Acta Univ. Palacki. Olomuc., Fac. rer. nat.,
Mathematica **34** (1995) 127–136

# Associated Linear Differential Equations to Linear Second Order Differential Equations of Sturm, Jacobi and General Form

Jitka LAITOCHOVÁ

*Deptartment of Mathematics, Pedagogical Faculty, Palacký University,*
*Žižkovo nám. 5, 771 00 Olomouc, Czech Republic*
*E-mail: laitochova@risc.upol.cz*

(Received December 13, 1994)

**Abstract**

In the theory of linear second order differential equations of Jacobi form

$$y'' = q(t)y, \tag{q}$$

where $q \in C_2(\mathcal{J})$, the associated linear second order differential equation of Jacobi form and associated linear second order differential equation with parameters $\alpha$, $\beta \in \mathbb{R}$ are considered together with equation (q). The associated equations are used in the theory of dispersions and phases of equation (q). In this paper we will extend the notions of the first associated equation and the associated equation with parameters $\alpha$, $\beta$ to the equation (q) to the case of the linear second order differential equations of Sturm form

$$(p(t)y')' - q(t)y = 0 \tag{pq}$$

and to equations of general form

$$y'' + a(t)y' + b(t)y = 0. \tag{ab}$$

**Key words:** Homogeneous linear second order differential equation, associated linear second order differential equation.

**MS Classification:** 34C20

# 1 Introduction

We will deal with homogeneos second order linear differential equations

$$(p(t)y')' - q(t)y = 0 \quad \text{resp.} \quad y'' - q(t)y = 0 \quad \text{resp.} \quad y'' + a(t)y' + b(t)y = 0,$$

where the first equation is called of Sturm form, the second one of Jacobi form and the third one of general form. We will define the associated equation with parameters $\alpha$, $\beta \in \mathbb{R}$ to each of them. Then the first associated equation is obtained by a special choice of parameters $\alpha$, $\beta$. The results are summarized in two theorems.

The first associated equation and the associated equation with parameters to the equation (q) were introduced in [1], [2].

A mapping of the space of solutions of the equation (ab) resp. (q) onto the space of solutions of the equation

$$Y'' + A(t)Y' + B(t)Y = 0 \tag{AB}$$

resp.

$$Y'' = Q(t)Y, \tag{Q}$$

which is given by the formula

$$Y(t) = \alpha(t)y + \beta(t)y',$$

$\alpha$, $\beta \in C_2(\mathcal{J})$, where $y$ is a solution of (ab) resp. (q), shows a context to the associated equation with parameters $\alpha$, $\beta \in \mathbb{R}$ resp. of the first associated equation. The mapping was studied for example in [3], [4], [5], [6]. While in [5] the coefficients $A$, $B$ of the differential equation (AB) depending of the functions $\alpha$, $\beta$ and the coefficients $a$, $b$ of the equation (ab) are determined in respect of the above mentioned mapping, we will deal both with the problem to find the solution of (AB) in the form

$$Y = \rho\left(\alpha y + \beta y' e^{\int_{t_0}^{t} a(s)ds}\right),$$

where $\alpha$, $\beta \in \mathbb{R}$ and $\rho = \rho(\alpha, \beta, a, b, A)$ and with the problem to find an explicite formula for the coefficient $B$ of the equation (AB) for a given coefficient $A$.

An analogous problem is solved also in the case of the equation (pq) of Sturm form.

# 2 Linear second order differential equations of Sturm form

**Theorem 1** *Consider two homogeneous linear second order differential equations of Sturm form*

$$(p(t)y')' - q(t)y = 0, \tag{pq}$$

$$(P(t)Y')' - Q(t)Y = 0, \tag{PQ}$$

*where* $p, q, P \in C_2(\mathcal{J})$, $p \neq 0$, $P \neq 0$. *Let* $\alpha, \beta \in \mathbb{R}$ *be parameters,* $\alpha^2 + \beta^2 > 0$. *Let*

$$K = \beta^2 q - \alpha^2 \frac{1}{p} \neq 0 \tag{1}$$

*for* $t \in \mathcal{J}$. *Let*

$$Q = \frac{q}{p}P - \frac{1}{2}P'' - \frac{1}{2}\left(P\frac{K'}{K}\right)' + \frac{P}{4}\left(\frac{P'}{P} + \frac{K'}{K}\right)^2 - \alpha\beta\left(\frac{q}{p}\right)'\frac{P}{K}. \tag{2}$$

*Then* $Q \in C_0(\mathcal{J})$.

