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METHOD OF LOWER AND UPPER SOLUTIONS FOR A THIRD-ORDER THREE-POINT REGULAR BOUNDARY VALUE PROBLEM

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Abstract. This paper is concerned with the existence of solutions of the problem

$$u''' = f(t,u,u',u'')$$

$$u'(0) = u'(1) = u(\eta) = 0, \quad 0 \le \eta \le 1 .$$

The method of lower and upper solutions is used here.

Key words: Boundary value problems, lower and upper solutions, a priori bounds.

MS Classification : 34B10

1. Introduction. In this paper we are concerned with the existence of solutions of the boundary value problem (BVP)

$$u'''=f(t,u,u',u'') \tag{1.1}$$

$$u'(0)=u'(1)=u(\eta)=0, \quad 0\le\eta\le1, \tag{1.2}$$

where f satisfies the local Carathéodory conditions on $(0,1)\times R^3$. This problem is regular in the sense that the associated linear problem has only the trivial solution. This problem models the static deflection of a three-layered elastic beam. In [18] there is proved an existence result for BVP (1.1), (1.2) without requiring a growth condition on the whole interval and some uniqueness theorems are given there to.

Multi-point BVPs for differential equations of the n-th order have been studied by many authors (see References). For $n\geq 2$ and $2\leq k\leq n$,the question of existence and unqueness of solutions of k-point BVPs Cauchy-Nicoletti , de la Valeé-Poussin or similar ones, in which the values of a solution or the values of its derivatives are given, have been solved e.g. in [10,11, 12-15].

We consider equation (1.1) with three-point boundary conditions. In this case the Valeé-Poussin conditions have the form

$$u(a)=A, u(c)=C, u(b)=B \quad , \tag{1.3}$$

where $-\infty<a<c<b<+\infty$, $A,B,C\in R$.
BVP (1.1), (1.3) has been investigated e.g. in [1,2,5,19]. Replacing function values by its derivatives, we obtain

$$u'(a)=A \ , \ u(c)=C \ , \ u'(b)=B. \tag{1.4}$$

In [4] , the subfunction method is used for the existence of solutions of BVP (1.1), (1.4) and in [16] , the necessary and sufficient conditions for solvability of this problem are proved by means of lower and upper functions.

BVP (1.1) ,

$$u(c)=0 \ , \ u'(a)=u'(b) \ , \ u''(a)=u''(b) \tag{1.5}$$

where $-\infty<a\leq c\leq b<+\infty$, has been investigated in [17] by a method very similar to the method used in this paper.

C.P.Gupta [7] studied the questions of the existence and uniqueness of solutions of the equation

$$-u'''-\pi^2u+g(x,u,u',u'')=e(x) \tag{1.6}$$

or

$$u'''+\pi^2u+g(x,u,u',u'')=e(x) \tag{1.7}$$

satisfying (1.2). The existence of a solution for the resonance problem (1.6),(1.2) was obtained when e was a Lebesgue-intgrable function with $\int_0^1 e(x)\sin\pi x dx=0$ and g was a Carathéodory function, bounded on $[0,1]\times B^2\times R$ (for every bounded B of R) and

$$g(x,u,v,w)v\geq 0, \quad \text{for } x\in[0,1] \ , \ u,v,w\in R.$$

For the existence of a solution for (1.7), (1.2) g, in adition,

$$\lim_{v\to\infty} \sup \frac{g(x,u,v,w)}{v} = \beta<3\pi^2 \ .$$

These results were proved by means of the method using second-order integro-differential BVPs and the Leray-Schauder continuation theorem.

In contrast to this, here we defined lower and upper solutions for (1.1), (1.2) directly not transforming the BVP on to an integro-diferential problem.

2. Notations and definitions.

In what follows we suppose that $p,q\in[1,+\infty)$, where $1/p+1/q=1$, X is the set of all real functions with one real argument ,
$C^m(a,b)=\{f\in X:\ f^{(m)}$ is continuous on $[a,b]\}$, $m\in N$,
$L^p(a,b)=\{f\in X: |f|^p$ is Lebesgue integrable on $(a,b)\}$ with a norm
$\|f\|_{L^p(a,b)}=(\int_a^b|f(t)|^p\,dt)^{1/p}$ for $p<+\infty$,
$L^\infty(a,b)=\{f\in X:\ \operatorname{ess\,sup}_{a<t<b}|f(t)|<+\infty\}$, with a norm
$\|f\|_{L^\infty(a,b)}=\operatorname{ess\,sup}_{a<t<b}|f(t)|$,
$AC^m(a,b)=\{f\in X:\ f^{(m)}$ is absolutely continuous on $[a,b]\}$.

