

$$\begin{aligned}\frac{\partial^2\vartheta^{(1)}}{\partial x^2} &- \left(12 + 3\frac{\alpha_h + \alpha_d}{k}H\right)\frac{1}{H^2}\vartheta^{(1)} - \frac{6}{H^2}\frac{\alpha_h - \alpha_d}{k}\vartheta^{(0)} \\ &+ \frac{6}{H^2}\frac{\alpha_h\vartheta_h - \alpha_d\vartheta_d}{k} + r^{(1)} + \theta_0\frac{E\alpha}{k}\frac{\partial^2}{\partial x^2}\left(\frac{\partial u}{\partial t}\right) = a\frac{\partial\vartheta^{(1)}}{\partial t}\end{aligned} \tag{14}$$

or after final simplification in the form

$$\begin{aligned}\frac{\partial^2\vartheta^{(1)}}{\partial x^2} &- \left(12 + 6\frac{\overline{\alpha}}{k}H\right)\frac{1}{H^2}\vartheta^{(1)} + \frac{6\overline{\alpha}}{kH^2}(\vartheta_h - \vartheta_d) + r^{(1)} \\ &+ \theta_0\frac{E\alpha}{k}\frac{\partial^2}{\partial x^2}\left(\frac{\partial u}{\partial t}\right) = a\frac{\partial\vartheta^{(1)}}{\partial t} ,\end{aligned} \tag{15}$$

where we supposed that coefficients of heat exchange on upper and down surface are equal ($\alpha_h = \alpha_d = \overline{\alpha}$).

Remark 4 In the case $\alpha_h = \alpha_d = \overline{\alpha}$ equations (13) and (14) (or (15)) are not coupled and can be solved independently. More, as the component $\vartheta^{(0)}$ describes constant distribution of the temperature along the height of beam (plate) and "works" only on normal component of displacement (in the direction of beam axis), it has no sense in our model and therefore it can be "neglected" in our model problem.

Due to last remark we see that the problem described by the set of two equations (13), (15) can be still further simplified: we can formulate our problem for only second component of unknown couple of functions—for $\vartheta^{(1)}$, while the equation (13) for first component has no sense now (within framework of our *MP* and linearized theory, see [25] for details, for example).

Thus obtained govern equation of "heat convection" in our *MP* has the resulting simple form

$$a_1 D_t\vartheta = D^2\vartheta - a_2\vartheta + a_3 D^2(D_t u) + r , \tag{16}$$

where we used following notation $\vartheta \equiv \vartheta^{(1)}$, $r = r^{(1)} + \frac{6\overline{\alpha}}{kH^2}(\vartheta_h - \vartheta_d)$, $a_1 = a$, $a_2 = \frac{12}{H^2} + \frac{6\overline{\alpha}}{kH}$, $a_3 = \theta_0\frac{E\alpha}{k}$ and $D = \frac{\partial}{\partial x}$, $D_t = \frac{\partial}{\partial t}$ (these two latter notations of derivatives will be systematically used in what follows).

2.1.2 Equlibrium equation

In this paragraph we just recall standard direct derivation of the equation of beam (plate strip) bending within framework of so called "technical theory" of thermoelastic beam (another acceptable approach is to use dimensional reduction for equation (9)). Normal component of stress tensor (in direction of axis of the beam) $\sigma \equiv \sigma_x$ is given by

$$\sigma = \sigma(x, y, z, t) = -ED^2u(x,t)y - E\alpha\vartheta(x, y, z, t)$$

and thus bending moment has the form

$$M = \int_{-\frac{H}{2}}^{\frac{H}{2}} \sigma y dy = -EJD^2u - E\alpha \int_{-\frac{H}{2}}^{\frac{H}{2}} \vartheta y\, dy \ .$$

Using assumption (11) and then again the same notation $\vartheta \approxeq \vartheta^{(1)}$ we obtain

$$M = -EJD^2u - EJ\alpha\vartheta \ ,$$

and after that from "standard" condition of equlibrium (see [29], [16], [10] or [3], for example) we get

$$D^2(EJD^2u) + D^2(\alpha EJ\vartheta) = \tilde{q}.$$

or (due to our assumptions)

$$D^4u + \alpha D^2\vartheta = q \ ,$$

where $q = \frac{\tilde{q}}{EJ}$ and J is moment of innertia of cross section.

Resulting set of govern equations of our *model problem (MP)* for couple of unknown functions $u = u(x,t)$, $\vartheta = \vartheta(x,t)$ has the final form

$$D^4u + \alpha D^2\vartheta = q \ , \tag{17}$$

$$a_1 D_t\vartheta = D^2\vartheta - a_2\vartheta + a_3 D^2 D_t u + r \ . \tag{18}$$

2.2 Method of factorization—"uncoupled case"

Classical boundary conditions for our *MP* can be divided into two disjoint groups: the first one—without any posibility to transform the problem into uncoupled one and the second one which enables to simplify and study problem as an uncoupled one. In this paragraph we briefly mention only the second group of boundary conditions (see [25] for more details).

Boundary conditions representing simple support of the both ends of the beam as well as one clamped end of the beam and the other free end belong into this group. For the sake of brevity we restrict ourselves in this paper only to the first case—simple support of the beam.

Thus the *MP* for unknown couple $\{u, \vartheta\}$ can be in this case formulated as follows

$$(P_{ss}) \begin{cases} D^4u + \alpha D^2\vartheta = q & \text{in } Q \ , \\ a_1 D_t\vartheta = D^2\vartheta - a_2\vartheta + a_3 D_t D^2 u + r & \text{in } Q \ , \\ \gamma(u) = \hat{u},\ \gamma_N(M) = \hat{M},\ \gamma(\vartheta) = \hat{\vartheta} & \text{on } \Gamma \ , \\ \vartheta = \vartheta_0 & \text{on } \Omega_0 \ , \end{cases}$$

where $M = -EJ(D^2u + \alpha\vartheta)$ and $\hat{u}, \hat{\vartheta}, \hat{M}$ are given vertical displacements of support, prescribed temperatures and moment loads at the ends of the beam. Here, we used following notation

$$I = (0,T),\ T \in \mathbf{R}^+,\ T > 0,\ \Omega = (0,L),\ L \in \mathbf{R}^+,\ L > 0,\ \partial\Omega = \{0, L\},$$
$$\Gamma = \partial\Omega \times I,\ \Omega_0 = \Omega \times \{0\},\ Q = \Omega \times I,$$

and γ, γ_N means standard and Neumann trace operator (see [6] for example).

