Ivan Chajda

Regular lattices

Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica, Vol. 32 (1993), No. 1, 17--20

ACTA UNIVERSITATIS PALACKIANAE OLOMUCENSIS FACULTAS RERUM NATURALIUM
1993 Mathematica XXXII Vol. 110

REGULAR LATTICES

IVAN CHAJDA

(Received February 4, 1992)

Abstract

It was proven by O. M. Mamedov that a variety of algebras is congruence-regular if and only if it has n-transferable congruences for some natural number n. We show that in the case of lattices, this result is valid also for a single algebra and $n = 2$. If a lattice is relatively complementary, we can take $n = 1$.

Key words: congruence regularity, n-transferable congruences, lattice, relatively complementary lattice.

MS Classification: 06B10, 08A30

The concept of transferable principal congruences was introduced in [1]: an algebra A has *tranferable principal congruences* if for each a, b, c of A there exists an element $d \in A$ with $\Theta(a,b) = \Theta(c,d)$. As it was shown in [1], this condition implies regularity of A (recall that A is *regular* if $\Theta = \Phi$ for every two $\Theta, \Phi \in ConA$ whenever they have a congruence class in common).

O.M.Mamedov [2] generalized the concept of transferability in this way:

Definition 1 Let A be an algebra and $a, b \in A$. A principal congruence $\Theta(a,b)$ is *n-transferable* if for any $c \in A$ there exist elements $d_1, \ldots, d_n$ of A with

$$\Theta(a,b) = \Theta(c, d_1, \ldots, d_n). \qquad (*)$$

An algebra A has *n-transferable principal congruences* if for every a, b, c of A there exist $d_1, \ldots, d_n$ of A such that $(*)$ holds.

It is easy to show that if $\Theta(a,b)$ (or an algebra) is n-transferable (has n-transferable principal congruences), then $\Theta(a,b)$ (or A, respectively) has this property for each $n' \geq n$.

The following generalization of our theorem of [1] was proven in [2]:

Lemma 1 *If an algebra A has n-transferable principal congruences for some integer $n \geq 1$, then A is regular.*

With a slight modification of Proposition 3 in [2], we obtain:

Lemma 2 *Let A be a regular algebra and a, b, c be elements of A. Then there exist an integer $n \geq 1$ and elements $d_1, \ldots, d_n$ of A such that*

$$\Theta(a,b) = \Theta(c, d_1, \ldots, d_n).$$

Proof The regularity of A implies

$$\Theta(a,b) = \Theta([c]_{\Theta(a,b)}),$$

because both of these congruences have a common class $[c]_{\Theta(a,b)}$. Hence

$$\langle a,b\rangle \in \Theta([c]_{\Theta(a,b)}),$$

thus there exists a finite subset $F \subseteq [c]_{\Theta(a,b)}$ with $\langle a,b\rangle \in \Theta(F)$. We obtain

$$\Theta(a,b) \subseteq \Theta(F) \subseteq \Theta(\{c\} \cup F) \subseteq \Theta([c]_{\Theta(a,b)}) = \Theta(a,b)$$

whence

$$\Theta(a,b) = \Theta(c, d_1, \ldots, d_n)$$

for

$$F = \{d_1, \ldots, d_n\}.$$

□

Mamedov [2] has shown that for varieties of algebras, regularity is equivalent to n-transferability (for some $n \in \mathbb{N}$). We are going to show that for lattices, this result can be generalized also for a single algebra instead of a variety and, moreover, this n can be uniform:

Theorem 1 *For a lattice L, the following conditions are equivalent:*

(i) L is regular;

(ii) L has 2-transferable principal congruences.

Proof The implication $(ii) \Rightarrow (i)$ follows by Lemma 1. Prove $(i) \Rightarrow (ii)$. Let L be a regular lattice and $a, b, c \in L$. By Lemma 2, there exist an integer n and elements $d_1, \ldots, d_n \in L$ such that

$$\Theta(a, b) = \Theta(c, d_1, \ldots, d_n).$$

Put $e_1 = d_1 \wedge \ldots \wedge d_n$, $e_2 = d_1 \vee \ldots \vee d_n$ in the lattice L. Then $e_1 \leq d_i \leq e_2$ for $i = 1, 2, \ldots, n$, thus

$$\langle d_i, d_i \rangle \in \Theta(c, e_1, e_2) \qquad \langle c, e_1 \rangle \in \Theta(c, e_1, e_2) \qquad \langle c, e_2 \rangle \in \Theta(c, e_1, e_2)$$