We say that some property is satisfied on D (resp. D′), if it is satisfied for a.e. $t\in(0,1)$ (resp. $t\in(a,b)$) and for each $x,y,z\in R$.

Let $s_1,s_2\in C^0(0,1)$, $s_1(t)\le s_2(t)$ on $[0,1]$ and S_1, S_2 be such that $S_1'(t)=s_1(t)$, $S_2'(t)=s_2(t)$ on $(0,1)$ and $S_1(\eta)=S_2(\eta)=0$. Then we say that some property is satisfied on $D(s_1,s_2)$, if it is satisfied for a.e $t\in(0,1)$ and for each $x,y,z\in R$, where $|z|\ge1$, $s_1(t)\le y\le s_2(t)$, $\min\{S_1(t),S_2(t)\}\le x\le\max\{S_1(t),S_2(t)\}$.

Let $D'=((a,b)\times R^3)$. We say that $f:D'\to R$ satisfies the local Carathéodory conditions on D′ ($f\in Car_{loc}(D')$), if
$f(.,x,y,z):\ (a,b)\to R$ is measurable on (a,b) for each $x,y,z\in R$,
$f(t,.,.,.):\ R^3\to R$ is continuous for a.e. $t\in(a\ b)$
and $\sup\{|f(t,x,y,z)|:|x|+|y|+|z|\le\rho\}\in L^1(a,b)$ for any $\rho\in(0,+\infty)$.

A function $u\in AC^2(0,1)$ satisfying (1.1) for a.e. $t\in(0,1)$ and fulfilling (1.2), will be called a solution of BVP (1.1), (1.2).

Functions $\sigma_1,\sigma_2\in AC^2(0,1)$ satisfying

$$\sigma_1'''\ge f(t,x,\sigma_1'(t),\sigma_1''(t)), \tag{1.8}$$

for a.e. $t\in(0,1)$
and for $x\in[\min\{\sigma_1(t),\sigma_2(t)\},\max\{\sigma_1(t),\sigma_2(t)\}]$,

$$\sigma_1(\eta)=0,\ \sigma_1'(0)\le0,\ \sigma_1'(1)\le0, \tag{1.9}$$

$$\sigma_2'''\le f(t,x,\sigma_2',\sigma_2''), \tag{1.10}$$

for a.e. $t\in(0,1)$

and for $x\in[\min\{\sigma_1(t),\sigma_2(t)\}, \max\{\sigma_1(t),\sigma_2(t)\}]$,

$$\sigma_2(\eta)=0,\ \sigma_2'(0)\geq 0,\ \sigma_2'(1)\geq 0, \qquad (1.11)$$

will be called a lower and an upper solution of BVP (1.1),(1.2).

For i = 0,1,2 we denote $c_i=\max\{|\sigma_1^{(i)}(t)|+|\sigma_2^{(i)}(t)|: 0\leq t\leq 1\}$.

3. **Lemmas.**

Lemma 1. *(generalized Fredholm alternative theorem [19])*

Let $D'=(a,b)\times R^n$, $\phi_i: C^{n-1}(a,b)\to R$, $i=1,2,\ldots,n$ are continuous linear functionals, $A_i\in R$ for $i=1,2,\ldots,n$. Let us put

$$Ly=y^n-\sum_{i=1}^{n} a_i y$$

$$Ny=f(t,y,y',\ldots,y^{(n-1)}),$$

where $a_i\in L(a,b)$, $i=0,1,2,\ldots,n$, $f\in Car_{loc}(D')$.

Let the BVP

$$Ly=0,$$

$$\phi_i(y)=0,\ i=1,2,\ldots,n$$

have only the trivial solution. If the absolute value of the function f is bounded by a Lebesgue integrable function on D', then the BVP

$$Ly=Ny,$$

$$\Phi_i(y)=A_i,\quad i=1,2,\ldots,n$$

has at least one solution.

Lemma 2. *Let σ_1 be a lower solution and σ_2 an upper solution of BVP (1.1),(1.2) and $\sigma_1'(t)\leq\sigma_2'(t)$ for every $t\in[0,1]$. Let there exist $h_0\in L(0,1)$ such that on D there is satisfied*

$$|f(t,x,y,z)|\leq h_0(t) \qquad (1.12)$$

for $\sigma_1'(t)\leq y\leq\sigma_2'(t)$.