*Let* $y \in$ (pq) *be a solution of the differential equation. Then the function*

$$Y = \rho(\alpha y + \beta p y') \tag{3}$$

*is a solution of equation* (PQ), *where the multiplier* $\rho$ *is given by the formula*

$$\rho = \frac{c}{\sqrt{|P(\beta^2 q - \frac{\alpha^2}{p})|}}, \tag{4}$$

*and* $c \in \mathbb{R}$ *is an arbitrary constant.*

**Proof** In the interval $\mathcal{J}$ we search for a function $\rho = \rho(t)$ so that the function $Y$ given by equation (3) for any solution $y \in$ (pq) and parameters $\alpha, \beta$, $\alpha^2 + \beta^2 > 0$, is a solution of the equation (PQ) for a given coefficient $P \in C_2(\mathcal{J})$.

By consecutive differentiation and by the help of (pq) we get

$$\begin{aligned} Y' &= \rho'(\alpha y + \beta p y') + \rho(\alpha y' + \beta q y), \\ PY' &= P\rho'(\alpha y + \beta p y') + P\rho\left(\frac{\alpha}{p}py' + \beta q y\right), \end{aligned} \tag{5}$$

and

$$\begin{aligned} QY &\equiv (PY')' \\ &= P'\rho'(\alpha y + \beta p y') + P\rho''(\alpha y + \beta p y') + P\rho'(\alpha y' + \beta q y) \\ &\quad + P'\rho(\alpha y' + \beta q y) + P\rho'(\alpha y' + \beta q y) \\ &\quad + P\rho\left[\alpha\left(\frac{1}{p}\right)' p y' + \frac{\alpha}{p} q y + \beta q' y + \beta q y'\right]. \end{aligned}$$

After a rearrangement we have

$$\begin{aligned} (PY')' &= P'\rho'(\alpha y + \beta p y') + P\rho''(\alpha y + \beta p y') \\ &\quad + P\rho\frac{q}{p}(\alpha y + \beta p y') + 2\rho'[P(\alpha y' + \beta q y)] \\ &\quad + \rho\left\{P'(\alpha y' + \beta q y) + P\left[\alpha\left(\frac{1}{p}\right)' p y' + \beta q' y\right]\right\}, \end{aligned} \tag{6}$$

where

$$\left\{P'(\alpha y' + \beta qy) + P\left[\alpha\left(\frac{1}{p}\right)' py' + \beta q'y\right]\right\} = \alpha\left(\frac{P}{p}\right)' py' + \beta(Pq)'y.$$

If we substitute into (PQ) terms (3), (5) and (6) we get

$$\begin{aligned} 0 &= (PY')' - QY \\ &= (\alpha P'\rho' + \alpha P\rho'' + \alpha\tfrac{q}{p}P\rho + 2\beta qP\rho' + \beta qP'\rho + \beta q'P\rho - \alpha Q\rho)y \\ &\quad + (\beta pP'\rho' + \beta pP\rho'' + \beta qP\rho + 2\alpha P\rho' + \alpha P'\rho - \alpha\tfrac{p'}{p}P - \beta pQ\rho)y'. \end{aligned}$$

This identity is valid for any solution $y \in (\mathrm{pq})$ provided the coefficients of $y$ and $y'$ are identically zeros:

$$\alpha P'\rho' + \alpha P\rho'' + \alpha\frac{q}{p}P\rho + 2\beta qP\rho' + \beta qP'\rho + \beta q'P\rho - \alpha Q\rho = 0,$$

$$\beta pP'\rho' + \beta pP\rho'' + \beta qP\rho + 2\alpha P\rho' + \alpha P'\rho - \alpha\frac{p'}{p}P - \beta pQ\rho = 0.$$

If we multiply the first equation by $\beta p$ and the second one by $\alpha$ and subtract them we get after a rearrangement

$$\rho'(2\beta^2 pqP - 2\alpha^2 P) + \rho\left(\beta^2 pqP' + \beta^2 q'pP - \alpha^2 P' + \alpha^2\frac{p'}{p}P\right) = 0$$

and dividing by $p$ we have

$$2\rho' PK + \rho(P'K + PK') = 0.$$

According to (1) we get

$$\frac{\rho'}{\rho} = -\frac{1}{2}\frac{(PK)'}{PK}, \tag{7}$$

and therefore

$$\rho = c\frac{1}{\sqrt{|PK|}},$$

where $c \in \mathbb{R}$ is an arbitrary constant. Thus formula (4) holds.

