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BOUNDED SOLUTIONS OF SECOND ORDER FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract: This paper deals with bounded (on R) solutions of a functional differential equation $y''(t) = 2\alpha y'(t) + Q[y,y'](t)y(t) + F[y,y'](t)$, where $\alpha \neq 0$ is a constant.

Key words: Second-order functional differential equation, bounded solution, Schauder linearization technique, Banach and Schauder-Tychonoff fixed point theorems, Ascoli theorem.

MS Classification: 34C11, 34K15.

1. Introduction

Let X be the Fréchet space of C^0-functions on R with the usual topology of local uniform convergence on R and let X_B be the set of bounded C^0-functions on R with the topology as in X.

Consider a functional differential equation

$$y''(t) = 2\alpha y'(t) + Q[y,y'](t)y(t) + F[y,y'](t)\,, \tag{1}$$

where $\alpha \neq 0$ is a constant and Q, $F: X_B \times X_B \longrightarrow X$ are continuous operators, that is $\lim_{n\to\infty} Q[y_n,z_n] = Q[y,z]$, $\lim_{n\to\infty} F[y_n,z_n] = F[y,z]$ for all convergent (in X_B) sequences $\{y_n\}$, $\{z_n\}$, $\lim_{n\to\infty} y_n = y$, $\lim_{n\to\infty} z_n = z$.

In the present paper using of the Schauder linearization technique and the Banach and Schauder-Tychonoff fixed point theorems there are given sufficient conditions on Q, F for the existence of bounded solutions of (1). A special case of (1) is the differential equation $y'' = 2\alpha y' + q(t,y,y')y + f(t,y,y')$ in which $g,f\colon R^3 \longrightarrow R$ are continuous functions.

The problem of bounded solutions on a halfline or on R for systems of differential equations, for classes of functional differential equations and for n-th order differential equations has been studied in many works by various methods (see e.g. [1], [3] - [10]).

2. Lemmas

Lemma 1. Let g, $h \in X_B$. Then any solution $y \in X_B$ of the differential equation

$$y'' = 2\alpha y' + g(t)y + h(t) \tag{2}$$

is a solution of the integral equation

$$y(t) = e^{\alpha t}\int_t^{\infty}\int_s^{\infty} e^{-\alpha\nu}[(g(\nu)+\alpha^2)y(\nu)+h(\nu)]d\nu ds \quad \text{for } \alpha > 0 \tag{3}$$

or

$$y(t) = e^{\alpha t}\int_{-\infty}^{t}\int_{-\infty}^{s} e^{-\alpha\nu}[(g(\nu)+\alpha^2)y(\nu)+h(\nu)]d\nu ds \quad \text{for } \alpha < 0 \tag{4}$$

in the space X_B and also reversally any solution of (3) or (4) in X_B is a solution of (2) in X_B.

P r o o f . Let $\alpha > 0$ and let $y \in X_B$ be a solution of (2). Then $y' \in X_B$ by Esclangon theorem (see [2] and e.g. [11]). From the equality $y''(t) = \alpha y'(t) + [\alpha y'(t) + g(t)y(t) + h(t)]$ it follows

$$y'(t) = e^{\alpha t}[c + \int_0^t e^{-\alpha s}(\alpha y'(s) + g(s)y(s) + h(s))ds]$$

with c being an appropriate constant. Since $y' \in X_B$ and $\lim_{t\to\infty} e^{\alpha t} =$ $= \infty$ we have $c = -\int_0^\infty e^{-\alpha s}[\alpha y'(s) + g(s)y(s) + h(s)]ds$, consequently,

$$y'(t) = -e^{\alpha t}\int_t^\infty e^{-\alpha s}(\alpha y'(s) + g(s)y(s) + h(s))ds \quad \text{for all } t \in R.$$

Similarly, from the equality

$$y'(t) = \alpha y(t) - \left[\alpha y(t) + e^{\alpha t}\int_t^\infty e^{-\alpha s}(\alpha y'(s) + g(s)y(s) + h(s))ds\right]$$

we get

$$y(t) = e^{\alpha t}\int_t^\infty\int_s^\infty e^{-\alpha \nu}(\alpha y'(\nu) + g(\nu)y(\nu) + h(\nu))d\nu ds + \alpha e^{\alpha t}\int_t^\infty e^{-\alpha s}y(s)ds$$

and using the equality

$$\alpha e^{\alpha t}\int_t^\infty\int_s^\infty e^{-\alpha\nu}y'(\nu)d\nu ds = -\alpha e^{\alpha t}\int_t^\infty e^{-\alpha s}y(s)ds + \alpha^2 e^{\alpha t}\int_t^\infty\int_s^\infty e^{-\alpha\nu}y(\nu)d\nu ds$$

we see that y is a solution of (2) in X_B.

