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A CONTRIBUTION TO THE PHASE THEORY OF A LINEAR SECOND-ORDER DIFFERENTIAL EQUATION IN THE JACOBIAN FORM

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Dedicated to Prof. Dr. O. Borůvka to his 93 birthday

Abstract

A canonical phase function is introduced by O. Borůvka in [1], [2]. It is connected closely with a character of a linear second-order differential equation in the Jacobian form

$$y'' = q(t)y. \qquad (q)$$

In this paper the algebraic structure of the set of phase functions or the set of first phases of oscillatory equations (q) in the interval $(-\infty, \infty)$ is investigated.

We shall deal with the situation when the differential equation (q) is of finite type (m) special in the interval $(-\infty, \infty)$.

Key words: Phase function, canonical phase function, first phase of a linear second-order differential equation in the Jacobian form of finite type special.

MS Classification: 34A30

1 Canonical phase function of a class D_m and phase function of a class D_m.

We shall be concerned with phase functions of a class D_m.

Definition 1 The phase function of a class D_m is a real function α with the following properties:

$$\alpha = \alpha(t), \qquad t \in (-\infty, \infty),$$
$$\alpha \in C_3(-\infty, \infty),$$
$$\alpha'(t) \neq 0$$

and for the numbers

$$c = \lim_{t \to -\infty} \alpha(t), \quad d = \lim_{t \to \infty} \alpha(t) \quad \text{it holds} \quad |c - d| = m\pi,$$

m positive integer.

The number $|c - d|$ is called an oscillation of the phase function α and its notation is $O(\alpha)$. So

$$O(\alpha) = |c - d|.$$

Definition 2 The canonical phase function of a class D_m is a function $X = X(t),\ t \in (-\infty, \infty)$ if, throughout the interval $(-\infty, \infty)$ hold:

$$X \in C_3, \qquad X'(t) > 0$$

and for the numbers

$$C = \lim_{t \to -\infty} X(t), \quad D = \lim_{t \to \infty} X(t) \quad \text{it holds} \quad C = 0,\ D = m\pi,$$

m positive integer.

It is known (see [1], p.206) that the carrier of every differential equation (q) can be defined by means of the canonical phase function X of a class D_m in the interval $j = (-\infty, \infty)$ as follows

$$q(t) = -\{X, t\} - X'^2(t), \tag{1}$$

when $\{X, t\}$ is the Schwarzian derivative of a function X, that is

$$\{X, t\} = \frac{1}{2}\frac{X'''(t)}{X'(t)} - \frac{3}{4}\frac{X''^2(t)}{X'^2(t)}.$$

If the differential equation (q) with the carrier q defined by (1) is of finite type (m) special in the interval $j = (-\infty, \infty)$ then the function X has properties of a canonical phase function of the class D_m.

We shall note the set of all phase functions α of a class D_m by G.

The following assertions are evident:

1. $X \in G$,

2. $\alpha \in G \Rightarrow (\alpha + l) \in G$ for every real number l,

3. $\alpha \in G \Rightarrow \alpha_{(0)} \in G$,
   where the function $\alpha_{(0)}$ is defined by

$$\alpha(t) = \alpha_{(0)}(t) + \kappa,$$

   κ is a suitable real number, and

$$\lim_{t \to -\infty} \alpha_{(0)}(t) = 0, \quad \lim_{t \to \infty} \alpha_{(0)}(t) = m\pi$$

   in the case that α increases on $j = (-\infty, \infty)$, and

$$\lim_{t \to -\infty} \alpha_{(0)}(t) = m\pi, \quad \lim_{t \to \infty} \alpha_{(0)}(t) = 0$$

   in the case that α decreases on $j = (-\infty, \infty)$.

4. $\alpha, \beta, x \in G, \quad \alpha = \alpha_{(0)} + \kappa, \quad \beta = \beta_{(0)} + \lambda \Rightarrow$ the composite function $\alpha_{(0)} X^{-1} \beta_{(0)} + \kappa + \lambda \in G$, where X^{-1} is the inverse of the canonical phase function X,

5. $\alpha \in G$, $\alpha = \alpha_{(0)} + \kappa \Rightarrow \hat{\alpha} \in G$, where $\hat{\alpha} = X \alpha_{(0)}{}^{-1} X - \kappa$ and $\alpha_{(0)}{}^{-1}$ is the inverse of the function $\alpha_{(0)}$.

Let us note:

We denote the set of all functions $\alpha + l$, where l is a real number, by a symbol $[\alpha]$ and call a complete phase system generated by the phase function α.