Transformation (3) assigns multiplier $\rho$ given by (4) to any coefficient $P \in C_2(\mathcal{J})$. The equation $(PQ)$ yields that the coefficient $Q$ is determined by the equation.

We will show now how to calculate the coefficient using functions $\rho$ and $\rho'$. From $(PQ)$ we get by the help of (6) and (3) that

$$\begin{aligned} Q &= \frac{(PY')'}{Y} = \frac{P'\rho'}{\rho} + \frac{P\rho''}{\rho} + \frac{q}{p}P \\ &\quad + \frac{2\rho'}{\rho}\frac{P(\alpha y' + \beta qy)}{\alpha y + \beta py'} + \frac{\alpha(\frac{P}{p})'py' + \beta(Pq)'y}{\alpha y + \beta py'}. \end{aligned} \tag{8}$$

Substituting (7) for $\rho'/\rho$ we get

$$\begin{aligned}
&\frac{2\rho'}{\rho}\frac{P(\alpha y'+\beta q y)}{\alpha y+\beta p y'}+\frac{\alpha(\frac{P}{p})'py'+\beta(Pq)'y}{\alpha y+\beta p y'}\\
&\quad=-\frac{P'}{P}\frac{P(\alpha y'+\beta q y)}{\alpha y+\beta p y'}-\frac{\beta^2 q'+\alpha^2\frac{p'}{p^2}}{\beta^2 q-\alpha^2\frac{1}{p}}\frac{P(\alpha y'+\beta q y)}{\alpha y+\beta p y'}\\
&\quad+\frac{\alpha(\frac{P'}{p}-\frac{Pp'}{p^2})py'+\beta(P'q+Pq')y}{\alpha y+\beta p y'}=\frac{\alpha\beta P(q\frac{p'}{p^2}+q'\frac{1}{p})}{\alpha^2\frac{1}{p}-\beta^2 q}.
\end{aligned}$$

Since $\frac{P'\rho'}{\rho}+\frac{P\rho''}{\rho}=\frac{(P\rho')'}{\rho}$, we calculate by the help of (7) that

$$\begin{aligned}
(P\rho')' &= \left(-\frac{1}{2}P'\rho-\frac{1}{2}\frac{\beta^2 q'+\alpha^2\frac{p'}{p^2}}{\beta^2 q-\alpha^2\frac{1}{p}}P\rho\right)'\\
&= \left[-\frac{1}{2}P''-\frac{1}{2}\left(\frac{\beta^2 q'+\alpha^2\frac{p'}{p^2}}{\beta^2 q-\alpha^2\frac{1}{p}}\right)'P-\frac{1}{2}\frac{\beta^2 q'+\alpha^2\frac{p'}{p^2}}{\beta^2 q-\alpha^2\frac{1}{p}}P'\right]\rho\\
&\quad+\left(-\frac{1}{2}P'-\frac{1}{2}\frac{\beta^2 q'+\alpha^2\frac{p'}{p^2}}{\beta^2 q-\alpha^2\frac{1}{p}}P\right)\rho'.
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{(P\rho')'}{\rho} &= -\frac{1}{2}P''-\frac{1}{2}\left(\frac{\beta^2 q'+\alpha^2\frac{p'}{p^2}}{\beta^2 q-\alpha^2\frac{1}{p}}\right)'P\\
&\quad-\frac{1}{2}\frac{\beta^2 q'+\alpha^2\frac{p'}{p^2}}{\beta^2 q-\alpha^2\frac{1}{p}}P'+P\left(\frac{1}{2}\frac{P'}{P}+\frac{1}{2}\frac{\beta^2 q'+\alpha^2\frac{p'}{p^2}}{\beta^2 q-\alpha^2\frac{1}{p}}\right)^2
\end{aligned}$$

and from (8) we get

$$\begin{aligned}
Q &= -\frac{1}{2}P''-\frac{1}{2}\left(\frac{\beta^2 q'+\alpha^2\frac{p'}{p^2}}{\beta^2 q-\alpha^2\frac{1}{p}}P\right)'\\
&\quad+\frac{P}{4}\left(\frac{P'}{P}+\frac{\beta^2 q'+\alpha^2\frac{p'}{p^2}}{\beta^2 q-\alpha^2\frac{1}{p}}\right)^2+\frac{q}{p}P+\alpha\beta\frac{P(q\frac{p'}{p^2}+q'\frac{1}{p})}{\alpha^2\frac{1}{p}-\beta^2 q}.
\end{aligned}$$

Therefore formula (2) holds. The assumptions on coefficients $p, q, P$ and condition (1) imply $Q \in C_0(\mathcal{J})$.

