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The asymptotic properties of solutions of differential system of the form $g_i(x)y_i' = u_i(y_i) + f_i(x, y_1, \cdots, y_n)$, $i = 1, 2, \cdots, n$ in some neighbourhood of a singular point

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THE ASYMPTOTIC PROPERTIES OF SOLUTIONS OF DIFFERENTIAL SYSTEM OF THE FORM $g_i(x)y_i' = u_i(y_i) + f_i(x, y_1, \ldots, y_n), \quad i = 1, 2, \ldots, n$ IN SOME NEIGHBOURHOOD OF A SINGULAR POINT

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Abstract

The paper deals with asymptotic properties of solutions of non-linear differential system in some neighbourhood of the singular point. The paper contains sufficient conditions for existence of a solution which enter the singular point.

Key words: non-linear differential system, singular point, point of exit (strict exit, input, strict input), scalar product, integral curve, normal vector, directional field, curve without contact.

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1 Introduction

In the present paper we shall consider asymptotic properties of solutions of non-linear differential systems of the form

$$g_i(x)y_i' = u_i(y_i) + f_i(x, y_1, \ldots, y_n), \qquad i = 1, 2, \ldots, n \tag{1.1}$$

in some neighbourhood of the singular point $O = (0, 0, \ldots, 0)$ of the system (1.1). We intend to establish sufficient conditions for the functions $g_i(x)$, $u_i(y_i)$ and $f_i(x, y_1, \ldots, y_n)$, $i = 1, 2, \ldots, n$ that there exists on some interval $(0, \delta)$ at least one continuously differentiable solution $Y(x) = [y_1(x), y_2(x), \ldots, y_n(x)]$

such that $\lim_{x\to 0+} y_i(x) = 0$. This is done in Section 3. The proof is based on the topological method of Ważewski, which is described e.g. in [2]. This method was used for example in [5], where is considered a system $y' = f(x,y)$ with the restriction $f(x,y) > 0$, what is not required in the present paper.

The following notations will be used. R^n denotes the n-dimensional Euclidean space, $C^m(I)$ denotes the space of m-times differentiable real functions on an interval I. Let Ω be an open region, by Ω_e (Ω_{se}), Ω_i (Ω_{si}) we denote the points of exit (strict exit) from Ω and input (strict input) to Ω (see e.g.[2]). If $\vec{N}$ and $\vec{T}$ are vectors, $(\vec{N},\vec{T})$ means their scalar product. By $pro_x\vec{N}$, $pro_y\vec{N}$ we denote a projection of vector $\vec{N}$ on the x-axis and on the y-axis.

2 Auxiliary results

In this section we state theorem on existence of a function given implicitly by the equation

$$(F(x,y)\equiv)\ \Phi(x)+\Psi(y)=0 \tag{2.1}$$

on some domain $D = I_x \times I_y$, where $I_x = (x_0, x_0+\Delta_1)$, $I_y = (y_0, y_0+\Delta_2)$, $0 < \Delta_k$ =const., $k = 1,2$ and on existence its derivation. Because the proofs are similar to the proofs of Theorems I., II. from [1, pp. 447–453] we only state the results.

Theorem 2.1 *Suppose*

(1) $F: D \to R$ *is continuous;*

(2) F'_x, F'_y *exist and are continuous on* D*;*

(3) $\lim_{\substack{x\to x_0^+\\ y\to y_0^+}} F(x,y) = 0$*;*

(4) $\forall\ \varepsilon_1,\varepsilon_2 > 0$ *is* $F'_x(x_0+\varepsilon_1, y_0+\varepsilon_2)\cdot F'_y(x_0+\varepsilon_1,y_0+\varepsilon_2) < 0$ *on* D.

Then

(a) there is determined uniquely on some domain $\tilde{D}=\tilde{I}_x\times I_y$*, where* $\tilde{I}=(x_0,x_0+\delta)$, $0<\delta<\Delta_1$*, by (2.1) a function* $f:\tilde{I}_x\to R$*;*

(b) $\lim_{x\to x_0^+} f(x) = y_0$*;*

(c) f *is continuous;*

(d) f *is monotonic and has a continuous derivative:* $f'(x) = -F'_x\cdot(F'_y)^{-1}$.