Let $y \in X_B$ be a solution of (3). Then $y \in C^2(R)$ and one can easily check by the standard calculations that y is a bounded solution of (2).

In the case $\alpha < 0$ is the proof analogous as above.

Notation. On X_B define a functional $\|.\|$ by $\|x\| =$ $= \sup\{|x(t)|;\ t \in R\}$.

Lemma 2. Let $g, h \in X_B$ and let $\inf\{g(t);\ t \in R\} < 0$, $\|g\| < 2\alpha^2$. Then there is the unique solution of (2) in X_B.

Proof. In view of Lemma 1 it is sufficient to prove that equation (3) or (4) admits the unique solution in X_B.

Let Y be the Banach space of bounded C^0-functions on R with the norm $\|.\|$. Let $T: Y \longrightarrow Y$ be an operator defined by

$$(Ty)(t) = e^{\alpha t}\int_t^{\infty}\int_s^{\infty} e^{-\alpha\nu}[(g(\nu)+\alpha^2)y(\nu)+h(\nu)]d\nu ds \quad \text{for } \alpha > 0$$

or

$$(Ty)(t) = e^{\alpha t}\int_{-\infty}^{t}\int_{-\infty}^{s} e^{-\alpha\nu}[(g(\nu)+\alpha^2)y(\nu)+h(\nu)]d\nu ds \quad \text{for } \alpha < 0.$$

The assumptions of Lemma 2 imply the existence of a positive constant $\varepsilon > 0$ such that $-\varepsilon \geqq g(t) \geqq -2\alpha^2 + \varepsilon$ for all $t \in R$. Then $\|g + \alpha^2\| \leqq \alpha^2 - \varepsilon$ and for $y,z \in Y$ we have

$$|(Ty)(t) - (Tz)(t)| \leqq (1 - \frac{\varepsilon}{\alpha^2})\|y - z\| \quad \text{for all } t \in R,$$

consequently,

$$\|Ty - Tz\| \leqq (1 - \frac{\varepsilon}{\alpha^2})\|y - z\| \quad \text{for all } y,z \in Y.$$

Hence T is a contraction and by the Banach fixed point theorem there is the unique solution of (3) or (4) in Y.

Lemma 3. Let $g, h \in X_B$ and let $-\varepsilon \geqq g(t) \geqq -2\alpha^2 + \varepsilon$ be fulfilled for all $t \in R$ with a positive constant ε. If y is a bounded solution of (2) then

$$\|y\| \leqq \frac{\|h\|}{\varepsilon}, \qquad \|y'\| \leqq \frac{2|\alpha|\,\|h\|}{\varepsilon}. \tag{5}$$

P r o o f . Let y be a (and then the unique by Lemma 2) solution of (2) and thus also the unique bounded solution of (3) or (4). Substituting y(t) into (3) or (4) we get for an evident calculation the following estimate $|y(t)| \leqq \frac{1}{\alpha^2}[\|g + \alpha^2\|\,\|y\| + \|h\|]$ for all $t \in R$.

Hence $\|y\| \leqq (1 - \frac{\varepsilon}{\alpha^2})\|y\| + \frac{\|h\|}{\varepsilon}$ and $\|y\| = \frac{\|h\|}{\varepsilon}$. Since

$$y'(t) = \alpha y(t) - e^{\alpha t}\int_t^{\infty} e^{-\alpha s}[(g(s) + \alpha^2)y(s) + h(s)]ds \quad \text{for all } t \in R \text{ and } \alpha > 0$$

and

$$y'(t) = \alpha y(t) + e^{\alpha t}\int_{-\infty}^{t} e^{-\alpha s}[(g(s) + \alpha^2)y(s) + h(s)]ds \quad \text{for all } t \in R \text{ and } \alpha < 0,$$

we have

$$|y'(t)| \leqq |\alpha| \, \|y\| + \frac{1}{|\alpha|}\left[\|g+\alpha^2\| \, \|y\| + \|h\| \right]$$

for all $t \in R$,

consequently,

$$\|y\| \leqq \frac{2|\alpha| \, \|h\|}{\varepsilon}$$

3. Bounded solutions of (1)

Say that Q, F satisfy the assumption (A) if:

There are positive constants k, ε such that $-\varepsilon \geqq$ $\geqq Q[y,y'](t) \geqq -2\alpha^2 + \varepsilon$ for all $t \in R$ and $y \in \{y \in C^1(R),$ $\|y\| \leqq \frac{k}{\varepsilon}, \ \|y'\| \leqq \frac{2|\alpha|k}{\varepsilon}\}$ and $\sup\{\|F[y,y']\|;$ $y \in C^1(R), \ \|y\| \leqq \frac{k}{\varepsilon}, \ \|y'\| \leqq \frac{2|\alpha|k}{\varepsilon}\} \leqq k$. (A)

Theorem 1. Let assumption (A) be fulfilled. Then equation (1) admits a bounded solution y and the inequalities

$$\|y\| \leqq \frac{k}{\varepsilon}, \quad \|y'\| \leqq \frac{2|\alpha|k}{\varepsilon} \tag{6}$$

hold.

Proof. Assume $\alpha > 0$. Let Z be the Fréchet space of C^1-functions on R with the topology of local uniform convergence on R of functions and their derivatives. Setting $K = \{y; \ y \in Z,$ $\|y\| \leqq \frac{k}{\varepsilon}, \ \|y'\| \leqq \frac{2|\alpha|k}{\varepsilon}\}$ then K is a bounded closed convex subset of Z. Let $\varphi \in K$ and consider the differential equation

$$y'' = 2\alpha y' + Q[\varphi, \varphi'](t)y + F[\varphi, \varphi'](t). \tag{7}$$

By Lemma 2 there is the unique bounded solution y of (7) and Lemma 3 implies $y \in K$. Putting $T(\varphi) = y$ we obtain an operator $T: K \longrightarrow K$. To prove T is continuous operator suppose $\{y_n\} \subset K$ is a convergent sequence and $\lim_{n\to\infty} y_n = y$ that is $\lim_{n\to\infty} y_n^{(i)}(t) =$ $= y^{(i)}(t)$ locally uniformly on R for $i = 0,1$. Let $z_n = T(y_n)$ and $z = T(y)$. Then

$$z_n(t) = e^{\alpha t}\int_t^{\infty}\int_s^{\infty} e^{-\alpha\nu}[(Q[y_n,y_n'](\nu)+\alpha^2)z_n(\nu)+F[y_n,y_n'](\nu)]\,d\nu\,ds$$

and

$$z(t) = e^{\alpha t} \int_t^{\infty} \int_s^{\infty} e^{-\alpha \nu}[(Q[y,y'](\nu) + \alpha^2) z(\nu) + F[y,y'](\nu)] d\nu \, ds$$

for all $t \in R$ and $n \in N$. Using the equalities $z_n''(t) = 2\alpha z_n'(t) + Q[y_n, y_n'](t) z_n(t) + F[y_n, y_n'](t)$ we have $\|z_n''\| \leq \frac{6\alpha^2 k}{\varepsilon}$ for all $n \in N$. Therefore the Ascoli theorem implies that from every subsequence $\{\bar{z}_n\}$ of $\{z_n\}$ one may select a convergent (in K) subsequence $\{\bar{\bar{z}}_n\}$ such that $\{\bar{\bar{z}}_n(t)\}$ and $\{\bar{\bar{z}}_n'(t)\}$ are locally uniformly convergent on R. Let $\lim_{n\to\infty} \bar{\bar{z}}_n = \hat{z}$. Using the Lebesgue theorem on the dominanted convergence we conclude $\hat{z}$ satisfies the equality

$$\hat{z}(t) = e^{\alpha t} \int_t^{\infty} \int_s^{\infty} e^{-\alpha \nu}[(Q[y,y'](\nu) + \alpha^2) z(\nu) + F[y,y'](\nu)] d\nu \, ds$$

for all $t \in R$. Consequently, $\hat{z}$ is a bounded solution of the differential equation

$$w'' = 2\alpha w' + Q[y,y'](t) w + F[y,y'](t) .$$

Since this equation admits the unique bounded solution (by Lemma 2) it is necessary $z = \hat{z}$ and therefore all selected convergent subsequences of $\{z_n\}$ have the same limit equal to z. This proves $\{z_n\}$ is a convergent sequence, $\lim_{n\to\infty} z_n = z$ and, consequently, T is a continuous operator.