**Definition 1** **The differential equation (PQ) with a given coefficient $P \in C_2(\mathcal{J})$ and coefficient $Q$ given by (2) is called the** ***associated linear differential equation with parameters*** **$\alpha$, $\beta \in \mathbb{R}$ to linear differential equation (pq).**

**Special cases:**
**1.** Let $p(t) \equiv P(t) \equiv 1$, $\alpha^2 + \beta^2 > 0$. Then equations (pq) and (PQ) are of *Jacobi form*. Transformation (3) is of the form

$$Y = \rho(\alpha y + \beta y')$$

and with multiplier $\rho$, which is given in this case by the formula

$$\rho = c\frac{1}{\sqrt{|\alpha^2 - \beta^2 q|}},$$

transforms a solution $y$ of the differential equation (q) into a solution $Y$ of the differental equation (Q), where

$$Q(t) = q(t) + \frac{1}{2}\frac{\beta^2 q''(t)}{\alpha^2 - \beta^2 q(t)} + \frac{3}{4}\frac{\beta^4 q'^2(t)}{[\alpha^2 - \beta^2 q(t)]^2} + \alpha\beta\frac{q'(t)}{\alpha^2 - \beta^2 q(t)}.$$

It is the associated equation with parameters $\alpha$, $\beta$ according to the differential equation (q) (see [2]).
**2.** Let $p(t) \equiv P(t) \equiv 1$, $\alpha = 0$, $\beta = 1$. Then transformation (3) is of the form

$$Y = \rho y'$$

with multiplier $\rho$

$$\rho = c\frac{1}{\sqrt{|q|}},$$

and it transforms a solution $y$ of the diferential equation (q) into a solution Y of the differential equation (Q) with coefficient

$$Q(t) = q(t) - \frac{1}{2}\frac{q''(t)}{q'(t)} + \frac{3}{4}\frac{q'^2(t)}{q^2(t)}.$$

It is the first associated equation according to the differential equation (q) (see [1]).

# 3 Linear second order differential equation in general form

**Theorem 2** *Consider two homogeneous linear second order differential equations in general form*

$$y'' + a(t)y' + b(t)y = 0, \qquad \text{(ab)}$$

$$Y'' + A(t)Y' + B(t)Y = 0, \qquad \text{(AB)}$$

*where* $a, b, A \in C_2(\mathcal{J})$. *Let* $\alpha, \beta \in \mathbb{R}$ *be parameters,* $\alpha^2 + \beta^2 > 0$. *Let*

$$L = \beta^2 b e^{\mathcal{I}} + \frac{\alpha^2}{e^{\mathcal{I}}}(\neq 0),$$

*where*

$$\mathcal{I} = \int_{t_0}^{t} a(t)\,dt \tag{9}$$

*and* $t_0 \in \mathcal{J}$ *is an arbitrary number.*

*Let*

$$B = b + \frac{1}{2}\frac{L''}{L} - \frac{3}{4}\frac{L'^2}{L^2} + \frac{\alpha\beta b'}{L} + \frac{1}{4}A^2. \tag{10}$$

*Then* $B \in C_0(\mathcal{J})$.

*Let* $y \in$ (ab) *be a solution of the differential equation. Then the function*

$$Y = \rho(\alpha y + \beta e^{\mathcal{I}} y') \tag{11}$$

*is a solution of* (AB) *with multiplier* $\rho$

$$\rho = \frac{1}{\sqrt{|\beta^2 b e^{\mathcal{I}} + \frac{\alpha^2}{e^{\mathcal{I}}}|}} \exp\left(-\frac{1}{2}\int_{t_0}^{t} A(t)\,dt\right), \tag{12}$$

*where* $t_0 \in \mathcal{J}$ *is an arbitrary number.*

**Proof** In the interval $\mathcal{J}$ we search for a function $\rho = \rho(t)$ such that for any solution $y \in$ (ab) and given parameters $\alpha, \beta \in \mathbb{R}$ the function given by (11) is a solution of (AB) for a given coefficient $A \in C_2(\mathcal{J})$.