3 Main results

Let us consider systems (1.1) on some domain $Q = I_x \times I_{y_1} \times \ldots \times I_{y_n}$, where $I_x = (0, x_0)$, $I_{y_i} = (0, y_i^{(0)})$, $i = 1, 2, \ldots, n$, $x_0, y_1^{(0)}, \ldots y_n^{(0)}$ are positive constants, and the following conditions are assumed to hold without further mention:

(3.2) $g_i \in C^2(I_x)$, $u_i \in C^2(I_{y_i})$, $g_i(x) > 0$, $u_i(y_i) > 0$, $i = 1, 2, \ldots, n$;

(3.3) $f_i \in C^1(Q)$, $i = 1, 2, \ldots, n$;

(3.4) $\lim_{x \to 0+} g_i'(x) = 0$, $\lim_{y_i \to 0+} u_i'(y_i) = 0$, $i = 1, 2, \ldots, n$;

(3.5) there exists (finite or infinite) limit

$$\lim_{\substack{x \to 0+ \\ y \to 0+}} \frac{g_i''(x)}{u_i''(y_i)}, \qquad i = 1, 2, \ldots, n;$$

We remark that $O = (0, 0, ..., 0)$ is a point of the boundary Q and as it can be seen there are no conditions for this one. We shall see that there exists the integral curve of (1.1) in a sufficiently small neighbourhood of the origin which enters this point.

Definition 3.1 A curve $y_i = \varphi_i(x)$, $i = 1, 2, \ldots, n$ is said to be a curve without contact in view of the integral curves of (1.1) if all points

$$(x, \varphi_1(x), \ldots, \varphi_n(x)) \in Q$$

are points of strict exit (or strict input).

Theorem 3.1 *Suppose*

(1) $g_i'(x) > 0$, $g_i''(x) > 0$ *on* I_x;

(2) $u_i'(y_i) < 0$, $u_i''(y_i) < 0$, $(u_i'(y_i) > 0$, $u_i''(y_i) > 0)$ *on* I_{y_i};

(3) there exist

$$\lim_{x \to 0+} g_i''(x) g_i(x) = M_i > 0, \qquad \lim_{y_i \to 0+} u_i''(y_i) u_i(y_i) = N_i < 0 \ (> 0);$$

$i = 1, 2, \ldots, n$.

Then there exists on some interval $(0, \delta_0)$ *at least one continuously differentiable solution* $Y(x) = [y_1(x), y_2(x), \ldots, y_n(x)]$ *such that* $\lim_{x \to 0+} y_i(x) = 0$, $i = 1, 2, \ldots, n$.

Proof It follows from (3.2) and (3.3) that trough the each point of Q there goes only one integral curve which is determined in a sufficiently small neighbourhood of the initial point.

First, we want to establish, that there exist curves without contact in view of the projection of the integral curves on the xy_i-planes, $i = 1, 2, \ldots, n$. We prove, that there are curves

$$(3.7) \qquad (F_{ki}(x, y_i) \equiv)\ A_k g_i'(x) + u_i'(y_i) = 0, \quad k = 1, 2 \quad i = 1, 2, \ldots, n,$$

where $A_1 = b - \delta$, $A_2 = b + \delta$, $0 < \delta < b$.

From (3.2) and (3.7) it is clear that the supposition (1) of Theorem 2.1 holds for all functions F_{ki}, $k = 1, 2$, $i = 1, 2, \ldots, n$.

Let us calculate partial derivatives of functions F_{ki} :

$$(3.8) \qquad \frac{\partial F_{ki}}{\partial x} = A_k g_i''(x) \qquad \frac{\partial F_{ki}}{\partial y_i} = u_i''(y_i) \qquad k = 1, 2 \quad i = 1, 2, \ldots, n.$$

Hence and from (3.2) and the from assumptions of the theorem it follows that the supposition (2) and (4) of Theorem 2.1 are held too. From (3.4) we have $\lim_{x\to 0+, y_i \to 0+} F_{ki}(x, y_i) = 0$ which shows that the supposition (3) of Theorem 2.1 is held, too.