By using method of factorization (see [12] for example), we can equivalently reformulate the original problem (P_{ss}) to the three following 2[nd] order problems for unknown triple of functions $\{M, u, \vartheta\}$:

- Starting problem for the bending moment $M = M(\tilde{q}, \hat{M})$

$$(P_M)\begin{cases} -D^2 M = \tilde{q} & \text{in } Q\ , \\ \gamma(M) = \hat{M} & \text{on } \Gamma\ , \end{cases}$$

- Coupled problem for the vertical displacement $u = u(\vartheta, M, \hat{u})$

$$(P_u)\begin{cases} -EJ(D^2 u + \alpha\vartheta) = M & \text{in } Q\ , \\ \gamma(u) = \hat{u} & \text{on } \Gamma\ , \end{cases}$$

- Coupled problem for the temperature $\vartheta = \vartheta(u, r, \hat{\vartheta}, \vartheta_0)$

$$(P_\vartheta)\begin{cases} a_1 D_t\vartheta = D^2\vartheta - a_2\vartheta + a_3 D_t D^2 u + r & \text{in } Q\ , \\ \gamma(\vartheta) = \hat{\vartheta} & \text{on } \Gamma\ , \\ \vartheta = \vartheta_0 & \text{on } \Omega_0\ , \end{cases}$$

and this system can be now easily simplified by eliminating of the term $a_3 D_t D^2 u$ from the problem (P_ϑ) (see [25], for example). After that we obtain resulting uncoupled set of two problems of the 2[nd] order which reads as follows

- Independent (uncoupled) problem for the temperature $\vartheta = \vartheta(M, r, \hat{\vartheta}, \vartheta_0)$

$$(\tilde{P}_\vartheta)\begin{cases} (a_1 + \alpha a_3)D_t\vartheta = D^2\vartheta - a_2\vartheta - \frac{a_3}{EJ} D_t M + r & \text{in } Q\ , \\ \gamma(\vartheta) = \hat{\vartheta} & \text{on } \Gamma\ , \\ \vartheta = \vartheta_0 & \text{on } \Omega_0\ , \end{cases}$$

- Directly dependent problem (it depends only on former one) for the displacement $u = u(\vartheta, \hat{u}, M)$

$$(P_u)\begin{cases} -EJ(D^2 u + \alpha\vartheta) = M & \text{in } Q\ , \\ \gamma(u) = \hat{u} & \text{on } \Gamma\ , \end{cases}$$

where the function $M = M(x,t)$ can be find by solving the problem (P_M) but this is a simple task and can be done even explicitly.

Analogical approach can be used to simplification of the beam problem with following boundary conditions: "clamped end" $(\gamma(u) = \hat{u}, \gamma(Du) = \hat{u}_1)$ and "free end" $(\gamma_N(D^2u) = \hat{u}_2, \gamma_N(D^3u) = \hat{u}_3)$.

3 Weak formulation of the problem

3.1 Classical formulation—continuous case

First, we recall in this section a definition of classical solution of our *MP* but just for the case of homogeneous boundary conditions and homogeneous initial condition. Thus we start here with following

Definition 1 Let us suppose $q, r \in C(Q)$ are given continuous functions. Then couple of abstract functions

$$\begin{aligned}\{u,\vartheta\} \ \in \ & C(\bar{I}; C^{(4)}(\Omega) \cap C^{(2)}((0,L\rangle) \cap C^{(1)}(\bar{\Omega})) \\ & \cap C^{(1)}(\bar{I}; C^{(2)}(\Omega)) \times C^{(1)}(\bar{I}; C^{(2)}(\Omega) \cap C(\bar{\Omega}))\end{aligned}$$

such that

$$(P)\left\{\begin{array}{cl} D^4u + \alpha D^2\vartheta = q & \text{in } Q \ , \\ a_1 D_t\vartheta = D^2\vartheta - a_2\vartheta + a_3 D_t D^2 u + r & \text{in } Q \ , \\ \gamma(u) = 0, \ \gamma(\vartheta) = 0 & \text{on } \Gamma \ , \\ Du = 0 & \text{on } \{0\} \times I \ , \\ D^2u = 0 & \text{on } \{L\} \times I \ , \\ \vartheta = 0 & \text{in } \Omega_0 \end{array}\right.$$

holds is being to said *classical solution* of the beam problem within framework of linear theory of coupled thermoelasticity.

As it is well known, the classical solution does not exist even for very frequent and practical problems due to too restrictional requirements on smootheness of the solution itself, data of the problem and so on. This is the reason why we use more general notion of the solution.

3.2 Weak formulation—continuous case

In this paragraph we introduce new definition of the term solution of the *MP*. For this purpose we firstly give some useful notation which will be used systematically in what follows:

- Linear space of kinematically admissable functions

  $$\mathcal{V} = \{v \in H_0^1(\Omega) \cap H^2(\Omega) \mid Dv(0) = 0\} \ ,$$

  where $H^k(\Omega)$ denotes standard Sobolev space (see [9], [6] or [14]),

- Cartesian product of test functions spaces

  $$\mathcal{H} = \mathcal{V} \times H_0^1(\Omega) \ ,$$

- Bilinear forms defined on $H^k(\Omega) \times H^k(\Omega)$ for $k = 2$ and $k = 1$

$$\begin{aligned} \mathbf{a}(u,v) &= \textstyle\int_\Omega D^2u(x)D^2v(x)\mathrm{d}x \quad \text{for } u,v \in H^2(\Omega)\ , \\ \mathbf{b}(u,v) &= \textstyle\int_\Omega Du(x)Dv(x)\mathrm{d}x \quad \text{for } u,v \in H^1(\Omega)\ , \end{aligned}$$

