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On Jensen's Inequality for Self-Adjoint Operators in Hilbert Space

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Abstract

Some inequalities, related to Jensen's discrete inequality, are given for self-adjoint operators in Hilbert space.

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Preliminaries

Let X be a linear space and C a convex subset in X. If $f : C \to \mathbb{R}$ is convex on C, then the following inequality is well known in the literature as Jensen's discrete inequality:

$$f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) \le \frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i)$$

where x_i are n-elements in C, $p_i \ge 0$ for $i = 1, \dots, n$ and $P_n = \sum_{i=1}^{n} p_i > 0$.

For some refinements of this classical result as well as certain applications in the theory of inequalities connected with the arithmetic–geometric mean inequality, generalized triangle inequality, Ky Fan's and other inequalities, we refer to the recent papers [1–7] and [11–12].

Now, let $(H;(\ ,\ ))$ be a Hilbert space and $A : H \to H$ a self-adjoint operator on H satisfying the inequality

$$mI \leq A \leq MI, \quad \text{i.e.} \quad m\|x\|^2 \leq (Ax, x) \leq M\|x\|^2 \quad \text{for all } x \text{ in } H.$$

To the real valued function $g : [m, M] \to \mathbb{R}$, there is associated in a natural way a self-adjoint operator on H denoted by $g(A)$ (see e.g. [13, pp. 265–273]).

We shall make use of the following [13, p. 271].

Lemma 1 *Suppose that $g_1, g_2 : [m, M] \to \mathbb{R}$ are continuous and that $g_2(\lambda) \geq g_1(\lambda)$ for all $\lambda \in [m, M]$, then also $g_2(A) \geq g_1(A)$.*

By the use of this lemma we shall give some analogues of Jensen's inequality for self-adjoint operators in Hilbert space. Some natural applications for convex functions are also given.

Results

First we shall note that the following result is a simple consequence of Lemma 1 and the definition of convex functions.

Theorem 1 *Let $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$ be a continuous convex function, $x, y \in [a, b]$ and A a self-adjoint operator in Hilbert space H with $0 \leq A \leq I$. Then*

$$f(xA + y(I - A)) \leq Af(x) + (I - A)f(y)$$

in the order of $A(H)$, ($A(H)$ denotes the linear subspace of self-adjoint operators on H).

Theorem 2 *Suppose that $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$ is continuous convex on $[a, b]$, $p_i \geq 0$, $x_i \in [a, b]$ $(i = 1, \ldots, n)$ with $P_n > 0$, and A is a self-adjoint operator on a Hilbert space H with $0 \leq A \leq I$. Then*

$$f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) I \leq \frac{1}{P_n}\sum_{i=1}^{n} p_i f\left\{x_i A + \left(\frac{1}{P_n}\sum_{j=1}^{n} p_j x_j\right)(I - A)\right\} \leq$$
$$\leq \left(\frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i)\right) I \tag{1}$$

in the order of $A(H)$.

Proof Consider the mappings $g_1, g_2, g_3 : [0,1] \to \mathbb{R}$ given by

$$g_1(t) = f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right), \quad g_2(t) = \frac{1}{P_n}\sum_{i=1}^{n} p_i f\left[t x_i + (1-t)\frac{1}{P_n}\sum_{j=1}^{n} p_j x_j\right]$$

and

$$g_3(t) = \frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i).$$

Since f is continuous convex on $[a,b]$, g_2 is also convex and continuous on $[0,1]$. The mapping g_1 is continuous on $[0,1]$ (being constant on $[0,1]$) and by Jensen's inequality one has

$$g_2(t) \geq f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i[t x_i + (1-t)\frac{1}{P_n}\sum_{j=1}^{n} p_j x_j]\right) = g_1(t)$$

for all $t \in [0,1]$.

Using Lemma 1 for g_2 and g_1 defined above, we get the first inequality in (1).

To prove the second inequality, we observe that

$$g_2(t) \leq t\,\frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i) + (1-t) f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) \leq g_3(t)$$

for all $t \in [0,1]$. Applying Lemma 1 for g_2 and g_3 we deduce the desired result.

Corollary 1.1 *Suppose that $x_i > 0$, $p_i \geq 0$ with $P_n > 0$ $(i = 1, \ldots, n)$ and $p \geq 1$. Then for a self-adjoint operator A on Hilbert space H with $0 \leq A \leq I$, we have*

$$\left(\sum_{i=1}^{n} p_i x_i\right)^p I \leq P_n^{p-1}\sum_{i=1}^{n} p_i\left\{x_i A + \left(\frac{1}{P_n}\sum_{j=1}^{n} p_j x_j\right)(I-A)\right\}^p \leq$$
$$\leq P_n^{p-1}\left(\sum_{i=1}^{n} p_i x_i^p\right) I$$

in the order of $A(H)$.

