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ERROR ESTIMATE FOR QUADRATIC SPLINE INTERPOLATING THE FIRST DERIVATIVES

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Abstract. For the quadratic spline interpolating the given values of the first derivative the estimates of interpolation error are studied.

Key words: splines, quadratic splines, error estimates

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1. Simple set of spline knots

Let us have the growing sequence of simple knots

$$(\Delta x)=\{x_i;\ i=0(1)n+1\}\ ,\ \text{with}\quad h_i=x_{i+1}-x_i\ ,$$

and the given values $\{m_i\ ;\ i=0(1)n+1\}$. Denote $\mathcal{S}_2(\Delta x)$ the linear space of quadratic splines with knots (Δx). It is wellknown that any quadratic spline $s(x)\in\mathcal{S}_2(\Delta x)$ can be piecewise written as (see [1])

$$(1)\qquad s(x)=(1-t^2)s_i+t^2s_{i+1}+h_it(1-t)m_i$$

for $x\in[x_i,x_{i+1}]$, $i=0(1)n$, $t=(x-x_i)/h_i$, $s_j=s(x_j)$.

Such a spline can be uniquely determined by the conditions of interpolation of the first derivatives

(2) $$s'(x_i)=m_i \ , \qquad i=0(1)n+1$$

and by one additional (initial) condition

(3) $$s(x_k)=s_k \qquad \text{with } k\in\{0,\dots,n+1\} \ .$$

The spline values $s_i=s(x_i)$ and the values of its derivatives $m_i=s'(x_i)$ are connected together by continuity conditions $s\in C^1[x_0,x_{n+1}]$, which can be expressed as

(4) $$s_{i+1}-s_i=h_i(m_{i+1}+m_i)/2 \ , \qquad i=0(1)n, \qquad \text{(see [1],[2])}.$$

In case of $m_i=g'(x_i)$ with some known function g, we can ask for the interpolation error $e(x)=g(x)-s(x)$. For equidistant knot mesh (Δx) we can find the error estimate in [2] .

Theorem 1

Let the spline knots (Δx) *and values* $m_i=g'(x_i)=g'_i$, $i=0(1)n+1$ *with* $g\in C^3[x_0,x_{n+1}]$ *are given.*

Denote $s(x)\in S_2(\Delta x)$ *the quadratic spline determined by the conditions*

(5) $$s(x_0)=s_0=g(x_0), \quad s'(x_i)=m_i \ , \quad i=0(1)n+1$$

and let $e(x)=g(x)-s(x)$, $e_j=e(x_j)$.

Then the following error estimates are valid for $x\in[x_j,x_{j+1}]$:

1. $|e_j| \le \frac{1}{12} H_j^2(x_j-x_0)||g'''||_{0j}$ *with* $H_j=\max\{h_i;i\le j\}$;

2. $|e(x)|\le \frac{1}{12} H_{j+1}^2(x_{j+1}-x_0)||g'''||_{0,j+1}+\frac{2}{81} h_j^3 \ ||g'''||_j$;

3. $|e'(x)| \le \frac{1}{8} h_j^2 \ ||g'''||_j \le \frac{1}{8} H_j^2 \ ||g'''||_{0,j+1}$;

4. $|e''(x)|\le \frac{1}{2} h_j[\ ||g'''||_j + \omega_j(g''',h_j)] \le$

$$\le \frac{1}{2} H_j[||g'''||_{0,j+1}+\omega_{0j}(g''',H_j)],$$

where $||..||$ *denotes the maximum norm,* $||..||_i$ *and* ω_i *the maximum norm and modulus of continuity with respect to the interval* $[x_i,x_{i+1}]$, $||..||_{0j}$ *and* ω_{0j} *- with respect to the interval* $[x_0,x_j]$.

Proof

1. Adding (4) with $i=0(1)j-1$, we obtain

$$s_j - s_0 = \frac{1}{2} \sum_{i=0}^{j-1} h_i (m_i + m_{i+1})$$

Using the trapezoidal rule of numerical integration with known error term we have

$$g_j - g_0 = \int_{x_0}^{x_j} g'(x)dx = \frac{1}{2} \sum_{i=0}^{j-1} h_i (m_i + m_{i+1}) - \frac{1}{12} \sum_{i=0}^{j-1} h_i^3 g'''(z_i) ,$$

with $z_i \in [x_i, x_{i+1}]$. As $s_0 = g_0$, we obtain by substraction

$$g_j - s_j = - \frac{1}{12} \sum_{i=0}^{j-1} h_i^3 g'''(z_i)$$

and then

$$|e_j| \le \frac{1}{12} H_j^2 \, ||g'''||_{0j} \sum_{i=0}^{j-1} h_i = \frac{1}{12}(x_j - x_0) H_j^2 ||g'''||_{0j}$$

follows.