- Bilinear forms defined on $\mathcal{H} \times \mathcal{H}$

$$\begin{aligned} \mathcal{A}(\mathbf{U},\mathbf{V}) &= \mathbf{a}(u,v) + \mathbf{b}(\vartheta,\eta) + a_2(\vartheta,\eta)_{L_2(\Omega)} && \text{for } \mathbf{U},\mathbf{V} \in \mathcal{H}\ , \\ \mathcal{B}(\mathbf{U},\mathbf{V}) &= a_1(\vartheta,\eta)_{L_2(\Omega)} + a_3\mathbf{b}(u,\eta) && \text{for } \mathbf{U},\mathbf{V} \in \mathcal{H}\ , \\ \mathcal{C}(\mathbf{U},\mathbf{V}) &= \alpha\mathbf{b}(\vartheta,v) && \text{for } \mathbf{U},\mathbf{V} \in \mathcal{H}\ , \end{aligned}$$

where we used following notation

$$\begin{aligned} \mathbf{U} &= \{u,\vartheta\},\ u \in \mathcal{V},\ \vartheta \in H_0^1(\Omega)\ , \\ \mathbf{V} &= \{v,\eta\},\ v\ \in \mathcal{V},\ \eta \in H_0^1(\Omega)\ , \end{aligned}$$

- Linear form defined on $\mathcal{H}$

$$\mathcal{F}(\mathbf{V}) = (q,v) + \langle r,\vartheta\rangle \quad \text{for } \mathbf{V} \in \mathcal{H}\ ,$$

where $(.,.)$ and $\langle .,.\rangle$ denote duality pairing on $\mathcal{V}^* \times \mathcal{V}$ and $H^{-1}(\Omega) \times H_0^1(\Omega)$, respectively (see [17] for example).

Definition 2 Let us suppose the couple of abstract functions

$$\{q,r\} \in L_2(I;\mathcal{V}^*) \times L_2(I;H^{-1})$$

is given. Then an abstract function $\mathbf{U} = \mathbf{U}(t) : I \to \mathcal{H}$ that holds

$$\mathbf{U} \in L_2(I;\mathcal{H}) \cap AC(I;\mathcal{V} \times L_2(\Omega)) \tag{19}$$

$$D_t\mathbf{U} \in L_2(I;\mathcal{V} \times L_2(\Omega)) \tag{20}$$

$$(P_2\mathbf{U})(0) = 0 \ \text{ in } \ C(I;L_2(\Omega)) \tag{21}$$

and such that

$$\begin{aligned} &\int_I \mathcal{A}(\mathbf{U}(t),\mathbf{V}(t))\mathrm{d}t - \int_I \mathcal{C}(\mathbf{U}(t),\mathbf{V}(t))\,\mathrm{d}t + \int_I \mathcal{B}(D_t\mathbf{U}(t),\mathbf{V}(t))\,\mathrm{d}t \\ &\qquad = \int_I \mathcal{F}(\mathbf{V}(t))\,\mathrm{d}t \qquad \forall \mathbf{V} \in L_2(I;\mathcal{H}) \end{aligned} \tag{22}$$

holds, is said to be a weak solution of the problem (P).

4 Results

Now we have prepared everything to able to formulate main result concerning existence a uniqueness of the weak solution of the problem (P). As we mentioned before we restrict oureselves here only to the steady case of heat source and load of the beam.

Theorem 1 *Let us suppose functions* $\{q, r\} \in \mathcal{V}^* \times L_2(\Omega)$ *are given. Then there exists unique weak solution of the problem (P) (defined by (19)-(22)), that is in the sense of the* Definition 2.

Theorem 2 *Under the assumption of Theorem 1 the solution depends continuously on the given data. More precisely: If functions* $\mathbf{U}^1$ *and* $\mathbf{U}^2$ *are two weak solutions of the problem (P) for the functionals* $\mathcal{F}^1$ *and* $\mathcal{F}^2$*, respectively, then following estimation holds*

$$||\mathbf{U}^1 - \mathbf{U}^2||_{C(\bar{I};H^1(\Omega)\times L_2(\Omega))} \leq C||\mathcal{F}^1 - \mathcal{F}^2||_{H^*}$$

where C *is constant independent on* $\mathcal{F}^{\,i}$, $i = 1, 2$.

5 Proof of the Theorem 1

5.1 Existence of a weak solution

In this paragraph, we prove just existence of the weak solution of (P). For this purpose we use Rothe method. At first, in the following subparagraph, we introduce another formulation of the model problem—so called "instant" semidiscrete formulation. Then we show existence and uniqueness of the semidiscrete solution and later on we show also its a-priory estimations. Finally, in the next subparagraph, we use these semidiscrete solutions for construction of approximation of weak solution.

5.1.1 Semidiscrete formulation—Rothe method

Starting point for our method of proof of Theorem 1 is the method of discretization in time. Thus we need to introduce so called "instant" semidiscrete formulation and then we formulate and proof theorem on existence and uniqueness of this "instant" weak solution. This is used later on for construction of Rothe functions.

Definition 3 Suppose $p \in \mathbf{N}$ is given and define $p^{(n)} = 2^{(n-1)}p$, $n \in \mathbf{N}$, and

$$\mathcal{D}^{(n)} = \{t_j^{(n)}\}_{j=1}^{p^{(n)}}, \qquad t_j^{(n)} = jl^{(n)}, \quad j = 0, 1, \ldots, p^{(n)}, \quad l^{(n)} = \frac{T}{p^{(n)}}.$$

Then a couple of functions $\{z_j^{(n)}, \xi_j^{(n)}\} \in \mathcal{V} \times H_0^1(\Omega)$ ($z_j^{(n)} = z(x, t_j^{(n)})$, $\xi_j^{(n)} = \xi(x, t_j^{(n)})$) such that

$$(\mathcal{P}_j^{(n)}) \begin{cases} \mathbf{a}(z_j^{(n)}, v) - \alpha\mathbf{b}(\xi_j^{(n)}, v) = (q_j^{(n)}, v) \qquad \forall v \in \mathcal{V} \\ a_1 \left(\frac{\xi_j^{(n)} - \xi_{j-1}^{(n)}}{l^{(n)}}, \eta\right)_{L_2(\Omega)} + \mathbf{b}(\xi_j^{(n)}, \eta) + a_2(\xi_j^{(n)}, \eta)_{L_2(\Omega)} \\ \quad + a_3\mathbf{b}\left(\frac{z_j^{(n)} - z_{j-1}^{(n)}}{l^{(n)}}, \eta\right) = \langle r_j^{(n)}, \eta\rangle \qquad \forall \eta \in H_0^1(\Omega) \end{cases}$$

holds, where $\{q_j^{(n)}, r_j^{(n)}\} \in \mathcal{V}^* \times L_2(\Omega)$, $q_j^{(n)} = q(x, t_j^{(n)})$, $r_j^{(n)} = r(x, t_j^{(n)})$, $j = 1, \dots, p^{(n)}$, is said to be "instant" weak semidiscrete solution in time $t_j^{(n)} \in \bar{I}$. For $j = 0$ we put $\{z_0, 0\}$, where $a(z_0, v) = (q, v)\ \forall v \in \mathcal{V}$.