Corollary 1.2 *Suppose that $x_i > 0$, $p_i \geq 0$ with $P_n > 0$ $(i = 1, \ldots, n)$ and A is as above. Then one has the inequality:*

$$\left(\prod_{i=1}^{n} x_i^{p_i}\right)^{1/P_n} I \leq \left[\prod_{i=1}^{n}\left[x_i A + \left(\frac{1}{P_n}\sum_{j=1}^{n} p_j x_j\right)(I-A)\right]^{p_i}\right]^{1/P_n} \leq \left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) I,$$

in the order of $A(H)$.

Proof By a similar argument as in Theorem 2 for the convex mapping $f(x) = -\ln x$ $(x > 0)$ we get the following refinement of the arithmetic–geometric mean inequality:

$$\left(\sum_{i=1}^{n} x_i^{p_i}\right)^{1/P_n} \leq \left[\prod_{i=1}^{n}[tx_i + (1-t)\frac{1}{P_n}\sum_{j=1}^{n} p_j x_j]^{p_i}\right]^{1/P_n} \leq \frac{1}{P_n}\sum_{i=1}^{n} p_i x_i$$

for all $x_i > 0,\ p_i \geq 0$ $(i = 1, \ldots, n)$ with $P_n > 0$ and $t \in [0,1]$. Now, applying Lemma 1, we get the desired inequality.

Theorem 3 *Let f, x_i, p_i $(i = 1, \ldots, n)$, A be as in Theorem 2. Thus, one has the inequalities*

$$\left(\frac{1}{P_n^2}\sum_{i,j=1}^{n} p_i p_j f\left(\frac{x_i + x_j}{2}\right)\right) I \leq \frac{1}{p_n^2}\sum_{i,j=1}^{n} p_i p_j f[x_i A + x_j(I - A)] \leq$$
$$\leq \left(\frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i)\right) I \tag{2}$$

in the order of $A(H)$.

Proof We consider the mappings $g_1, g_2, g_3 : [0,1] \to \mathbb{R}$ given by

$$g_1(t) = \frac{1}{P_n^2}\sum_{i,j=1}^{n} p_i p_j f\left(\frac{x_i + x_j}{2}\right), \qquad g_2(t) = \frac{1}{P_n^2}\sum_{i,j=1}^{n} p_i p_j f(tx_i + (1-t)x_j)$$

and

$$g_3(t) = \frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i).$$

Now, let us observe that $g_i (i = 1,2,3)$ are continuous on $[0,1]$ (note that g_2 is also convex on $[0,1]$). By the convexity of f one has

$$\frac{1}{2}[f(tx_i + (1-t)x_j) + f((1-t)x_i + tx_j)] \geq f\left(\frac{x_i + x_j}{2}\right)$$

for all $t \in [0,1]$ and $i, j \in \{1, \ldots, n\}$. By multiplying this inequality with $p_i p_j \geq 0$ and summing over i and j from 1 to n, we deduce that

$$\frac{1}{2P_n^2}\left[\sum_{i,j=1}^{n} p_i p_j f(tx_i + (1-t)x_j) + \sum_{i,j=1}^{n} p_i p_j f((1-t)x_i + tx_j)\right] \geq$$
$$\geq \frac{1}{P_n^2}\sum_{i,j=1}^{n} p_i p_j f\left(\frac{x_i + x_j}{2}\right)$$

and since

$$\sum_{i,j=1}^{n} p_i p_j f(tx_i + (1-t)x_j) = \sum_{i,j=1}^{n} p_i p_j f((1-t)x_i + tx_j)$$

we get $g_2(t) \geq g_1(t)$ for all $t \in [0,1]$.

Now, applying Lemma 1 for g_1 and g_2 we deduce the first inequality in (2).

For the second part of (2), we have, by the convexity of f, that

$$g_2(t) \leq t\frac{1}{P_n^2}\sum_{i,j=1}^{n} p_i p_j f(x_i) + (1-t)\frac{1}{P_n}\sum_{i,j=1}^{n} p_i p_j f(x_j) = g_3(t)$$

for all $t \in [0,1]$. By Lemma 1, applied to g_2 and g_3, we get the desired result.

Remark 1 Jensen's inequality for double sums gives that

$$\frac{1}{P_n^2}\sum_{i,j=1}^{n} p_i p_j f\left(\frac{x_i + x_j}{2}\right) \geq f\left(\frac{1}{P_n^2}\sum_{i,j=1}^{n} p_i p_j \left(\frac{x_i + x_j}{2}\right)\right) = f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right)$$

which shows that inequality (2) is also an improvement of Jensen's inequality.