By consecutive differentiation of (11) and by the help of equation (ab) we get

$$\begin{aligned} Y' &= \rho'(\alpha y + \beta e^{\mathcal{I}} y') + \rho(\alpha y + \beta e^{\mathcal{I}} y')' \\ &= \rho'(\alpha y + \beta e^{\mathcal{I}} y') + \rho(-\beta b e^{\mathcal{I}} y + \alpha y'), \end{aligned}$$

and

$$\begin{aligned} Y'' &= \rho''(\alpha y + \beta e^{\mathcal{I}} y') + 2\rho'(-\beta b e^{\mathcal{I}} y + \alpha y') \\ &\quad + \rho[(-\beta b' e^{\mathcal{I}} - \beta a b e^{\mathcal{I}} - \alpha b)y - (\beta b e^{\mathcal{I}} + \alpha a)y'], \end{aligned} \tag{13}$$

$$AY' = A\rho'(\alpha y + \beta e^{\mathcal{I}} y') + A\rho(-\beta b e^{\mathcal{I}} y + \alpha y'), \tag{14}$$

$$BY = B\rho(\alpha y + \beta e^{\mathcal{I}} y'). \tag{15}$$

Substituting into equation (AB) we get

$$\begin{aligned} 0 &= Y'' + AY' + BY \\ &= [\alpha\rho'' - 2\beta b e^{\mathcal{I}}\rho' + (-\beta b' e^{\mathcal{I}} - \beta a b e^{\mathcal{I}} - \alpha\beta)\rho + \alpha A\rho' - \beta b A e^{\mathcal{I}}\rho + \alpha B\rho]y \\ &\quad + [\beta e^{\mathcal{I}}\rho'' + 2\alpha\rho' - (\beta b e^{\mathcal{I}} + \alpha a)\rho + \beta A e^{\mathcal{I}}\rho' + \alpha A\rho + \beta B e^{\mathcal{I}}\rho]. \end{aligned}$$

This identity is valid for any solution $y \in$ (ab) provided the coefficients of $y$ and $y'$ are identically zero. So we have conditions

$$\begin{aligned} \alpha\rho'' - 2\beta b e^{\mathcal{I}}\rho' - \beta b' e^{\mathcal{I}}\rho - \beta a b e^{\mathcal{I}}\rho - \alpha b\rho + \alpha A\rho - \beta b A e^{\mathcal{I}}\rho + \alpha B\rho = 0 \\ \beta e^{\mathcal{I}}\rho'' + 2\alpha\rho' - \beta b e^{\mathcal{I}}\rho - \alpha a\rho + \beta A e^{\mathcal{I}}\rho' + \alpha A\rho + \beta B e^{\mathcal{I}}\rho = 0. \end{aligned}$$

Multiply the first equation by $\beta e^{\mathcal{I}}$ and the second one by $\alpha$. Then subtract the second one from the first one to get

$$\rho'(-2\beta^2 b\exp^2\mathcal{I} - 2\alpha^2) + \rho(-\beta^2 b'\exp^2\mathcal{I} - \beta^2 ab\exp^2\mathcal{I} - \beta^2 bA\exp^2\mathcal{I} + \alpha^2 a - \alpha^2 A) = 0.$$

Therefore after dividing by $e^{\mathcal{I}}$ we get

$$-2\rho'\left(\beta^2 be^{\mathcal{I}} + \frac{\alpha^2}{e^{\mathcal{I}}}\right) = \rho\left[\beta^2\left(b'e^{\mathcal{I}} + abe^{\mathcal{I}}\right) - \alpha^2\frac{a}{e^{\mathcal{I}}} + A\left(\beta^2 be^{\mathcal{I}} + \frac{\alpha^2}{e^{\mathcal{I}}}\right)\right].$$

From here

$$\frac{2\rho'}{\rho} = -\frac{(\beta^2 be^{\mathcal{I}} + \frac{\alpha^2}{e^{\mathcal{I}}})'}{\beta^2 be^{\mathcal{I}} + \frac{\alpha^2}{e^{\mathcal{I}}}} - A$$

and

$$\rho = \frac{1}{\sqrt{|\beta^2 be^{\mathcal{I}} + \frac{\alpha^2}{e^{\mathcal{I}}}|}}\exp\left(-\frac{1}{2}\int_{t_0}^{t} A\,dt\right),$$

which corresponds to formula (12).