Then BVP (1.1), (1.2) has a solution u satisfying

$$\sigma_1'(t)\leq u'(t)\leq\sigma_2'(t) \qquad (1.13)$$

for $t\in[0,1]$.

Proof. Let us choose $m\in N$ and put (on D)

$$s_1(t)=\min\{\sigma_1(t),\sigma_2(t)\},\qquad s_2(t)=\max\{\sigma_1(t),\sigma_2(t)\},$$

$$p(t,x)=\begin{cases} s_1(t) & \text{for } x\leq s_1(t)\\ x & \text{for } s_1(t)\leq x\leq s_2(t)\\ s_2(t) & \text{for } x\geq s_2(t)\,,\end{cases}$$

$$w_1(t,x,y,z)=-m(y-\sigma_1')(f(t,p(t,x),\sigma_1'(t),\sigma_1''(t))-f(t,p(t,x),\sigma_1'(z),z)),$$
$$w_2(t,x,y,z)=m(y-\sigma_2')(f(t,p(t,x),\sigma_2'(t),\sigma_2''(t))-f(t,p(t,x),\sigma_2'(t),z)),$$

$$f_m=\begin{cases} f(t,p(t,x),\sigma_1'(t),\sigma_1''(t)) & \text{for } y\le\sigma_1'(t)-1/m,\\ f(t,p(t,x),\sigma_1'(t),z)+w_1(t,x,y,z) & \text{for } \sigma_1'(t)-1/m<y<\sigma_1'(t),\\ f(t,p(t,x),y,z) & \text{for } \sigma_1'(t)\le y\le\sigma_2'(t),\\ f(t,p(t,x),\sigma_2'(t),z)+w_2(t,x,y,z) & \text{for } \sigma_2'(t)<y<\sigma_2'(t)+1/m,\\ f(t,p(t,x),\sigma_2'(t),\sigma_2''(t)) & \text{for } \sigma_2'(t)+1/m\le y. \end{cases} \qquad (1.14)$$

From (1.12) and (1.14) it follows that on D it is

$$|f_m(t,x,y,z)|\le h_0(t). \qquad (1.15)$$

Let us consider the differential equation

$$u'''=f_m(t,u,u',u''). \qquad (1.16)$$

According to Lemma 1 BVP (1.16),(1.2) has a solution u_m.We shall show that u_m satisfies

$$\sigma_1'(t)-1/m\le u_m'(t)\le\sigma_2'(t)+1/m \qquad (1.17)$$

for every $t\in[0,1]$. Put

$$v(t)=(-1)^i(u_m'(t)-\sigma_i'(t))-1/m$$

for $t\in[0,1]$ and $i\in\{1,2\}$.

Then by (1.2),(1.9)and (1.11) we get $v(0)\le0$, $v(1)\le0$.

Let there exist $t_0\in(0,1)$ such, that $v(t_0)>0$. Then there exists an interval (a_0,b_0), where $0\le a_0<t_0<b_0\le1$, such that $v(t)>0$ for $t\in(a_0,b_0)$, $v(a_0)=v(b_0)=0$, $v'(a_0)\ge0$, $v'(b_0)\le0$. From (1.8),(1.10) and (1.14) it follows that

$$v''(t)=(-1)^i(f_m(t,u_m,u_m',u_m'')-\sigma_i'''(t))\ge0 \qquad (1.18)$$

for a.e. $t\in(a_0,b_0)$, for $i\in\{1,2\}$. Integrating (1.18) from t_1 to t_2, where $a_0<t_1<t_2<b_0$, we get

$$v'(t_2)-v'(t_1)\ge0.$$

The last inequality implies, that the function $v'(t)$ is nondecreasing in (a_0,b_0). Let $v(t_3)=\max\{v(t);\ t\in(a_0,b_0)\}$, then $v'(t_3)=0$ and $v'(t)$ is nondecreasing in (t_3,b_0). Since $v(t_3)>0$ we get $v(b_0)>0$ which contradicts to $v(b_0)=0$. Hence (1.17) is proved. From (1.17) and (1.2) it follows that

$$|u_m'(t)|\le c_1+1/m \quad \text{for } t\in[0,1] \qquad (1.19)$$

and

$$|u_m(t)|\le c_1+1/m \quad \text{for } t\in[0,1]. \tag{1.20}$$

Integrating (1.16), where $u=u_m$, from t to α , where $t,\alpha\in(0,1)$ and α is such that $u_m''(\alpha)=0$ we get