Now we may conclude that equations (3.7) determine for all $i = 1, 2, \ldots, n$ exactly two monotonic, continuously differentiable functions $y_i = \varphi_{ki}(x)$, $x \in (0, \delta_1)$, where $\delta_1 < x_0$.

From (d) in view of (3.8) we have

$$\varphi_{ki}'(x) = -\frac{A_k g_i''(x)}{u_i''(y_i)}, \quad k = 1, 2 \quad i = 1, 2, \ldots, n;$$

hence and from the assumptions of the theorem it follows that functions φ_{ki} are increasing, and for all $i = 1, 2, \ldots, n$ is function $y_i = \varphi_{2i}(x)$ increasing quicker than function $y_i = \varphi_{1i}(x)$ ($A_2 > A_1 > 0$). From (3.5) it follows existence of derivative from the right of functions φ_{ki} in the point 0, too (fig.1).

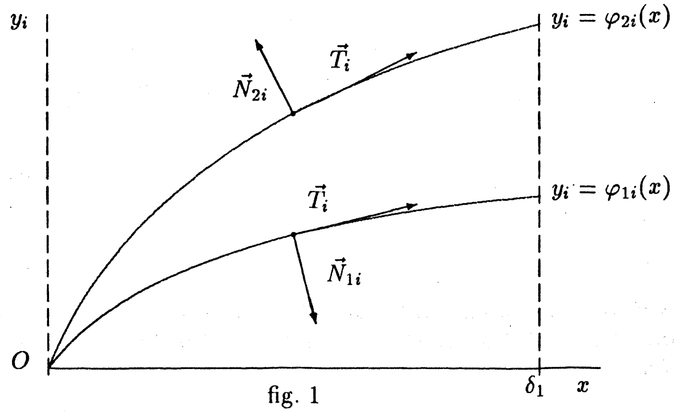

fig. 1

Our task is, further, to show that the curves $y_i = \varphi_{ki}(x)$, $k = 1, 2$, $i = 1, 2, \dots, n$ are curves without contact in view of the projection of the integral curves of (1.1) on xy_i-planes for all $i = 1, 2, \dots, n$.

We find, therefore, the scalar product $(\vec{N}_{ki}, \vec{T}_i)$ where $\vec{N}_{ki}$ is the normal vector of φ_{ki} and $\vec{T}_i$ is the projection of the directional field on xy_i-plane, and we show that this one is unequal to zero.

Using the relation for calculation a normal vector [1, p.524] and (3.8) we obtain

$$\begin{aligned} \vec{N}_{1i} &= \{-(\delta - b)g_i''(x),\ u_i''(y_i)\}, \\ \vec{N}_{2i} &= \{-(\delta + b)g_i''(x),\ -u_i''(y_i)\}, \end{aligned} \qquad i = 1, 2, \dots, n.$$

Because

$pro_{O_x}\vec{N}_{1i} > 0$, $pro_{O_{y_i}}\vec{N}_{1i} < 0$, $pro_{O_x}\vec{N}_{2i} < 0$, $pro_{O_{y_i}}\vec{N}_{2i} > 0$, $i = 1, 2, \dots, n$, direction of the vectors $\vec{N}_{ki}$ is as we can see on fig.1.