To be short we shall write X instead of $X_{(0)}$ even if according to the definition there is $X = X_{(0)}$.

The fact that $\alpha = \alpha_{(0)}(t) + \kappa$ will be written by means of an index: $\alpha = \alpha_{(\kappa)}$. So that $\alpha_{(\kappa)} = \alpha_{(0)} + \kappa$.

2 Group $\mathcal{G}$ of phase functions of a class D_m

Let G be the set of all phase functions of a class D_m; $X \in G$ be a canonical phase function. Let $\alpha = \alpha_{(\kappa)}$, $\beta = \beta_{(\lambda)} \in G$ be any elements.

We introduce a binary operation $\circ$ into G by the following equation

$$\alpha \circ \beta = \alpha_{(0)} X^{-1} \beta_{(0)} + \kappa + \lambda,$$

where X^{-1} is the inverse function to X.

Theorem 1 *The set G with the binary operation $\circ$ form a group.*

Proof The binary operation $\circ$ is associative as for any

$$\alpha = \alpha_{(\kappa)}, \quad \beta = \beta_{(\lambda)}, \quad \gamma = \gamma_{(\mu)} \in G$$

it holds

$$\begin{aligned} \alpha \circ (\beta \circ \gamma) &= \alpha_{(0)} X^{-1}(\beta_{(0)} X^{-1} \gamma_{(0)}) + \kappa + (\lambda + \mu) = \\ &= (\alpha_{(0)} X^{-1} \beta_{(0)}) X^{-1} \gamma_{(0)} + (\kappa + \lambda) + \mu = (\alpha \circ \beta) \circ \gamma. \end{aligned}$$

The canonical phase function X is the unit element as for any element $\alpha = \alpha_{(\kappa)} \in G$ we have

$$\begin{aligned} \alpha \circ X &= \alpha_{(0)} X^{-1} X + \kappa = \alpha_{(0)} + \kappa = \alpha_{(\kappa)} = \alpha, \\ X \circ \alpha &= X X^{-1} \alpha_{(0)} + \kappa = \alpha_{(0)} + \kappa = \alpha_{(\kappa)} = \alpha. \end{aligned}$$

To every element $\alpha \in G$ there is the inverse element $\hat{\alpha} \in G$, where

$$\hat{\alpha} = X \alpha_{(0)}^{-1} X - \kappa$$

as

$$\begin{aligned} \alpha \circ \hat{\alpha} &= \alpha_{(0)} X^{-1} X \alpha_{(0)}^{-1} X + \kappa - \kappa = X, \\ \hat{\alpha} \circ \alpha &= X \alpha_{(0)}^{-1} X X^{-1} \alpha_{(0)} - \kappa + \kappa = X. \end{aligned}$$

Thus G is a group, the group operation is the binary operation $\circ$, the canonical phase function X is the unit element of the group and the elemnt $\hat{\alpha}_{(-\kappa)}$ is the inverse to the element $\alpha_{(\kappa)} \in G$.

Definition 3 The group from the above theorem will be denoted $\mathcal{G}$ and called the group of phase functions of a class D_m.

It is evident that:

The product $\alpha \circ \beta$, $\alpha, \beta \in \mathcal{G}$, is an increasing (decreasing) phase function if both phase functions increase or decrease (one of them increases and the other decreases).

The inverse element $\hat{\alpha}$ corresponding to any element $\alpha \in \mathcal{G}$ represents an increasing (decreasing) phase function according as α is an increasing (decreasing) phase function.

Theorem 2 *Be $\mathcal{N}$ a set of all increasing phase functions of a class D_m. Then $\mathcal{N}$ is a normal divisor of the group $\mathcal{G}$.*

Proof It holds $\hat{\alpha} \circ \mathcal{N} \circ \alpha = \mathcal{N}$, $\alpha \in \mathcal{G}$.

Theorem 3 *The factor group $\mathcal{G}/\mathcal{N}$ consists of two elements, namely $\mathcal{N}$ and the class $\mathcal{A}$ of all decreasing phase functions of a class D_m.*

3 Differential equations (q) of finite type (m) special and their first phase.

We shall consider now a linear second-order differential equation of Jacobian form

$$y'' = q(t)y, \tag{q}$$

where $q \in C_0(-\infty, \infty)$.

The coefficient q of this differential equation is called a carrier of the differential equation (q).