$$|u_m''(t)|\le\int_0^1 h_0(t)dt. \tag{1.21}$$

From (1.19), (1.20) and (1.21) it follows that the sequences $(u_m)_{m=1}^{\infty}$, $(u_m')_{m=1}^{\infty}$ are uniformly bounded and equi-continuous on [0,1] and that the sequence $(u_m'')_{m=1}^{\infty}$ is uniformly bounded. From (1.16) and by the theory of the Lebesgue integral we get that the sequence $(u_m'')_{m=1}^{\infty}$ is equi-continuous on [0,1]. By the Arzela-Ascoli lemma without loss of generality, we may suppose that all the three sequences are uniformly converging on [0,1]. By Lebesgue theorem and by (1.14),(1.16),(1.17) the function $u(t)=\lim_{m\to\infty} u_m(t)$ on [0,1] is a solution of BVP (1.1), (1.2) and fulfils (1.13). Lemma is proved.

Lemma 3. *(On a priori estimates) Let* $r_1,r_2\in R$, $r_1<r_2$, $r_1\le 0\le r_2$, $g\in Car_{loc}((0,1)\times R)$, $h\in L^q(r_1,r_2)$ *and* $\omega\in C^0(0,\infty)$ *is a positive function satisfying*

$$\int_0^{\infty} \frac{ds}{\omega(s)} = +\infty. \tag{1.22}$$

Then there exists $r^*\in(1,\infty)$ *such that for any function* $u\in AC^0(0,1)$ *the conditions (1.2),*

$$r_1\le u'(t)\le r_2 \quad \text{for every } t\in[0,1], \tag{1.23}$$

$$|u'''|\le\omega(|u''|)g^{1/p}(t,u)h(u')(1+|u''|)^{1/q} \tag{1.24}$$

for a.e. $t\in(0,1)$, $|u''(t)|\ge 1$,

imply the estimate

$$|u''(t)|\le r^* \quad \text{for every } t\in[0,1]. \tag{1.25}$$

Proof. Let $G=\{ v\in AC^2(0,1)$: v satisfies (1.2) and (1.23)$\}$. If $v\in G$, then $|v(t)|\le\rho$, where $\rho=\max\{|r_1|,r_2\}$ and

$g_0(t)=\sup\{|g(t,v)|: v\in G\}\in L^1(0,1)$.

Put

$$k_0=2\|g_0^{1/p}\|_{L^p(0,1)}\|h\|_{L^p(r_1,r_2)}, \tag{1.26}$$

$$\Omega(x)=\int_0^x \frac{ds}{\omega(|s|)} \quad \text{for} \quad x\in R. \tag{1.27}$$

From (1.22) and (1.27) it follows that Ω is an odd function, $\Omega(R)=R$ and there exists the inverse mapping Ω^{-1}. Let $u\in AC^0(0,1)$ satisfy (1.2), (1.23) and (1.24) then there exists $a_0\in(0,1)$ such that $u''(a_0)=0$. Let us suppose that there exists $t_1\in(a_0,1]$ such that

$$|u''(t_1)|>r^*, \tag{1.28}$$

where

$$r^*=\Omega^{-1}(\Omega(1)+k_0). \tag{1.29}$$

Let $[a_1,b_1]\subset[a_0,1]$ be the maximal interval containing t_1, in which $|u''(t)|\geq 1$. Let $s_1\in(a_1,b_1]$ be such a point that

$$|u''(s_1)|=\rho_1=\max\{|u''(t)|:a_1\leq t\leq b_1\}.$$

From (1.24) and from the Hölder inequality we can obtain

$$\int_{a_1}^{s_1} \frac{u'''(t)}{\omega(|u''(t)|)}\,dt\leq k_0.$$

In the case that $u''(t)\geq 1$ on $[a_1,s_1]$ we get $\Omega(\rho_1)-\Omega(1)\leq k_0$, which implies by (1.26),(1.29) that $\rho_1\leq r^*$. The last inequality contradicts (1.28). We can obtain a similar contradiction in the case $u''(t)\leq -1$ on $[a_1,s_1]$. Therefore we have $|u''(t)|\leq r^*$ for every $t\in[a_0,1]$. If we suppose that $t_1\in[0,a_0]$, we can get in a similar way as above that $|u''(t)|\leq r^*$ for $t\in[0,a_0]$ and this comletes the proof.