imply

$$\langle c, d_i \rangle = \langle c \wedge (d_i \vee c), e_2 \wedge (d_i \vee e_1) \rangle \in \Theta(c, e_1, e_2)$$

for $i = 1, 2, \ldots, n$, whence

$$\Theta(a, b) = \Theta(c, d_1, \ldots, d_n) \subseteq \Theta(c, e_1, e_2).$$

Conversely,

$$\langle c, e_1 \rangle = \langle c \wedge \ldots \wedge c, d_1 \wedge \ldots \wedge d_n \rangle \in \Theta(c, d_1, \ldots, d_n)$$
$$\langle c, e_2 \rangle = \langle c \vee \ldots \vee c, d_1 \vee \ldots \vee d_n \rangle \in \Theta(c, d_1, \ldots, d_n),$$

i.e.

$$\Theta(c, e_1, e_2) = \Theta(c, e_1) \vee \Theta(c, e_2) \subseteq \Theta(c, d_1, \ldots, d_n) = \Theta(a, b)$$

proving

$$\Theta(a, b) = \Theta(c, e_1, e_2).$$ □

It is well-known that every boolean lattice is regular. However, there exist also non-distributive regular lattices, see e.g.:

Example The lattice L whose diagram is visualized in Fig.1 is regular (it has only three congruences, namely the least ω, the greatest $\iota = L \times L$ and that Θ given by congruence classes $\{0, b, c, x, r\}$, $\{a, p, q, 1\}$, see Fig.1).

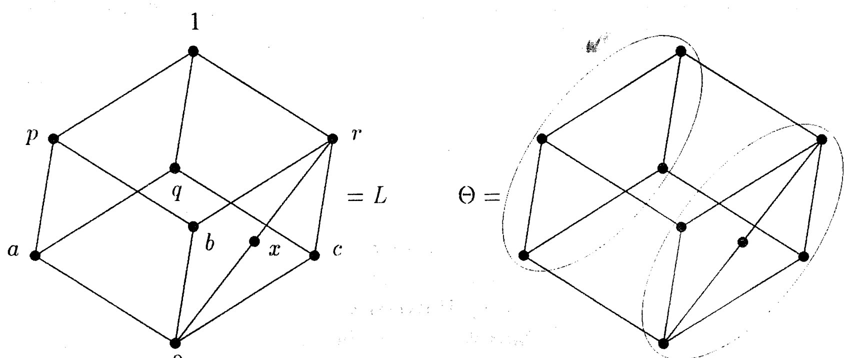

Fig. 1

Theorem 2 *Every relatively complementary lattice L has 1-transferable principal congruences.*

Proof It is well-known that every relatively complementary lattice is regular. By Lemma 2, for each $a, b, c \in L$ there exist $d_1, \dots, d_n$ with

$$\Theta(a,b) = \Theta(c, d_1, \dots, d_n).$$

Let d be a relative complement of c in the interval $[x, y]$, where

$$\begin{aligned} x &= c \wedge d_1 \wedge d_2 \wedge \dots \wedge d_n \\ y &= c \vee d_1 \vee d_2 \vee \dots \vee d_n \end{aligned} \qquad (**)$$

Then $c \vee d = y$, and $c \wedge d = x$.
Moreover, $x \leq c \leq y$, $x \leq d_i \leq y$ for $i = 1, 2, \dots, n$, and $x \leq d \leq y$. Hence

$$\langle c, d_i \rangle \in \Theta(x, y) = \Theta(c, d) \qquad \text{for} \quad i = 1, 2, \dots, n,$$

thus

$$\Theta(c, d_1, \dots, d_n) = \Theta(c, d_1) \vee \dots \vee \Theta(c, d_n) \subseteq \Theta(c, d).$$

By $(**)$, we obtain $\langle x, y \rangle \in \Theta(c, d_1, \dots, d_n)$ proving

$$\Theta(a, b) = \Theta(c, d_1, \dots, d_n) = \Theta(c, d).$$

□

References

[1] Chajda, I.: *Transferable principal congruences and regular algebras,* Math. Slovaca 34 (1984), 97–102.

[2] Mamedov, O.M.:*Characterizations of varieties with n-transferable principal congruences* (Russian), VINITI Akad. Nauk Azerbaid. SR, Inst. Matem. i Mech. (Baku) 1989, 2–12.

Author's address: Department of Algebra and Geometry
Faculty of Science
Palacký University
Tomkova 38, Hejčín
779 00 Olomouc
Czech Republic