Now we can formulate following

Theorem 3 *Let us suppose functions $\{q, r\} \in \mathcal{V}^* \times L_2(\Omega)$ are given. Then for any given partition $\mathcal{D}^{(n)} = \{t_j^{(n)}\}_{j=1}^{p^{(n)}}$ of the interval $\bar{I}$ there exists unique finite set $\{\{z_j^{(n)}, \xi_j^{(n)}\}_{j=1}^{p^{(n)}}\} \in [\mathcal{V} \times H_0^1(\Omega)]^{p^{(n)}}$ of weak solutions $\{z_j^{(n)}, \xi_j^{(n)}\}$ of the problems $(\mathcal{P}_j^{(n)})$, $j = 1, \dots, p^{(n)}$ (that is in the sense of the* Definition 3*).*

Sketch of the proof: this statement follows directly from Lax–Milgram theorem. We just need to check if all assumptions of this theorem are satisfied. At first, we rewrite formulation of the problem $(\mathcal{P}_j^{(n)})$ as follows

$$(\tilde{\mathcal{P}}_j^{(n)}) \begin{cases} \mathcal{A}(\mathbf{Z}_j^{(n)}, \mathbf{V}) + \frac{1}{l(n)}\mathcal{B}(\mathbf{Z}_j^{(n)}, \mathbf{V}) - \mathcal{C}(\mathbf{Z}_j^{(n)}, \mathbf{V}) = \\ = \frac{1}{l(n)}\mathcal{B}(\mathbf{Z}_{j-1}^{(n)}, \mathbf{V}) + \mathcal{F}_j^{(n)}(\mathbf{V}) \qquad \forall \mathbf{V} \in \mathcal{H}\ , \end{cases}$$

where $\{\mathbf{Z}_j^{(n)}\} = \{z_j^{(n)}, \xi_j^{(n)}\}$ and $\mathbf{V} = \{v, \eta\}$ and then we need to prove ellipticity of the composed billinear form A given by $\mathsf{A} = \mathcal{A} + \mathcal{B} - \mathcal{C}$ (while all other assumptions are clear). But this can be easily done through equivalent reformulation of the problem $(\tilde{\mathcal{P}}_j^{(n)})$ by multiplying both equations in $(\mathcal{P}_j^{(n)})$ by positive constants $\frac{a_3}{l(n)} > 0, \alpha > 0$ and then by their summation. Resulting bilinear form of the eqiuvalent problem is then $\mathcal{H}-$ elliptic (under assumption $\alpha > 0$, see [25] for all details).

5.1.2 Construction of Rothe's Functions

For definition of Rothe vector functions we firstly use finite set of "instant" semidiscrete solutions (independent variable is $x \in \Omega$) and then piecewise linear interpolation in time variable $t \in \bar{I}$. Latter theorem implies that for any $\mathcal{D}^{(n)}$ we have the unique finite set of "instant" semidiscrete solutions

$$\{\{\mathbf{Z}_j^{(n)}\}_{j=1,\dots,p^{(n)}}\} = \{\{z_j^{(n)}, \xi_j^{(n)}\}_{j=1,\dots,p^{(n)}}\}, \qquad \mathbf{Z}_j^{(n)} \in \mathcal{H}$$

and our goal is to use this set for construction of a sequence of vector abstract functions $\mathbf{U}^{(n)} : \bar{I} \ \to \mathcal{H}$ approximating, in a suitable sense a weak solution of the problem (P). Thus, for any $\mathcal{D}^{(n)}$, we can construct Rothe functions $\mathbf{U}^{(n)}$ in the following way

$$\mathbf{U}^{(n)}(t)\ \big|_{I_j^{(n)}} = \mathbf{Z}_{j-1}^{(n)} + \frac{\mathbf{Z}_j^{(n)} - \mathbf{Z}_{j-1}^{(n)}}{l(n)}(t - t_{j-1}^{(n)}), \quad t \in I_j^{(n)}$$

where we used notation

$$\mathbf{U}^{(n)} = \{u^{(n)}, \vartheta^{(n)}\}, \quad I_j^{(n)} = \langle t_{j-1}^{(n)}, t_j^{(n)} \rangle, \quad j = 1, \dots, p^{(n)}\ .$$

5.1.3 A-priory estimations of "instant" semidiscrete solutions

We start proving of theorem 1 through a-priory estimations of "instant" semidiscrete solutions. Using some special "symetrization" formulas (see [27] or [25]) and following "1[st] fundamental relation" in semidiscrete form

$$\mathbf{a}(z_j^{(n)} - z_{j-1}^{(n)}, z_j^{(n)} - z_{j-1}^{(n)}) = \alpha \mathbf{b}(\xi_j^{(n)} - \xi_{j-1}^{(n)}, z_j^{(n)} - z_{j-1}^{(n)}) \;, \qquad (23)$$

for $j = 1, \ldots, p^{(n)}$ will be our basic tool for finding of estimations of function values of $\mathbf{Z}_j^{(n)}$ (resp. $\mathbf{U}_j^{(n)}$). In the first step we choose as test functions $\mathbf{V} = \mathbf{Z}_{\mathrm{i}}^{(\mathrm{n})} - \mathbf{Z}_{\mathrm{i-1}}^{(\mathrm{n})}$ in problem $(\tilde{\mathcal{P}}_i^{(n)})$ thus we obtain set of following identities

$$\begin{aligned}\mathcal{A}(\mathbf{Z}_{\mathrm{i}}^{(\mathrm{n})}, \mathbf{Z}_{\mathrm{i}}^{(\mathrm{n})} - \mathbf{Z}_{\mathrm{i-1}}^{(\mathrm{n})}) + \tfrac{1}{l^{(n)}}\mathcal{B}(\mathbf{Z}_{\mathrm{i}}^{(\mathrm{n})}, \mathbf{Z}_{\mathrm{i}}^{(\mathrm{n})} - \mathbf{Z}_{\mathrm{i-1}}^{(\mathrm{n})}) - \mathcal{C}(\mathbf{Z}_{\mathrm{i}}^{(\mathrm{n})}, \mathbf{Z}_{\mathrm{i}}^{(\mathrm{n})} - \mathbf{Z}_{\mathrm{i-1}}^{(\mathrm{n})}) = \\ = \tfrac{1}{l^{(n)}}\mathcal{B}(\mathbf{Z}_{\mathrm{i-1}}^{(\mathrm{n})}, \mathbf{Z}_{\mathrm{i}}^{(\mathrm{n})} - \mathbf{Z}_{\mathrm{i-1}}^{(\mathrm{n})}) + \mathcal{F}_{\mathrm{i}}^{(\mathrm{n})}(\mathbf{Z}_{\mathrm{i}}^{(\mathrm{n})} - \mathbf{Z}_{\mathrm{i-1}}^{(\mathrm{n})}) \;, \qquad i = 1, \ldots, j.\end{aligned}$$