4. Theorems

Theorem 4. *Let σ_1 be a lower solution and σ_2 an upper solution of BVP (1.1) ,(1.2) and $\sigma_1'(t)\leq\sigma_2'(t)$ for each $t\in[0,1]$. Let on the set $D(\sigma_1',\sigma_2')$ the inequality*

$$|f(t,x,y,z)|\leq\omega(|z|)g^{1/p}(t,x)h(y)(1+|z|)^{1/q}, \tag{1.30}$$

be satisfied, where $h\in L^q(-c_1,c_1)$, $g\in Car_{loc}((0,1)\times R)$ are nonnegative and $\omega\in C^0(0,1)$ is a positive function satisfying (1.22). Then BVP (1.1),(1.2) has a solution such that

$$\sigma_1'(t)\leq u'(t)\leq\sigma_2'(t) \quad \text{for each } t\in[0,1]. \tag{1.31}$$

Proof. **Without loss of generality we may suppose** $c_1>0$. Let r^* be the constant found by **Lemma 3** for $r_1=-c_1$, $r_2=c_1$. Put

$$\rho_0=r^*+c_0+c_1+c_2,$$

$$\chi(\rho_0,s)=\begin{cases} 1 & \text{for } 0\le s\le\rho_0 \\ 2-s/\rho_0 & \text{for } \rho_0<s<2\rho_0 \\ 0 & \text{for } s\ge 2\rho_0, \end{cases}$$

$$l(t,x,y,z)=\chi(\rho_0,|x|+|y|+|z|)f(t,x,y,z) \quad \text{on } D. \tag{1.32}$$

Since $\max\{|\sigma_i(t)|+|\sigma_i'(t)|+|\sigma_i''(t)|;\ 0\le t\le 1\}<\rho_0$, for $i=1,2$, σ_1 is a lower solution and σ_2 is an upper solution of BVP

$$u'''=l(t,u,u',u''), \tag{1.33}$$

(1.2). Further $|l(t,x,y,z)|\le g^*(t)$ on D, where

$$g^*(t)=\sup\{|f(t,x,y,z)|:\ |x|+|y|+|z|\le 2\rho_0\}\in L^1(0,1).$$

By Lemma 2 BVP (1.33),(1.2) has a solution u satisfying (1.13). Consequently u fulfils (1.23) for $r_1=-c_1$, $r_2=c_1$. According to (1.30) and (1.32) we have

$$|u'''|\le\omega(|u''|)g^{1/p}(t,u)h(u')(1+|u''|)^{1/q}$$

for a.e $t\in(0,1)$, $|u''(t)|\ge 1$. Therefore by Lemma 3 $|u''(t)|\le r^*$ for $t\in[0,1]$. Consequently according to this estimate and to (1.2),(1.23) we get

$$|u(t)|+|u'(t)|+|u''(t)|\le\rho_0 \quad \text{for } t\in[0,1]. \tag{1.34}$$

In view of (1.32),(1.33) and (1.34) u is a solution of **BVP** (1.1),(1.2). Theorem is proved.

Note. If $\sigma_1'(t)=\sigma_2'(t)$ on [0,1] then $\sigma_1(t)=\sigma_2(t)$ on [0,1] and BVP (1.1),(1.2) has a solution $u(t)=\sigma_1(t)=\sigma_2(t)$.

Theorem 5. *Let there exist* $r_1,r_2\in R$ *such that* $r_1<r_2$, $r_1\le 0\le r_2$ *and*

$$f(t,x,r_1,0)\le 0, \qquad f(t,x,r_2,0)\ge 0 \tag{1.35}$$

for a.e. $t\in(0,1)$, $x\in[\min\{r_1(t-\eta),r_2(t-\eta)\},\ \max\{r_1(t-\eta),r_2(t-\eta)\}]$. *Further let (1.30) be fulfilled on* $D(r_1,r_2)$, *where* $h\in L^q(r_1,r_2)$, g,ω *are the functions from Theorem 4. Then BVP (1.1),(1.2) has a solution u such that*

$$r_1\le u'(t)\le r_2 \quad \text{for each } t\in[0,1].$$

Proof. Let us put $\sigma_1(t)=r_1(t-\eta)$, $\sigma_2(t)=r_2(t-\eta)$, then σ_1 is a lower solution and σ_2 is an upper solution of BVP (1.1) , (1.2) and $\sigma'_1<\sigma'_2$ on [0,1]. Thus Theorem 5 follows from Theorem 4.

Example. Theorem 5 (and also Theorem 4) is applicable for example to the function

$f(t,x,y,z)=(y^3+e^t)(1+z^2)g(t)+ze^x$, where g is a nonnegative function of C(0,1).

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