2. Using the Hermite interpolation and its error term for the function $g(x)$, we can write with $t=(x-x_i)/h_i \in [0,1]$

(6) $$g(x)=(1-t^2)g_j + t^2 g_{j+1} + h_j t(1-t)m_j + \frac{1}{6}g'''(z_i)(x-x_i)^2(x-x_{i+1}),$$
$x, z_i \in [x_i, x_{i+1}]$.

For the error $e(x)=g(x)-s(x)$ we obtain then

$$e(x)=(1-t^2)e_i + t^2 e_{i+1} + \frac{1}{6} g'''(z_i)(x-x_i)^2(x-x_{i+1}) .$$

So, for $x \in [x_j, x_{j+1}]$ we have

$$|e(x)| \le \max\{|e_j|, |e_{j+1}|\} + \frac{1}{6} h_j^3 |t^3 - t^2| \, |g'''(z_j)| \le$$

$$\le \max\{|e_j|, |e_{j+1}|\} + \frac{2}{81} h_j^3 \, ||g'''||_j \quad .$$

With the help of the proved first assertion of our theorem we get

$$|e(x)| \le \frac{1}{12} |x_{j+1} - x_0| \, H_{j+1} ||g'''||_{0,j+1} + \frac{2}{81} h_j^3 \, ||g'''||_j \quad .$$

3. The derivatives $s'(x), g'(x)$ for $x \in [x_j, x_{j+1}]$ can be expressed as

(7) $$s'(x)=(1-t)m_j + t m_{j+1} \quad ,$$
$$g'(x)=(1-t)m_j + t m_{j+1} + \frac{1}{2} g'''(z_j)(x-x_j)(x-x_{j+1}) .$$

Substracting the equalities in (7),

$$|g'(x)-s'(x)|=|e'(x)|\le \frac{1}{2}(x-x_j)(x_{j+1}-x)|g'''(z_j)|\le$$
$$\le \frac{1}{8} h_j^2 \|g'''\|_j \le \frac{1}{8} H_j^2 \|g'''\|_{0,j+1}$$

follows.

4. The Taylor's expansion of g'_{j+1} gives

$$g'_{j+1}=m_{j+1}=m_j+h_jg_j''+\frac{1}{2}h_j^2 g'''(y_j) , \quad y_j\in[x_j,x_{j+1}] ;$$

we have then

$$(m_{j+1}-m_j)/h_j=g_j''+\frac{1}{2}h_jg'''(y_j) .$$

For the second derivatives $s''(x)$, $g''(x)$, $x\in[x_j,x_{j+1}]$,

$$s''(x)=(m_{j+1}-m_j)/h_j=g_j''+\frac{1}{2}h_jg'''(y_j) ,$$
$$g''(x)=g_j''+h_jg'''(z_j)$$

hold .

The error $e(x)$ obeys then the relation

$$e''(x)=g''(x)-s''(x)=h_j[g'''(z_j)-\frac{1}{2}g'''(y_j)]=$$
$$=\frac{1}{2}h_j[g'''(z_j)+g'''(z_j)-g'''(y_j)] .$$

Using local modulus of continuity $\omega_i(g''',h_i)$, we obtain the estimate

$$|e''(x)|\le \frac{1}{2}h_j[\|g'''\|_j+\omega_j(g''',h_j)] .$$

2. Separated mesh $(\Delta x\Delta t)$

2.1 *Spline representation*

Let us consider the mesh with separated spline knots x_i and points of interpolation t_i

$$(\Delta x\Delta t)\equiv x_0\le t_0<x_1<t_1<\ldots<t_n\le x_{n+1} \quad \text{with} \quad h_i=x_{i+1}-x_i,\ d_i=(t_i-x_i)/h_i.$$