In the second step we substite (23) and all mentioned auxiliary relations into these identities and after some calculations and their final summation we obtain following estimations for components of semidiscrete solutions

$$\begin{aligned}||\xi_j^{(n)}||_{H^1(\Omega)} &\leq \frac{2}{\min(1, a_2)}||r||_{H^{-1}} \\ ||D^2 z_j^{(n)}||_{L_2(\Omega)} &\leq \frac{2\alpha}{\min(1, a_2)}||r||_{H^{-1}} + \bar{c}||q||_{\mathcal{V}^*} \qquad \text{for } j = 1, 2, \ldots\end{aligned}$$

and from equivalence of the norms on $\mathbf{H}(\Omega) = H_0^1(\Omega) \cap H^2(\Omega)$ we also have estimation

$$||z_j^{(n)}||_{H^2(\Omega)} \leq \frac{2\alpha\bar{c}}{\min(1, a_2)}||r||_{H^{-1}} + (\bar{c})^2||q||_{\mathcal{V}^*}$$

and thus we finally have for function values of $\mathbf{Z}_j^{(n)}$ following

$$\begin{aligned}||\mathbf{Z}_j^{(n)}||_{[L_2(\Omega)]^2} &\leq ||\mathbf{Z}_j^{(n)}||_{\mathcal{H}} \leq (\bar{c})^2||q||_{\mathcal{V}^*} + \frac{2(1+\alpha\bar{c})}{\min(1, a_2)}||r||_{H^{-1}} \\ &\leq \max\left((\bar{c})^2, \frac{2(1+\alpha\bar{c})}{\min(1, a_2)}\right)||\mathcal{F}||_{H^*} = C_1 \qquad (24)\end{aligned}$$

where $\bar{c} > 0$ is constant from equivalence of norms $||z||_{H^2(\Omega)}$ and $||D^2 z||_{L_2(\Omega)}$ on $\mathbf{H}(\Omega)$ and $C_1 > 0$ is generic constant independent on $\mathcal{D}^{(n)}$, $n \in \mathbf{N}$.

Thus we have obtained not only a-priory estimation of $\mathbf{Z}_j^{(n)}$ (see (24)) but in addition to it also (see definition of $\mathbf{U}^{(n)}$ and [13] or [27], for example) following statement

Lemma 1 *Suppose functions $\{q, r\} \in \mathcal{V}^* \times L_2(\Omega)$ are given. Then, for Rothe functions corresponding to the problem (19)–(22), following estimation holds*

$$\mathbf{U}^{(n)} \in L_\infty(I; H^2(\Omega) \times H^1(\Omega)) \qquad \forall \mathcal{D}^{(n)} \;, \; n \in \mathbf{N} \;.$$

Similarly we obtain estimation for difference quotient of semidiscrete solutions. First, we substract problem $(\tilde{\mathcal{P}}_{i-1}^{(n)})$ from $(\tilde{\mathcal{P}}_{i}^{(n)})$ for $i = 2, \ldots, j$. Then, in the next step, we use as a test function $\mathbf{V} = \{0, \frac{\xi_i^{(n)} - \xi_{i-1}^{(n)}}{l^{(n)}}\}$ in resulting identities

$$\mathcal{A}(\mathbf{Z}_{\mathrm{i}}^{(\mathrm{n})} - \mathbf{Z}_{\mathrm{i-1}}^{(\mathrm{n})}, \mathbf{V}) + \frac{1}{l^{(n)}}\mathcal{B}(\mathbf{Z}_{\mathrm{i}}^{(\mathrm{n})} - \mathbf{Z}_{\mathrm{i-1}}^{(\mathrm{n})}, \mathbf{V}) - \mathcal{C}(\mathbf{Z}_{\mathrm{i}}^{(\mathrm{n})} - \mathbf{Z}_{\mathrm{i-1}}^{(\mathrm{n})}, \mathbf{V}) =$$
$$= \frac{1}{l^{(n)}}\mathcal{B}(\mathbf{Z}_{\mathrm{i-1}}^{(\mathrm{n})} - \mathbf{Z}_{\mathrm{i-2}}^{(\mathrm{n})}, \mathbf{V}) \ , \qquad \text{for } i = 2, \ldots, j$$

and for further calculation in last set of identities, similar to (23), we use "2$^{\text{nd}}$ fundamental relation" in semidiscrete form (playing crucial role for estimation of diference quotient of $\mathbf{U}_j^{(n)}$) in the form

$$\mathbf{a}\left(\frac{z_i^{(n)} - z_{i-1}^{(n)}}{l^{(n)}} - \frac{z_{i-1}^{(n)} - z_{i-2}^{(n)}}{l^{(n)}}, \frac{z_i^{(n)} - z_{i-1}^{(n)}}{l^{(n)}}\right) =$$
$$= \alpha\mathbf{b}\left(\frac{\xi_i^{(n)} - \xi_{i-1}^{(n)}}{l^{(n)}}, \frac{z_i^{(n)} - z_{i-1}^{(n)}}{l^{(n)}} - \frac{z_{i-1}^{(n)} - z_{i-2}^{(n)}}{l^{(n)}}\right) \ , \tag{25}$$

for $i = 2, \ldots, j$. After similar procedure like in previous and after some simplification we obtain

$$\left\| \frac{\mathbf{Z}_j^{(n)} - \mathbf{Z}_{j-1}^{(n)}}{l^{(n)}} \right\|_{H^2 \times L_2} \leq \left(\frac{\bar{c}}{a_1} \frac{\sqrt{(a_1 + \alpha a_3)\alpha}}{\sqrt{a_3}} + \frac{\sqrt{a_1 + \alpha a_3}}{\sqrt{a_1^3}} \right) \|r\|_{L_2(\Omega)} = C_2 \tag{26}$$

where C_2 is again constant independent on $\mathcal{D}^{(n)}, n \in \mathbf{N}$. Now, similarly like in previous case, besides the a-priory estimation of diference quotient, we have obtained also easily provable (see (26) and definition of *RvF*) following additional result:

Lemma 2 *Suppose functions $\{q, r\} \in \mathcal{V}^* \times L_2(\Omega)$ are given. Then, for Rothe functions corresponding to the problem (19)-(22), following estimation holds*

$$\mathbf{U}^{(n)} \in H^1(I; H^2(\Omega) \times L_2(\Omega)) \qquad \forall \mathcal{D}^{(n)} ,\ n \in \mathbf{N} \ .$$

5.1.4 A-priory estimations of abstract vector Rothe functions

From estimation (24) and from definition of functions $\mathbf{U}^{(n)}$ we immediately have (see again [13], [27], for example)

$$\|\mathbf{U}^{(n)}(t)\|_{\mathcal{H}} \leq C_1$$

and because, as we can easily check, the Rothe functions are measurable and integrable in Bochner sense, we also have

$$\|\mathbf{U}^{(n)}\|_{L_2(I;\mathcal{H})}^2 = \int_0^T \|\mathbf{U}^{(n)}(t)\|_{\mathcal{H}}^2 \, \mathrm{d}t \leq (C_1)^2 T$$

and we see that $\mathbf{U}^{(n)} \in L_2(I;\mathcal{H})$ for any $n \in \mathbf{N}$. Then we can choose a weakly convergent subsequence $\{\mathbf{U}^{(n_k)}\}_{k=1}^{\infty}$ from sequence $\{\mathbf{U}^{(n)}\}_{n=1}^{\infty}$ to an element $\mathbf{U} \in L_2(I;\mathcal{H})$, that is

$$\mathbf{U}^{(n_k)} \xrightarrow{\mathrm{w}} \mathbf{U} \quad \text{in } L_2(I;\mathcal{H}) \qquad \text{for } k \to \infty. \tag{27}$$

Our goal is to show that just this "weak" limit element $\mathbf{U}$ is exactly weak solution of the problem (P).

For this purpose we first assign to function $\mathbf{U}^{(n)}$ a piecewise constant abstract function $\mathbf{W}^{(n)} : \bar{I} \to [L_2(\Omega)]^2$ which represents time derivative of function $\mathbf{U}^{(n)}$. Thus we define

$$\mathbf{W}^{(n)}(0) = \frac{1}{l^{(n)}}(\mathbf{Z}_1^{(n)} - \mathbf{Z}_0)\ , \quad \mathbf{Z}_0 = \{z_0, 0\}\ ,$$

and

$$\mathbf{W}^{(n)}(t)\ \Big|_{\tilde{I}_j^{(n)}} = \frac{1}{l^{(n)}}(\mathbf{Z}_j^{(n)} - \mathbf{Z}_{j-1}^{(n)}), \qquad t \in I, \quad j = 1, \ldots, p^{(n)}\ ,$$

where $\tilde{I}_j^{(n)} = (t_{j-1}^{(n)}, t_j^{(n)})$. We see that following estimations hold

$$\|\mathbf{W}^{(n)}(t)\|_{H^2(\Omega)\times L_2(\Omega)} \le C_2\ ,$$

$$\|\mathbf{W}^{(n)}\|_{L_2(I;H^2(\Omega)\times L_2(\Omega))} = \int_0^T \|\mathbf{W}^{(n)}(t)\|^2_{H^2(\Omega)\times L_2(\Omega)} \mathrm{d}t \le (C_2)^2 T.$$

Thus, on the top of just proved estimations, we obtained immediately statement concerning derivative of RvF.

Lemma 3 *Suppose functions $\{q, r\} \in \mathcal{V}^* \times L_2(\Omega)$ are given. Then, for derivatives in time of Rothe functions corresponding to the problem (19)-(22), following estimation holds*

$$D_t\mathbf{U}^{(n)} \cong \mathbf{W}^{(n)} \in L_\infty(I; H^2(\Omega) \times L_2(\Omega)) \qquad \forall \mathcal{D}^{(n)}\ ,\ n \in \mathbf{N}\ .$$

Last estimation also implies $\mathbf{W}^{(n)} \in L_2(I;\mathcal{V} \times L_2(\Omega))$ for any $n \in \mathbf{N}$. Thus we can again choose a convergent subsequence $\{\mathbf{W}^{(n_l)}\}_{l=1}^{\infty}$ from sequence $\{\mathbf{W}^{(n_k)}\}_{k=1}^{\infty}$ (see (27)) and an element $\mathbf{W} \in L_2(I;\mathcal{V} \times L_2(\Omega))$ such that

$$\mathbf{W}^{(n_l)} \xrightarrow{\mathrm{w}} \mathbf{W} \quad \text{in } L_2(I;\mathcal{V} \times L_2(\Omega)) \qquad \text{for } l \to \infty\ . \tag{28}$$

5.1.5 Properties of U, W and passing to the limit for $n \longrightarrow \infty$

From definition of antiderivative of abstract function (see [13], [27], for example) and from previous sections and properties of Rothe functions we see that integral $\int_0^t \mathbf{W}(\tau)\mathrm{d}\tau = \mathbf{w}(t)$ exists and $\mathbf{w} \in AC(\bar{I};\mathcal{V} \times L_2(\Omega))$ and we have

$$D_t\mathbf{w}(t) = \mathbf{W}(t) \quad \text{in } L_2(I;\mathcal{V} \times L_2(\Omega))\ .$$

Thus we also have

$$\int_0^t \mathbf{W}^{(l)}(\tau)\mathrm{d}\tau = \mathbf{U}^{(l)}(t) - \mathbf{U}^{(l)}(0), \quad \text{where} \quad \mathbf{U}^{(l)}(0) = \mathbf{Z}_0 = \{z_0, 0\}$$

and after that we obtain, after passing for $l \to \infty$, following identity

$$\mathbf{w}(t) = \mathbf{U}(t) - \mathbf{Z}_0 \quad \text{for a.e. } t \in I \qquad (\text{in the sense of } L_2(I; \mathcal{V} \times L_2(\Omega)).$$

Then we also see $\mathbf{U} \in AC(\bar{I}; \mathcal{V} \times L_2(\Omega)$ and such that

$$D_t\mathbf{U}(t) = \mathbf{W}(t) \quad \text{in} \quad \mathcal{V} \times L_2(\Omega) \quad \text{for a.e. } t \in I.$$

Finally we obtain relation

$$\mathbf{U}(t) = \mathbf{Z}_0 + \int_0^t \mathbf{W}(\tau)\mathrm{d}\tau$$

and from this relation we immediately obtain identity

$$\mathbf{U}(0) = \mathbf{Z}_0 \quad \text{in} \quad C(\bar{I}; \mathcal{V} \times L_2(\Omega))$$

representing initial condition of our problem.