Definition 4 The differential equation (q) is called of finite type (m) special in the interval $j = (-\infty, \infty)$ if it possesses solutions with m zeros but none with $(m+1)$ zeros and if there is a linearly independent solution with $(m-1)$ zeros.

Let u, v be independent solutions of a linear differential equation (q) in the interval $j = (-\infty, \infty)$, which form a basis (u, v) of a set of all solutions of the differential equation (q).

Definition 5 The function α ($\alpha \in C_3,\ \alpha'(t) \neq 0$) defined in the interval $j = (-\infty, \infty)$ by

$$\operatorname{tg}\, \alpha(t) = \frac{u(t)}{v(t)}, \tag{2}$$

with the exception of singularities on both sides is called the first phase of the base (u,v) of the differential equation (q).

Theorem 4 *Any phase function of a class D_m is the first phase of the differential equation (q) of finite type m special in $j = (-\infty, \infty)$ and the other way round.*

Proof To every phase function

$$\alpha \in G \quad (\alpha \in C_3,\ \alpha'(t) \neq 0,\ O(\alpha) = m\pi)$$

we associate the carrier q of the differential equation (q) by the follownig way

$$q(t) = -\{\alpha, t\} - \alpha'^2(t), \tag{3}$$

where the symbol { } notes Schwarzian derivative of the function α, that is

$$\{\alpha, t\} = \frac{1}{2}\frac{\alpha'''(t)}{\alpha'(t)} - \frac{3}{4}\frac{\alpha''^2(t)}{\alpha'^2(t)},$$

and the base (u, v) of the set of all solutions of the differential equaion (q), given by the formulas

$$u = \frac{1}{\sqrt{|\alpha'(t)|}} \sin \alpha(t), \qquad v = \frac{1}{\sqrt{|\alpha'(t)|}} \cos \alpha(t).$$

Thus the general solution of the differential equation (q) with the carrier q given by (3) is of the form

$$y = (c_1 \sin \alpha(t) + c_2 \cos \alpha(t))/\sqrt{|\alpha'(t)|} = k \sin(\alpha(t) + l)/\sqrt{|\alpha'(t)|},$$

where c_1, c_2 are real numbers and numbers k, l are given by equations

$$c_1 = k \cos l, \qquad c_2 = k \sin l.$$

As $O(\alpha) = m\pi$, it is also $O(\alpha(t) + l) = m\pi$ and every particular solution (contained in y) has m zeros in the interval $j = (-\infty, \infty)$ with the exception of the case that $\lim_{t \to -\infty}(\alpha(t) + l)$ is an integer multiple of the number π when solutions y (linearly dependent) have $(m-1)$ zeros.

So the differential equation (q) is of finite type (m) special in the interval $j = (-\infty, \infty)$. We form a quotient $\frac{u}{v}$ and we have

$$\frac{u(t)}{v(t)} = \operatorname{tg} \alpha(t)$$

and we can see that α is the first phase of the base (u, v).

On the contrary, let the differential equation (q) be of finite type (m) special in the interval $j = (-\infty, \infty)$, (u, v) be any base of (q) and α be the first phase of the base (u, v) of the set of all solutions of (q). Then the phase α satisfies (1).

From the equality (1) follows that

$$u = \rho(t) \sin \alpha(t), \qquad v = \rho(t) \cos \alpha(t).$$

The functions u, v form a base of solutions of the differential equation (q) if and only if

$$\rho(t) = \frac{1}{\sqrt{|\alpha'(t)|}}.$$

Then the general solution of the differential equation (q) is of the form

$$y = k \sin(\alpha(t) + l)/\sqrt{|\alpha'(t)|}.$$

It has to be

$$O(\alpha(t) + l) = O(\alpha(t)) = m\pi$$

for particular solutions obtained in y to have m zeros resp. for one indenpendent solution to have $(m-1)$ zeros.

Deriving the equality (2), we get

$$\alpha'(t)/\cos^2 \alpha(t) = -w/v^2(t),$$

where $w = uv' - u'v$ is the Wronskian of solutions u, v.

We have $\alpha'(t) > 0$ resp. $\alpha'(t) < 0$ if and only if $w < 0$ resp. $w > 0$, thus $\alpha'(t) \neq 0$ in the interval $j = (-\infty, \infty)$. As u, v are solutions of (q), $u, v \in C_2$ and $\alpha \in C_3$.

Every first phase of the differential equation (q) is a phase function of a class D_m.

4 Equivalence in the group $\mathcal{G}$

We introduce an equivalence relation into the group $\mathcal{G}$ now which we denote by a symbol $\sim$.