Quadratic spline $s(x)$ over the interval $[x_i,x_{i+1}]$ can be expressed as

(8) $$s(x)=A(t)s_i+B(t)s_{i+1}+h_iC(t)m_i,$$
$$t=(x-x_i)/h_i, \quad s_i=s(x_i), \quad m_i=s'(t_i),$$

where

$$A(t)=(t^2-2td_i)/(2d_i-1)=-B(t)+1,$$

$$B(t)=-t(t-2d_i)/(2d_i-1),$$

$$C(t)=t(t-1)/(2d_i-1)$$

in case of $d_i \neq \frac{1}{2}$, $i=0(1)n$. It can be uniquely determined

a) by the conditions of interpolation of derivatives

$$m_i=s'(t_i) \ , \ i=0(1)n \ ;$$

b) by two boundary conditions $s(x_0)=s_0$, $s(x_{n+1})=s_{n+1}$.

In case $d_i=\frac{1}{2}$, $i=0(1)n$, with $s'_i=s'(x_i)$ we can write

(9) $$s(x)=s_i+s'_i(x-x_i)+(s'_{i+1}-s'_i)(x-x_i)^2/(2h_i) \ .$$

Such a spline is uniquely determined

a) by the conditions of interpolation of derivatives

$$m_i=s'(t_i), \ i=0(1)n \ ,$$

b) by initial values $s(x_0)=s_0$, $s'(x_0)= s'_0$ (see [1],[2]).

2.2 *Error estimate in case* $d_i=\frac{1}{2}$, $i=0(1)n$

Let the values $m_i=s'(t_i)$, $i=0(1)n$, s_0, s'_0 be given . Denote $s_i=s(x_i)$, $s'_i=s'(x_i)$, $g_i=g(x_i)$, $g'_i=g'(x_i)$; $H_j=\max\{h_i;i<j\}$. Suppose that $g\in C^3[x_0,x_{n+1}]$, $g'(t_i)=m_i$, $g(x_0)=s_0$, $g'(x_0)=s'_0$.

There are simple relations between quantities m_j,s_j,s'_j :

(10) $$m_i=\tfrac{1}{2}(s'_i+s'_{i+1}) \ , \ s_{i+1}-s_i=h_im_i \ , \ i=0(1)n \ .$$

1° Summing the last equalities we obtain

(11) $$s_j-s_0=\sum_{i=0}^{j-1} h_im_i=\sum_{i=0}^{j-1} h_is'(t_i) \ .$$

On the other hand - by midpoint rule of numerical integration with known remainder term - we have

(12) $$\int_{x_0}^{x_j} g'(x)dx = g_j-g_0 =\sum_{i=0}^{j-1} h_im_i+ \frac{1}{24}\sum_{i=0}^{j-1} h_i^3g'''(z_i)$$

with some $z_i\in[x_i,x_{i+1}]$. Substracting (11) from (12) we obtain for the error $e(x)=g(x)-s(x)$ the relation

$$e_j=\frac{1}{24}\sum_{i=0}^{j-1} h_i^3 g''''(z_i) \quad ,$$

and finally the estimate

$$(13) \qquad |e_j| \le \frac{1}{24} H_j^2(x_j-x_0)\|g'''\|_{0j} \quad .$$

2° From Taylor's expansion of g' at $x=t_i$ we have

$$g'_{i+1}+g'_i=2g'(t_i)+\frac{1}{4}h_i^2\, g'''(z_i) \quad .$$

Following (10), there is also $s'_{i+1}+s'_i=2m_i$.

Substracting these relations, the recurrence relation

$$e'_{i+1}+e'_i=\frac{1}{4}h_i^2 g'''(z_i)$$

and the inequality

$$|e'_{i+1}|<|e'_i|+\frac{1}{4}h_i^2|g'''(z_i)|$$

follow .

By induction, using $e'_0=0$, we obtain the estimates

$$(14) \qquad |e'_j| \le \frac{1}{4}\sum_{i=0}^{j-1} h_i^2|g'''(z_i)| \le \frac{1}{4} H_j(x_j-x_0)\|g'''\|_{0j} \quad .$$

3° We have further (with $\tau_i,\eta_i\in[x_i,x_{i+1}]$)

$$g'_i=g'(t_i)-\frac{1}{2}h_i g''(t_i)+\frac{1}{8}h_i^2 g'''(\tau_i) \quad , \quad s''(t_i)=s''_i ,$$

$$g''(t_i)=g''_i+\frac{1}{2}h_i g'''(\eta_i) \quad , \quad s'_i=m_i-\frac{1}{2}h_i s''(t_i) \quad .$$