First we find the scalar product $(\vec{N}_{ki}, \vec{T}_i)$ for $k = 1$. Instead of the vector $\vec{T}_i$ we use the vector $g_i(x)\vec{T}_i$, $i = 1, 2, \dots, n$, to make calculation more simple with no influence on sign of the one. We have

$$(\vec{N}_{1i}, g_i(x)\vec{T}_i) = (b - \delta)g_i''(x)g_i(x) + u_i''(y_i)u_i(y_i) + u_i''(y_i) \cdot f_i(x, y_1, \dots, y_n).$$

then for $x \to 0^+$, $y_i \to 0^+$, $i = 1, 2, \dots, n$ and using (3.6) and suppose (3) of the theorem, the asymptotic equality is valid:

$$(\vec{N}_{1i}, g_i(x)\vec{T}_i) \approx -(\delta - b)M_i + N_i.$$

If we denote $-\frac{N_i}{M_i} = a_i > 0$, $i = 1, 2, \dots, n$, then we obtain

$$(\vec{N}_{1i}, g_i(x)\vec{T}_i) \approx -M_i[\delta - b + a_i] = -M_i[\delta - (b - a_i)], \qquad i = 1, 2, \dots, n.$$

If we require that $b > a_i$, $b - a_i < \delta < b$, which is possible to take, then the scalar product is negative.

In the case of $k = 2$ we obtain analogously

$$(\vec{N}_{2i}, g_i(x)\vec{T}_i) \approx -(b + \delta)M_i - N_i = -M_i[\delta + (b - a_i)], \qquad i = 1, 2, \dots, n,$$

which is negative too.

So as in the both cases we have

$$(\vec{N}_{ki}, g_i(x)\vec{T}_i) < 0 \qquad k = 1, 2,\ i = 1, 2, \dots, n,$$

which denotes that angle of these vectors is obtuse. Because $pro_{O_x}\vec{T}_i > 0$, $k = 1, 2$, $i = 1, 2, \dots, n$, direction of the vector $\vec{T}_i$ is as we can see on fig.1. Let us denote $\Omega_i = \{(x, y) \in R^2 / 0 < x < \delta_0,\ \varphi_{1i}(x) < y_i < \varphi_{2i}(x)\}$, $i = 1, 2, \dots, n$, where $\delta_0 = \frac{\delta_1}{2}$. Now it is easy to see that the curves $y_i = \varphi_{2i}(x)$, $k = 1, 2$, $i = 1, 2, \dots, n$, are curves without contact in view of the projection of the integral curves of (1.1) on xy_i-planes for all $i = 1, 2, \dots, n$, at which all points of these curves are points of strict input to Ω_i.

The cartesian product of Ω_i, $i = 1, 2, \dots, n$ we denote

$$\Omega^0 = \{(x, y_1, \dots, y_n) \in R^{n+1}/0 < x < \delta_0,\ \varphi_{1i}(x) < y_i < \varphi_{2i}(x),\ i = 1, 2, \dots, n\}$$

with boundary

$$\partial\Omega^0 = \bigcup_{j=0}^{n} \{(x, y_1, \dots, y_n) \in R^{n+1}/0 \le x \le \delta_0,\ \ y_j = \varphi_{1j}(x) \text{ or } \varphi_{2j}(x),$$
$$\varphi_{1i}(x) \le y_i \le \varphi_{2i}(x),\ \ i = 1, 2, \dots, j-1, j+1, \dots, n\}$$

From previous reasoning it is obviously that all points of $\partial\Omega^0$ are points of strict input to Ω^0.

Further we make use of the topological method of Ważewski. It is easy to prove [2, Theorem 2.1, p.333] in the case if all points of $\partial\Omega^0$ are points of strict input to Ω^0. It is obviously that $S \cap \partial\Omega^0$ is not a retract of S and is a retract of $\partial\Omega^0$, where

$$S = \{(x, y_1, \dots, y_n) \in R^{n+1}/x = \delta_0,\ \varphi_{1i}(x) \le y_i \le \varphi_{2i}(x),\ i = 1, 2, \dots, n\}.$$

The conclusion now follows.

Example 3.1 It is easy to verify that functions

$$g_i(x) = x^2 + c_i,\ c_i > 0,\ \ u_i(y_i) = k_i \frac{\sin y_i}{y_i},\ \left(u_i(y_i) = k_i \frac{\mathrm{tg} y_i}{y_i}\right),\ k_i > 0,$$

$i = 1, 2, \dots, n$,
satisfy the supposition of Theorem 3.1.