The transformation (11) assigns multiplier $\rho$ given by (12) to each coefficient $A \in C_2(\mathcal{J})$. Then equation (AB) implies that coefficient $B$ is determined by the equation. We will show how to calculate the coefficient using the functions $\rho$ and $\rho'$. From (AB) we get using (13), (14) and (15) that

$$-B = \frac{Y''}{Y} + A\frac{Y'}{Y}$$

and therefore

$$-B = \frac{\rho''}{\rho} + \frac{2\rho'}{\rho}\frac{(-\beta be^{\mathcal{I}}y + \alpha y')}{(\alpha y + \beta e^{\mathcal{I}}y')} - b - \frac{(\beta b'e^{\mathcal{I}} - \beta abe^{\mathcal{I}})y + \alpha ay'}{\alpha y + \beta e^{\mathcal{I}}y'} + A\frac{\rho'}{\rho} + A\frac{-\beta be^{\mathcal{I}}y + \alpha y'}{\alpha y + \beta e^{\mathcal{I}}y'}.$$

First we calculate the partial sum identity

$$\frac{2\rho'}{\rho}\frac{(-\beta be^{\mathcal{I}}y + \alpha y')}{(\alpha y + \beta e^{\mathcal{I}}y')} - \frac{(\beta b'e^{\mathcal{I}} - \beta abe^{\mathcal{I}})y + \alpha ay'}{(\alpha y + \beta e^{\mathcal{I}}y')} + A\frac{-\beta be^{\mathcal{I}}y + \alpha y'}{\alpha y + \beta e^{\mathcal{I}}y'} = -\frac{\alpha\beta b'}{\beta^2 be^{\mathcal{I}} + \frac{\alpha^2}{e^{\mathcal{I}}}}.$$

We have

$$\left(\frac{\rho'}{\rho}\right)' = \frac{\rho''\rho - \rho'^2}{\rho^2}$$

and therefore

$$\frac{\rho''}{\rho} = \left(\frac{\rho'}{\rho}\right)' + \left(\frac{\rho'}{\rho}\right)^2.$$

Since

$$\frac{\rho'}{\rho} = -\frac{1}{2}\frac{(\beta^2 be^{\mathcal{I}} + \frac{\alpha^2}{e^{\mathcal{I}}})'}{(\beta^2 be^{\mathcal{I}} + \frac{\alpha^2}{e^{\mathcal{I}}})} - \frac{1}{2}A,$$

we have

$$\left(\frac{\rho'}{\rho}\right)' = -\frac{1}{2}\frac{(\beta^2 be^{\mathcal{I}} + \frac{\alpha^2}{e^{\mathcal{I}}})''(\beta^2 be^{\mathcal{I}} + \frac{\alpha^2}{e^{\mathcal{I}}}) - (\beta^2 be^{\mathcal{I}} + \frac{\alpha^2}{e^{\mathcal{I}}})'^2}{(\beta^2 be^{\mathcal{I}} + \frac{\alpha^2}{e^{\mathcal{I}}})^2}$$

and

$$\left(\frac{\rho'}{\rho}\right)^2 = \frac{1}{4}\left(\frac{(\beta^2 be^{\mathcal{I}} + \frac{\alpha^2}{e^{\mathcal{I}}})'}{\beta^2 be^{\mathcal{I}} + \frac{\alpha^2}{e^{\mathcal{I}}}} + A\right)^2.$$