Definition 6 Two phase functions $\alpha, \gamma \in \mathcal{G}$ are equivalent in $\mathcal{G}$ and we write $\alpha \sim \gamma$ if the following equality holds in $j = (-\infty, \infty)$

$$\operatorname{tg} \gamma(t) = \frac{c_{11} \operatorname{tg} \alpha(t) + c_{12}}{c_{21} \operatorname{tg} \alpha(t) + c_{22}}, \tag{4}$$

where c_{ij} are real numbers, det $|c_{ij}| \neq 0$, $i = 1, 2$ with the exception of singularities of functions $\operatorname{tg} \alpha(t)$, $\operatorname{tg} \gamma(t)$. It is easy to see that the relation determined by (4) in the set G of all phase functions of a class D_m is reflexive ($\alpha \sim \alpha$), symmetric ($\alpha \sim \gamma \Rightarrow \gamma \sim \alpha$), and transitive ($\alpha \sim \beta, \beta \sim \gamma \Rightarrow \alpha \sim \gamma$), and consequently is an equivalent relation. There is a decomposition $\bar{G}$ of the set G onto classes of equivalent elements with respect to the relation $\sim$.

Theorem 5 *Every two equivalent phase functions of a class D_m determine the same carrier of the differential equation (q) and on the contrary every two first phase functions of the differential equations (q) are equivalent.*

Proof Let α, γ be two phase functions of a class D_m, let $\alpha \sim \gamma$. That means α, γ lie in the same element $\bar{a} \in \bar{G}$ and (4) holds. Let us denote q resp. p the carier given by (3) with the help of the phase function α resp. γ. Then we get

$$\begin{aligned} p(t) &= -\{\gamma, t\} - \gamma'^2(t) = \{\operatorname{tg} \gamma(t), t\} = -\left\{ \frac{c_{11} \operatorname{tg} \alpha(t) + c_{12}}{c_{21} \operatorname{tg} \alpha(t) + c_{22}}, t \right\} = \\ &= \{\operatorname{tg} \alpha(t), t\} = -\{\alpha, t\} - \alpha'^2(t) = q(t). \end{aligned}$$

Thus

$$p(t) = q(t) \quad \text{for} \quad t \in j = (-\infty, \infty).$$

On the contrary, we know [1] that for two first phases α, γ of the differential equation (q), $t \in j$, it holds (4). It yields an equivalence of first phases α, γ and the fact that they belong to the same element of the partition $\bar{a} \in \bar{G}$.

We can see that every element of the partition $\bar{a} \in \bar{G}$ consists of the first phases of just one carrrier $q(t)$. We get one-to-one mapping $\mathcal{A}$ of the partition $\bar{G}$ onto the set of differential equations (q) of finite type (m) special in the interval $j = (-\infty, \infty)$.

5 The fundamental subgroup $\mathcal{E}$

We shall deal with an algebraic structure of the partition $\bar{G}$ now.

Let us consider such an element $\mathcal{E} \in \bar{G}$ in which the unit element X of the group $\mathcal{G}$ lies. This element $\mathcal{E}$ of the partition $\bar{G}$ consist only of phase functions ζ equivalent to X that is of the phase functions which satisfy

$$\operatorname{tg} \zeta(t) = \frac{c_{11} \operatorname{tg} X(t) + c_{12}}{c_{21} \operatorname{tg} X(t) + c_{22}}, \tag{5}$$

$\det |c_{ij}| \neq 0$, $i, j = 1, 2$ on the interval $j = (-\infty, \infty)$.

Theorem 6 *The element $\mathcal{E} \in \bar{G}$ in which the canonical phase function X lies is a subgroup of the group $\mathcal{G}$.*

Proof We now show that when the phase functions $\xi, \eta \in \mathcal{E}$ then also $\xi \circ \eta \in \mathcal{E}$ and when $\xi \in \mathcal{E}$ then also the inverse phase function $\hat{\xi} \in \mathcal{E}$.