Then

$$g'_i-s'_i=\frac{1}{2}h_i(s''_i-g''_i)-\frac{1}{4}h_i^2 g'''(\eta_i)+\frac{1}{8}h_i^2 g'''(\tau_i)=$$

$$=\frac{1}{2}h_i(s''_i-g''_i)-\frac{1}{8}h_i^2[2g'''(\eta_i)-g'''(\tau_i)] \quad ,$$

and

$$s''_i-g''_i=\frac{2}{h_i}(g'_i-s'_i)+\frac{1}{4}h_i[g'''(\eta_i)+g'''(\eta_i)-g'''(\tau_i)].$$

With the local modulus of continuity $\omega_i(g''',h_i)$ we can write the recurrent estimate

$$|e''_i|\le\frac{2}{h_i}|e'_i|+\frac{1}{4}h_i[\|g'''\|_i+\omega_i(g''',h_i)] \quad .$$

Substitute the estimate (14), we obtain

$$|e''_i|\le\frac{1}{2}(H_j/h_i)(x_i-x_0)\|g'''\|_{0i}+\frac{1}{4}h_i[\|g'''\|_i+\omega_i(g''',h_i)].$$

4° For the function values $s(x)$, $g(x)$ with $x\in[x_i,x_{i+1}]$ we have the expansions ($\delta_i\in[x_i,x_{i+1}]$)

$$s(x)=s_i+s'_i(x-x_i)+(s'_{i+1}-s'_i)(x-x_i)^2/(2h_i) \ ,$$

$$g(x)=g_i+g'_i(x-x_i)+\frac{1}{2}g''_i(x-x_i)^2+\frac{1}{6}g'''(\delta_i)(x-x_i)^3 \ .$$

For the error $e(x)=g(x)-s(x)$ then the relations

$$e(x)=e_i+(x-x_i)e'_i+\frac{1}{2}(x-x_i)^2[g''_i-(s'_{i+1}-s'_i)/h_i]+\frac{1}{6}g'''(\delta_i)(x-x_i)^3 \ ,$$

$$|e(x)|\le |e_i|+h_i|e'_i|+\frac{1}{2}h_i^2|e''_i| + \frac{1}{6}h_i^3||g'''||_i$$

follow .

Substituting for $|e_i|,|e'_i|,|e''_i|$ from (13)-(15), we finally obtain the estimate

$$(16)\quad |e(x)|\le \frac{1}{24}H_i^2(x_i-x_0)||g'''||_{0i}+\frac{1}{2}h_iH_i(x_i-x_0)||g'''||_{0i}+$$

$$+\frac{1}{2}h_i^2\frac{1}{4}h_i\{||g'''||_i+\omega_i(g''',h_i)\}+\frac{1}{6}h_i^3||g'''||_i=$$

$$=\frac{1}{2}H_i(x_i-x_0)||g'''||_{0i}\{\frac{1}{12}H_i+h_i\}+\frac{1}{8}h_i^3[\frac{7}{3}||g'''||_i+\omega_i(g''',h_i)] \ .$$

We summarize our discussion from 2.2 in the following theorem.

Theorem 2

Let the function $g(x)\in C^3[x_0,x_{n+1}]$ *and* $s(x)$ *be the quadratic spline interpolating the derivatives* $m_i=g'(t_i)$ *on the mesh* $(\Delta x\Delta t)$ *with* $d_i=\frac{1}{2}$, $i=0(1)n$ *described by* (9).

Then the following estimates for $e(x)=g(x)-s(x)$ *hold*:

1. $|e_j|\le \frac{1}{24}H_j^2(x_j-x_0)\ ||g'''||_{0j}$;
2. $|e'_j|\le \frac{1}{4}H_j(x_j-x_0)\ ||g'''||_{0j}$;
3. $|e''_j|\le \frac{1}{2}(H_j/h_j)(x_j-x_0)\ ||g'''||_{0j}+\frac{1}{4}h_j\{||g'''||_j+\omega_j(g''',h_j)\}$;
4. $|e(x)|\le \frac{1}{2}H_j(x_j-x_0)||g'''||_{0j}(\frac{1}{12}H_j+h_j)+\frac{7}{24}h_j^3\ ||g'''||_j+$
$$+\frac{1}{8}h_j^3\omega_j(g''',h_j) \ .$$

(*with notation used in* Theorem 1).

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