Remark 3.1 The conclusion of Theorem 3.1 remains valid in the case when

(1) $g_i'(x) < 0,\ g_i''(x) < 0$ *on* I_x;

(2) $u_i'(y_i) < 0, u_i''(y_i) < 0,\ (u_i'(y_i) > 0, u_i''(y_i) > 0)$ *on* I_{y_i};

(3) there exist

$$\lim_{x \to 0+} g_i''(x) g_i(x) = M_i < 0,\ \ \lim_{y_i \to 0+} u_i''(y_i) u_i(y_i) = N_i < 0\ (> 0);$$

$i = 1, 2, \dots, n$.

Now we shall consider systems (1.1) on Q and further we shall assume that the conditions (3.2), (3.3) are held and following conditions too:

$$(3.4a) \qquad \lim_{x\to 0+} \frac{1}{g_i'(x)} = 0, \quad \lim_{y_i\to 0+} \frac{1}{u_i'(y_i)} = 0, \quad i = 1, 2, \ldots, n;$$

there exists (finite or infinite) limit

$$(3.5a) \qquad \lim_{\substack{x\to 0^+ \\ y\to 0^+}} \frac{g_i''(x)[u_i'(y_i)]^2}{[g_i'(x)]^2 u_i''(y_i)}, \qquad i = 1, 2, \ldots, n;$$

$$(3.6a) \qquad f_i(x, y_1, \ldots, y_n) = o\left(\frac{[u_i'(y_i)]^2}{u_i''(y_i)}\right), \quad y_i \to 0^+, \qquad i = 1, 2, \ldots, n.$$

This problem in the scalar case was studied in [4]. Analogously the results may be proved in the vector case. Because the proofs are similar to the proof of Theorem 3.1 we shall not accomplish it.

Theorem 3.2 *Suppose*

(1) $g_i'(x) > 0,\ g_i''(x) < 0$ *on* I_x;

(2) $u_i'(y_i) < 0, u_i''(y_i) > 0,\ (u_i'(y_i) > 0, u_i''(y_i) < 0)$ *on* I_{y_i};

(3) there exist

$$\lim_{x\to 0+} \frac{g_i''(x) g_i(x)}{[g_i'(x)]^2} = M_i < 0, \quad \lim_{y_i\to 0+} \frac{u_i''(y_i) u_i(y_i)}{[u_i'(y_i)]^2} = N_i > 0\ (< 0);$$

$i = 1, 2, \ldots, n.$

Then there exists on some interval $(0, \delta_0)$ *at least one continuously differentiable solution* $Y(x) = [y_1(x), y_2(x), \ldots, y_n(x)]$ *such that* $\lim_{x\to 0+} y_i(x) = 0$, $i = 1, 2, \ldots, n$.

Example 3.2 It is easy to verify that functions

$$g_i(x) = x^{\alpha_i}, \qquad u_i(y_i) = y_i^{-\beta_i} \exp\left(k_i y_i^{-\nu_i}\right),$$

where $\alpha_i, \beta_i, k_i, \nu_i$ are constants with the restrictions $0 < \alpha_i < 1$, $\beta_i > 0$, $k_i > 0$, $\nu_i > 0$, $i = 1, 2, \ldots, n$, satisfy the suppose of Theorem 3.2.

Remark 3.2 The conclusion of Theorem 3.2 remains valid in the case when

(1) $g_i'(x) < 0,\ g_i''(x) > 0$ *on* I_x;

(2) $u_i'(y_i) < 0, u_i''(y_i) > 0$, $(u_i'(y_i) > 0, u_i''(y_i) < 0)$ *on* I_{y_i};

(3) there exist

$$\lim_{x \to 0+} g_i''(x) g_i(x) = M_i > 0, \quad \lim_{y_i \to 0+} u_i''(y_i) u_i(y_i) = N_i > 0 \ (< 0);$$

$i = 1, 2, \ldots, n$.

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