Let $\zeta \in \mathcal{E}$. Then in view of (5) we have for suitable c_{ij}, $det|c_{ij}| \neq 0$, $i, j = 1, 2$

$$\operatorname{tg} \zeta = \frac{c_{11} \operatorname{tg} X + c_{12}}{c_{21} \operatorname{tg} X + c_{22}}.$$

Since for every real number l

$$\operatorname{tg} (\zeta + l) = \\ = \frac{\sin(\zeta + l)}{\cos(\zeta + l)} = \frac{\sin\zeta\cos l + \cos\zeta\sin l}{\cos\zeta\cos l - \sin\zeta\sin l} = \frac{\cos l \operatorname{tg} \zeta + \sin l}{\sin l \operatorname{tg} \zeta + \cos l}$$

we can see that $\zeta \sim \zeta + l$ and as $\zeta \sim X$ it yields $\zeta + l \sim X$ and thus

$$(\zeta(t) + l) \in \mathcal{E}.$$

Let $\xi, \eta \in \mathcal{G}$ and $\xi = \xi_{(\kappa)}$, $\eta = \eta_{(\lambda)}$. Then also $\xi_{(0)}, \eta_{(0)} \in \mathcal{G}$ and we have

$$\xi_{(0)} \sim X, \qquad \eta_{(0)} \sim X.$$

We have

$$\operatorname{tg} \xi_{(0)} = \frac{a_{11} \operatorname{tg} X + a_{12}}{a_{21} \operatorname{tg} X + a_{22}}, \qquad \operatorname{tg} \eta_{(0)} = \frac{b_{11} \operatorname{tg} X + b_{12}}{b_{21} \operatorname{tg} X + b_{22}} \tag{6}$$

where $\det |a_{ij}| \neq 0$, $\det |b_{ij}| \neq 0$, $t \in j = (-\infty, \infty)$.

If we replace t in formula (6) by the function $X^{-1}\eta_{(0)}$, we get

$$\operatorname{tg} (\xi_{(0)} \circ \eta_{(0)}) \equiv) \operatorname{tg} \xi_{(0)} X^{-1}\eta_{(0)} = \frac{a_{11} \operatorname{tg} XX^{-1}\eta_{(0)} + a_{12}}{a_{21} \operatorname{tg} XX^{-1}\eta_{(0)} + a_{22}} =$$

$$= \frac{a_{11}\frac{b_{11} \operatorname{tg} X + b_{12}}{b_{21} \operatorname{tg} X + b_{22}} + a_{12}}{a_{21}\frac{b_{11} \operatorname{tg} X + b_{12}}{b_{21} \operatorname{tg} X + b_{22}} + a_{22}} = \frac{(a_{11}b_{11} + a_{12}b_{21}) \operatorname{tg} X + (a_{11}b_{12} + a_{12}b_{22})}{(a_{21}b_{11} + a_{22}b_{21}) \operatorname{tg} X + (a_{21}b_{12} + a_{22}b_{22})}.$$

Thus

$$\xi_{(0)} \circ \eta_{(0)} \sim X$$

and also

$$\xi_{(0)} \circ \eta_{(0)} + \kappa + \lambda = \xi_{(\kappa)} \circ \eta_{(\lambda)} \sim X$$

or

$$\xi \circ \eta \in \mathcal{E}.$$

If we replace t by a composite function $\xi_{(0)}^{-1}X$ in the first equality of (6) we get

$$\text{tg } X \equiv) \text{ tg } \xi_{(0)}\xi_{(0)}^{-1}X = \frac{a_{11} \text{ tg } X\xi_{(0)}^{-1}X + a_{12}}{a_{21} \text{ tg } X\xi_{(0)}^{-1}X + a_{22}}.$$

From here we have

$$\text{tg } \hat{\xi}_{(0)} \equiv) \text{ tg } X\xi_{(0)}^{-1}X = \frac{-a_{22} \text{ tg } X + a_{12}}{a_{21} \text{ tg } X - a_{11}}$$

and thus

$$\hat{\xi}_{(0)} \sim X \quad \text{and also} \quad \hat{\xi}_{(0)} - \kappa = \hat{\xi}_{(-\kappa)} \sim X \quad \text{or} \quad \hat{\xi} \in \mathcal{E}.$$

We have shown above that $\mathcal{E}$ is a subgroup of the group $\mathcal{G}$.