So we can calculate

$$\begin{aligned} -B &= \frac{\rho''}{\rho} + A\frac{\rho'}{\rho} - b - \frac{\alpha\beta b'}{\beta^2 be^{\mathcal{I}} + \frac{\alpha^2}{e^{\mathcal{I}}}} \\ &= -\frac{1}{2}\frac{(\beta^2 be^{\mathcal{I}} + \frac{\alpha^2}{e^{\mathcal{I}}})''}{\beta^2 be^{\mathcal{I}} + \frac{\alpha^2}{e^{\mathcal{I}}}} + \frac{1}{2}\frac{(\beta^2 be^{\mathcal{I}} + \frac{\alpha^2}{e^{\mathcal{I}}})'^2}{(\beta^2 be^{\mathcal{I}} + \frac{\alpha^2}{e^{\mathcal{I}}})^2} + \frac{1}{4}\left(\frac{(\beta^2 be^{\mathcal{I}} + \frac{\alpha^2}{e^{\mathcal{I}}})'}{(\beta^2 be^{\mathcal{I}} + \frac{\alpha^2}{e^{\mathcal{I}}})} + A\right)^2 \\ &\quad + A\left(-\frac{1}{2}\frac{(\beta^2 be^{\mathcal{I}} + \frac{\alpha^2}{e^{\mathcal{I}}})'}{(\beta^2 be^{\mathcal{I}} + \frac{\alpha^2}{e^{\mathcal{I}}})} - \frac{1}{2}A\right) - b - \frac{\alpha\beta b'}{\beta^2 be^{\mathcal{I}} + \frac{\alpha^2}{e^{\mathcal{I}}}}, \end{aligned}$$

which agrees with formula (10). The assumptions on coefficients $a$, $b$, $A$ and condition (9) imply $B \in C_0(\mathcal{J})$.

**Definition 2** The differential equation (AB) with given coefficient $A \in C_2(\mathcal{J})$ and with coefficient $B$ given by (10) is called the *associated linear differential equation with parameters* $\alpha$, $\beta \in \mathbb{R}$ to the linear differential equation (ab).

**Special cases:**

**1.** Let $a(t) \equiv A(t) \equiv 0$, $\alpha^2 + \beta^2 > 0$. Then equations (ab) and (AB) are of Jacobi form. Since $e^{\mathcal{I}} = 1$, transformation (11) is of the form

$$Y = \rho(\alpha y + \beta y'),$$

where multiplier $\rho$ is given by

$$\rho = \frac{1}{\sqrt{|\beta^2 b + \alpha^2|}}$$

and it transforms a solution $y$ of the differential equation $y'' + by = 0$ into a solution $Y$ of the differential equation $Y'' + BY = 0$ with coefficient

$$B(t) = b(t) + \frac{1}{2}\frac{\beta^2 b''(t)}{\beta^2 b(t) + \alpha^2} - \frac{3}{4}\frac{\beta^4 b'^2(t)}{(\beta^2 b(t) + \alpha^2)^2} + \frac{\alpha\beta b'(t)}{\beta^2 b(t) + \alpha^2},$$

which is coefficient $-Q$ (with $q = -b$) calculated on page 132. As we have already noted here, it is the associated equation with parameters $\alpha$, $\beta$ according to the differential equation (q) (see [2]).

**2.** Let $a(t) \equiv A(t) \equiv 0$, $\alpha = 0$, $\beta = 1$. Transformation (11) is of the form

$$Y = \rho y',$$

where multiplier $\rho$ is given by the formula

$$\rho = \frac{1}{\sqrt{|b|}},$$

and it transforms a solution $y$ of the differential equation $(-b)$ into a solution $Y$ of the differential equation $(-B)$ with coefficient

$$B(t) = b(t) + \frac{1}{2}\frac{b''(t)}{b(t)} - \frac{3}{4}\frac{b'^2(t)}{b^2(t)}.$$

It is the first associated equation according to the differential equation (q) (see [1]).

# References

[1] Borůvka, O.: *Linear Differential Transformations of the Second Order.* The English University Press, London, 1971.

[2] Laitoch, M.: *L'équation associeé dans la théorie des transformation differentielles du second ordre.* Acta Univ. Palacki. Olomuc., Fac. rer. nat. T.12, 1963, 45–62.

[3] Laitochová, J.: *On transformations on two homogenous linear second order differential equations of general and Sturm forms.* Acta Univ. Palacki. Olomuc., Fac. rer. nat. **85**, Math. 25 (1986), 77–95.

[4] Fialka, M.: *On coincidence of the differential equation $y'' - q(t)y = r(t)$ with its associated equation.* Acta Univ. Palacki. Olomuc., Fac. rer. nat. **91**, Math. 27 (1988), 251–262.

[5] Staněk, S., Vosmanský, J.: *Transformations of linear second order ordinary differential equations.* Arch. Math. (Brno) **22**, 1 (1986), 55–60.

[6] Votava, M.: *Některé vlastnosti průvodních rovnic s váhovými funkcemi.* Acta Facultas Pedagogicae Ostraviensis **92**, Series A–20 (1985), 43–51.