Until now we only checked properties of limit elements $\mathbf{U}$, $\mathbf{W}$ but we also need to verify if the element $\mathbf{U}$ holds integral identity from Definition 2. For this purpose we need to go back to the problem $(\tilde{\mathcal{P}}_j^{(n_l)})$ and because it does not depend on variable $t \in I$ we start with its reformulation. Then, we define piecewise constant approximation of Rothe function $\tilde{\mathbf{U}}^{(l)} \in L_2(I; \mathcal{H})$, that is

$$\tilde{\mathbf{U}}^{(l)}(t) = \begin{cases} \mathbf{Z}_1^{(l)} & \text{if} \quad t = 0, \\ \mathbf{Z}_j^{(l)} & \text{if} \quad t \in (0, T\rangle \cap (t_{j-1}^{(l)}, t_j^{(l)}\rangle \end{cases}$$

(see also [27] or [25]) and later on, through definition of functions $\mathbf{U}^{(n)}$ and $\mathbf{W}^{(n)}$ and for any function $\mathbf{V} \in L_2(I; \mathcal{H})$ we obtain "instant" form of $(\tilde{\mathcal{P}}_j^{(n_l)})$

$$\mathcal{A}(\tilde{\mathbf{U}}^{(l)}(t), \mathbf{V}(t)) - \mathcal{C}(\tilde{\mathbf{U}}^{(l)}(t), \mathbf{V}(t)) + \mathcal{B}(\mathbf{W}^{(l)}(t), \mathbf{V}(t)) = \mathcal{F}(\mathbf{V}(t))$$

for a.e. $t \in I$, $l = 1, 2, \ldots$. After integration of this identity over I, we obtain following form of the integral identity (22) written for piecewise constant functions

$$\int_0^T \mathcal{A}(\tilde{\mathbf{U}}^{(l)}(t), \mathbf{V}(t))\,\mathrm{d}t - \int_0^T \mathcal{C}(\tilde{\mathbf{U}}^{(l)}(t), \mathbf{V}(t))\,\mathrm{d}t$$
$$+ \int_0^T \mathcal{B}(\tilde{\mathbf{W}}^{(l)}(t), \mathbf{V}(t))\,\mathrm{d}t = \int_0^T \mathcal{F}(\mathbf{V}(t))\,\mathrm{d}t \qquad \forall \mathbf{V} \in L_2(I; \mathcal{H}). \quad (29)$$

The proof of the Theorem 1 can be now finished as follows. First we note that following implication holds for $l \to \infty$ (that is for $\mu(\mathcal{D}^{(l)}) \to 0^+$; for exact proof of this statement see [27])

$$\mathbf{U}^{(l)} \xrightarrow{\mathrm{w}} \mathbf{U} \text{ in } L_2(I; \mathcal{H}) \quad \Longrightarrow \quad \tilde{\mathbf{U}}^{(l)} \xrightarrow{\mathrm{w}} \mathbf{U} \text{ in } L_2(I; \mathcal{H})$$

and then that all terms in last identity (29) are continous for passing $l \to \infty$. But this can be proved through some calculation (see [25]) and thus the proof of existence of weak solution of the problem (P) is completed.

5.2 Proof of uniqueness

For the proof of uniqueness of the solution we assume there exist two weak solutions $\mathbf{U}_1$ and $\mathbf{U}_2$ of the problem (P). Then we define $\overline{\mathbf{U}} = \mathbf{U}_1 - \mathbf{U}_2$ and we immediately see that relations (19)–(22) are also fulfilled for this function. Next, by choosing test function in the form $\mathbf{V} = \{v, 0\}$ we obtain following simplified form of equation (22) for function $\overline{\mathbf{U}}$

$$\int_0^T \mathbf{a}(\bar{u}(t), v(t))dt - \alpha \int_0^T \mathbf{b}(\bar{\vartheta}(t), v(t))dt = 0 \qquad \forall v \in L_2(I; \mathcal{V}) \tag{30}$$

and similarly by choosing $\mathbf{V} = \{0, \eta\}$ we have

$$\int_0^T \big(\mathbf{b}(\bar{\vartheta}(t), \eta(t)) + a_2(\bar{\vartheta}(t), \eta(t))_{L_2(\Omega)}\big)dt +$$
$$+\int_0^T \big(a_1(D_t\bar{\vartheta}(t), \eta(t))_{L_2(\Omega)} + a_3\mathbf{b}(D_t\bar{u}(t), \eta(t))\big)dt = 0 \;\; \forall \eta \in L_2(I; H_0^1(\Omega)). \tag{31}$$

Then, for arbitrary but fixed $t^* \in I$ we define special test function in the form

$$\mathbf{V}(t) = \begin{cases} \overline{\mathbf{U}}(t) = \{\bar{u}(t), \bar{\vartheta}(t)\} & \text{for } t \in \langle 0, t^*\rangle, \\ \mathbf{0} & \text{for } t \in (t^*, T\rangle, \end{cases}$$

and after its substitution into the previous identity we see that the first two terms are non-negative while for two last terms we have

$$\int_0^{t^*} \big(a_1(D_t\bar{\vartheta}(t), \bar{\vartheta}(t))_{L_2(\Omega)} + a_3\mathbf{b}(D_t\bar{u}(t), \bar{\vartheta}(t))\big)dt =$$
$$= \frac{a_1}{2}\int_0^{t^*} D_t(\|\bar{\vartheta}(t)\|^2_{L_2(\Omega)})\,dt + a_3\int_0^{t^*} \mathbf{b}(D_t\bar{u}(t), \bar{\vartheta}(t))\,dt\ .$$