Theorem 7 *The partition $\bar{G}$ coincides with the right class partition $\mathcal{G}/_r\mathcal{E}$ of the group $\mathcal{G}$ with respect to $\mathcal{E}$.*

Proof Let $\bar{a} \in \bar{G}$ be an arbitrary element and $\alpha \in \bar{a}$ a phase lying in it. We have to show that $\bar{a} = \mathcal{E} \circ \alpha$. For every element $\zeta \in \mathcal{E}$ there holds a formula such as (5) and for $\zeta_{(0)}$ a formula such as the first equality in (6). If, in that, we replace t by the composite function $X^{-1}\alpha_{(0)}$ than we have

$$\text{tg } \zeta_{(0)}X^{-1}\alpha_{(0)} = \frac{a_{11} \text{ tg } \alpha_{(0)} + a_{12}}{a_{21} \text{ tg } \alpha_{(0)} + a_{22}},$$

thus

$$\zeta_{(0)} \circ \alpha_{(0)} \sim \alpha_{(0)} \quad \text{and also} \quad \zeta \circ \alpha \sim \alpha \quad \text{or} \quad \zeta \circ \alpha \in \bar{a}$$

and we have

$$\mathcal{E} \circ \alpha \subset \bar{a}.$$

Moreover, for every element $\gamma \in \bar{a}$ there holds a formula such as (4)

$$\text{tg } \gamma_{(0)} = \frac{c_{11} \text{ tg } \alpha_{(0)} + c_{12}}{c_{21} \text{ tg } \alpha_{(0)} + c_{22}}.$$

If we replace t by the composite function $\alpha_{(0)}^{-1}X$, we get

$$\mathrm{tg}\ (\gamma_{(0)} \circ \hat{\alpha}_{(0)}) \equiv)\ \mathrm{tg}\ \gamma_{(0)}X^{-1}X\alpha_{(0)}^{-1}X = \frac{c_{11}\ \mathrm{tg}\ X + c_{12}}{c_{21}\ \mathrm{tg}\ X + c_{22}}$$

or

$$\gamma_{(0)} \circ \hat{\alpha}_{(0)} \sim X.$$

Hence $\gamma_{(0)} \circ \alpha_{(0)} \in \mathcal{E}$ that is

$$\gamma_{(0)}X^{-1}X\alpha_{(0)}^{-1}X \in \mathcal{E}$$

and from here

$$\gamma_{(0)} \in \mathcal{E}X^{-1}.\alpha_{(0)} = \mathcal{E} \circ \alpha_{(0)}.$$

For equivalent phase functions $\alpha \sim \alpha_{(0)}$, $\gamma \sim \gamma_{(0)}$ thus we have also

$$\gamma \in \mathcal{E} \circ \alpha, \quad \text{and} \quad \bar{a} \subset \mathcal{E} \circ \alpha.$$

We have shown that

$$\bar{a} = \mathcal{E} \circ \alpha.$$

We remark that the mappinng $\mathcal{A}$ maps the fundamental subgroup $\mathcal{E}$ onto the carrier

$$q = -\{\ \mathrm{tg}\ X, t\}.$$

Example An example of a canonical phase function of a class D_m is a function

$$X(t) = m\ \mathrm{arctg}\ t + \frac{m\pi}{2}, \tag{7}$$

$t \in (-\infty, \infty)$, $m \geq 1$ possitive integer.

That is to say

$$\lim_{t\to-\infty}(m\ \mathrm{arctg}\ t + \frac{m\pi}{2}) = 0, \qquad \lim_{t\to\infty}(m\ \mathrm{arctg}\ t + \frac{m\pi}{2}) = m\pi$$

and thus $O(X) = m\pi$ and moreover

$$X'(t) = \frac{m}{1+t^2} > 0.$$

It is easy to calculate with the help of (3) that the carrier q of the differential equation (7) is given by the formula

$$q(t) = -\frac{m^2+1}{(1+t^2)^2}.$$

Thus the differential equation

$$y'' = -\frac{m^2+1}{(1+t^2)^2}y$$

is of finite type (m) special on the interval $j = (-\infty, \infty)$. The basis (u, v) can be formed by functions

$$u = \sqrt{m}(1+t^2)^{\frac{1}{2}} \sin(m \text{ arctg } t + \frac{m\pi}{2}),$$

$$v = \sqrt{m}(1+t^2)^{\frac{1}{2}} \cos(m \text{ arctg } t + \frac{m\pi}{2}).$$

References

[1] Borůvka, O.: *Lineare Differential transformationen 2. Ordnung,* VEB DVW, Berlin 1967.

[2] Borůvka, O.: *Linear Differential Transformations of the Second Order*, The English University Press, London, 1971

[3] Laitoch, M.: *To the Theory of Linear Difference Equations.* Acta UPO, Fac. Rer. Nat., Vol. 79(1985), 11–24.

[4] Tesaříková, E.: *To the Theory of Central Disprsions for the Linear Differential Equations* $y'' = q(t)y$ *of a Finite Type, Special.* Acta UPO, Fac. Rer. Nat., Vol. 88(1987).

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