Corollary 2.1 *Suppose that $x_i \geq 0$, $p_i \geq 0$ $(i = 1,\ldots,n)$ with $P_n > 0$ and $p \geq 1$. Then, for all A as above, we have*

$$\left(\sum_{i,j=1}^{n} p_i p_j \left(\frac{x_i + x_j}{2}\right)^p\right) I \leq \sum_{i,j=1}^{n} p_i p_j [x_i A + x_j(I-A)]^p \leq P_n \left(\sum_{i=1}^{n} p_i x_i^p\right) I.$$

Corollary 2.2 *Let $x_i > 0$, $p_i \geq 0$ $(i = 1,\ldots,n)$ with $P_n > 0$ and A as above. Then*

$$\left[\prod_{i,j=1}^{n} \left(\frac{x_i + x_j}{2}\right)^{p_i p_j}\right]^{1/P_n^2} I \leq \left(\prod_{i,j=1}^{n} [x_i A + x_j(I-A)]^{p_i p_j}\right)^{1/P_n^2} \leq \left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) I.$$

Another result connected with Jensen's inequality is embodied in the next theorem.

Theorem 4 *Let $f : [a,b] \to \mathbb{R}$ be continuous convex on $[a,b]$, $x_i \in [a,b]$ and $p_i \geq 0$ $(i = 1,\ldots,n)$ with $P_n > 0$ and A as above. Then*

$$\frac{1}{P_n}\sum_{i,j=1}^{n} p_i p_j f[x_i A + x_j(I-A)] \geq \begin{cases} \sum_{i=1}^{n} p_i f\left[x_i A + \left(\frac{1}{P_n}\sum_{j=1}^{n} p_j x_j\right)(I-A)\right] \\ \sum_{i=1}^{n} p_i f\left[x_i(I-A) + \left(\frac{1}{P_n}\sum_{j=1}^{n} p_j x_j\right)A\right] \end{cases} \tag{3}$$

in the order of $A(H)$.

Proof It is sufficient to prove the first inequality in (3). We have, by Jensen's inequality, that

$$g_2(t)=\frac{1}{P_n}\sum_{i,j=1}^{n}p_ip_jf(tx_i+(1-t)x_j)=$$

$$=\sum_{i=1}^{n}p_i\left[\frac{1}{P_n}\sum_{j=1}^{n}p_jf(tx_i+(1-t)x_j)\right]\geq\sum_{i=1}^{n}p_if\left(\frac{1}{P_n}\sum_{j=1}^{n}p_j(tx_i+(1-t)x_j)\right)=$$

$$=\sum_{i=1}^{n}p_if\left(tx_i+(1-t)\frac{1}{P_n}\sum_{j=1}^{n}p_jx_j\right)=g_1(t)$$

for all $t\in[0,1]$.

Since the above mappings g_1 and g_2 are continuous convex on $[0,1]$ and $g_2(t)\geq g_1(t)$ for all $t\in[0,1]$, hence by Lemma 1, we get $g_2(A)\geq g_1(A)$. This completes the proof.

Corollary 3.1 *Suppose that $x_i\geq 0$, $p_i\geq 0$ $i=1,\ldots,n)$ with $P_n>0$ and $p\geq 1$. Then for all A as above one has:*

$$\sum_{i,j=1}^{n}p_ip_j(x_iA+x_j(I-A))^p\geq\begin{cases}P_n\sum\limits_{i=1}^{n}p_i\left[x_iA+(\frac{1}{P_n}\sum\limits_{j=1}^{n}p_jx_i)(I-A)\right]^p\\ P_n\sum\limits_{i=1}^{n}p_i\left[x_i(I-A)+(\frac{1}{P_n}\sum\limits_{j=1}^{n}p_jx_j)A\right]^p.\end{cases}$$

Corollary 3.2 *If $x_i>0$ and $p_i\geq 0$ $(i=1,\ldots,n)$ and A as above. Then*

$$\left[\prod_{i,j=1}^{n}(x_iA+x_j(I-A))^{p_ip_j}\right]^{1/P_n}\leq\begin{cases}\prod\limits_{i=1}^{n}\left[x_iA+(\frac{1}{P_n}\sum\limits_{j=1}^{n}p_jx_j)(I-A)\right]^{p_i}\\ \prod\limits_{i=1}^{n}\left[x_i(I-A)+(\frac{1}{P_n}\sum\limits_{j=1}^{n}p_jx_j)A\right]^{p_i}.\end{cases}$$

For other inequalities for self-adjoint operators in Hilbert space, see [8–9] and [10] where further references are given.

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