Last term on the right side of the latter relation can be expressed more exactly by special selection of the test function in (30) ($v = D_t\bar{u} \in L_2(I; \mathcal{V})$) as it can be is seen from following (3^{rd} "fundamental relation")

$$a_3\int_0^{t^*} \mathbf{b}(D_t\bar{u}(t), \bar{\vartheta}(t))\,dt = \frac{a_3}{\alpha}\int_0^{t^*} \mathbf{a}(D_t\bar{u}(t), \bar{u}(t))\,dt = \frac{a_3}{2\alpha}\int_0^{t^*} D_t(\|\bar{u}(t)\|^2_{H(\Omega)})\,dt\,.$$

From this and previous relation we finally get

$$\int_0^{t^*} \big(a_1(D_t\bar{\vartheta}(t), \bar{\vartheta}(t))_{L_2(\Omega)} + a_3\mathbf{b}(D_t\bar{u}(t), \bar{\vartheta}(t))\big)dt =$$
$$= \frac{a_1}{2}\|\bar{\vartheta}(t^*)\|^2_{L_2(\Omega)} + \frac{a_3}{2\alpha}\|D^2\bar{u}(t^*)\|^2_{L_2(\Omega)} \tag{32}$$

where we used initial condition and definition of the norm on $\mathbf{H}(\Omega)$. Finally, from (31) and (32) we obtain estimation

$$0 \geq \frac{a_1}{2}\|\bar{\vartheta}(t^*)\|^2_{L_2(\Omega)} + \frac{a_3}{2\alpha}\|D^2\bar{u}(t^*)\|^2_{L_2(\Omega)}$$

showing $\bar{\vartheta}(t^*) = 0$ and $\bar{u}(t^*) = 0$. The point $t^* \in I$ was arbitrary chosen and thus we have $\overline{\mathbf{U}}(t) \equiv 0$ for a.e. $t \in I$ and proof of uniqueness is finished.

5.2.1 Consequences of uniqueness of the solution

Now we can formulate some consquences of the statement of Theorem 1, namely of uniqueness of the solution of the problem (P).

First we introduce direct consequence of uniqueness of the solution—statement concerning convergence of the whole sequence of Rothe functions.

Corollary 1 *The whole sequence of the Rothe functions* $\{\mathbf{U}^{(n)}\}_{n=1}^{\infty}$ *converges weakly in* $L_2(I;\mathcal{H})$ *for* $n \to \infty$ *to the weak solution of* (P) .

This lemma can be proved by contradiction argument and thus it follows immediately from uniqueness of the solution of the problem (P). But we can obtain even more—that is following statement

Corollary 2 *The sequence of the Rothe functions* $\{\mathbf{U}^{(n)}\}_{n=1}^{\infty}$ *converges strongly in* $C(\bar{I};H^1(\Omega)\times L_2(\Omega))$ *for* $n \to \infty$ *to the weak solution of* (P) *(that is uniformly with respect to* $t \in \bar{I}$*).*

Proof: is based on one generalization of Arzelà–Ascoli Theorem (see [13], Theorem 1.6.9, p.42, for example) as well as on compactness of imbedding of space $\mathcal{V} \times H^1(\Omega)$ into space $H^1(\Omega) \times L_2(\Omega)$.

Let us first remind we have $\mathbf{U}^{(n)} \xrightarrow{w} \mathbf{U}$ in $L_2(I;\mathcal{H})$ for $n \to \infty$ and $\{\mathbf{U}^{(n)}\}_{n=1}^{\infty} \subset C(\bar{I};H^1(\Omega) \times L_2(\Omega))$. Moreever, from a-priory estimation we see that set of Rothe functions $\{\mathbf{U}^{(n)}\}_{n=1}^{\infty}$ is not only bounded but also equicontinuous and we also can easily verify that for any $t \in \bar{I}$ the set $\{\mathbf{U}^{(n)}(t)\}_{n=1}^{\infty}$ is relatively compact in $H^1(\Omega) \times L_2(\Omega)$. Thus, according to generalized Arzelà–Ascoli Theorem, there exists subsequence $\{\mathbf{U}^{(n_k)}\}_{k=1}^{\infty}$ such that $\mathbf{U}^{(n_k)} \to \tilde{\mathbf{U}}$ in $C(\bar{I};H^1(\Omega) \times L_2(\Omega))$ (in "strong" sense) and we immediately see that $\tilde{\mathbf{U}} = \mathbf{U}$. Next, from uniqueness of the solution, we see the same strong convergency holds also for whole sequence $\{\mathbf{U}^{(n)}\}_{n=1}^{\infty}$ and statement of Corollary 2 is proved.

Now, Theorem 2 can be also easily proved: its statement is just direct consequence of the Corollary 2, a-priory estimation (24) and definition of RvF. The constant C and estimation itself from Theorem 2 have thus exactly following form

$$||\mathbf{U}^1 - \mathbf{U}^2||_{\mathcal{H}} \le \max\left((\bar{c})^2, \tfrac{2(1+\alpha\bar{c})}{\min(1,a_2)}\right) ||\mathcal{F}^1 - \mathcal{F}^2||_{H^*} .$$

6 Conclusion and future generalization

Using of Rothe method of discretization in time we have proved existence and uniqueness of the weak solution for the problem representing evolution of the bending of beam or infinite plate strip. Besides, we have also proved some aproximative properties of the Rothe functions and continuous dependence of the solution on data of the problem.

The main result presented here can be generalized in many ways. First, some a-priory estimations of the semidiscrete solutions for special type of nonsteady heat source are given in [25], for example. Then we can also prove similar result for "full linearized model": instead of model problem with unknown couple of functions $\{u, \vartheta\}$ we can consider unknown quadruple of functions $\{\{u^{(1)}, u^{(2)}\}, \{\vartheta^{(1)}, \vartheta^{(2)}\}\}$ representing normal and vertical displacement and constant and linear distribution of temperature along height of cross section. Analogical result can be obtained for Riessner–Mindlin–Timoshenko model described by quintuplet of unknown functions $\{\{u^{(1)}, u^{(2)}, u^{(3)}\}, \{\vartheta^{(1)}, \vartheta^{(2)}\}\}$. Next, another possible way of generalization can be as follows: more general heat source, time evolution of the beam load, non-classical boundary condition representing rigid or elastic support and with or without friction, inner obstacles and so on (see [12] or [25], for example). Another interesting example of possible generalization of our result can be the special problem of two beams in contact which but without infleunce of heat was studied in [20]. And of course, one can take into consideration also such a type of boundary condition where both components of unknown abstract vector function $\mathbf{U}$ are mutually coupled: realisation of contact can lead to heat transfer, see also previous example of two beams, for example.

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