Since $T(K) \subset \{y;\ y \in K \cap C^2(R),\ \|y''\| \leq \frac{6\alpha^2 k}{\varepsilon}\}$, T(K) is a precompact subset of Z and by the Schauder-Tychonoff fixed point theorem there is a fixed point y of T in K. This y is a solution of (1) satisfying (6).

For $\alpha < 0$ the proof is analogical.

From Theorem 1 immediately follows

<u>Corollary 1</u>. Suppose there are positive constants ε, k such that $-\varepsilon \geq q(t,y,z) \geq -2\alpha^2 + \varepsilon$ and $|f(t,y,z)| \leq k$ for all $t \in R$, $|y| \leq \frac{k}{\varepsilon}$ and $|z| = \frac{2|\alpha|k}{\varepsilon}$. Then equation $y'' = 2\alpha y' + q(t,y,y')y + f(t,y,y')$ admits a solution y satisfying (6).

Example 1. Let n be a positive integer and let $\alpha \geq \widetilde{\pi} + \sqrt{1+\pi}$. Consider the functional differential equation

$$y''(t) = 2\alpha y'(t) - (1 + 2\alpha\pi + \int_t^{t^2} \frac{y(k_0(s))|y'(s)|}{1+s^2}\,ds)y(t) +$$

$$+ \frac{1}{2}\int_t^{t+1} y^n[y'(k_1(s)) + s]ds + p(t), \qquad (8)$$

where $k_0, k_1, p \in C^0(R)$ and $|p(t)| \leq \frac{1}{2}$ for all $t \in R$. The assumptions of Theorem 1 are fulfilled with $\varepsilon = k = 1$ consequently, there is a solution y of (8) satisfying $\|y\| \leq 1$, $\|y'\| \leq 2\alpha$.

Corollary 2. Let assumption (A) be fulfilled and let $\lim_{t\to\nu\infty} Q[y,y'](t) = -\alpha^2$, $\lim_{t\to\nu\infty} F[y,y'](t) = 0$ for all $y, y' \in X_B$, $\|y\| \leq \frac{k}{\varepsilon}$, $\|y'\| \leq \frac{2|\alpha|k}{\varepsilon}$ and some $\nu \in \{-1, 1\}$. Then $\lim_{t\to\nu\infty} y^{(i)}(t) = 0$ for every solution y of (1) satisfying (6), $i = 0, 1$.

Proof. Let y be a solution of (1) satisfying (6). Such a solution y exists by Theorem 1 and by Lemma 1 the equality (3) or (4) holds with $g(t) = Q[y,y'](t)$, $h(t) = F[y,y'](t)$ for all $t \in R$. Using the L'Hospital rule we obtain

$$\lim_{t\to\nu\infty} y(t) = \lim_{t\to\nu\infty} \frac{1}{\alpha^2}[(Q[y,y'](t) + \alpha^2)y(t) + F[y,y'](t)] = 0$$

and

$$\lim_{t\to\nu\infty} y'(t) = \alpha \lim_{t\to\nu\infty} y(t) + \frac{1}{|\alpha|}\lim_{t\to\nu\infty} [(Q[y,y'](t) + \alpha^2)y(t) +$$

$$+ F[y,y'](t)] = 0.$$

Example 2. Let n be a positive integer. Consider the functional differential equation

$$y''(t) = -2y'(t) + (-1 + \frac{[y'(y(t)+t)]^n}{2(1+t^2)}\,y(t) + \frac{1}{4}e^{-t^2}\cos y(t). \qquad (9)$$

The assumptions of Corollary 2 are fulfilled with $\alpha = -1$,

$\mathcal{E} = \frac{1}{2}$, $k = \frac{1}{4}$, $\nu = -1, 1$. Thus there is a solution y of (9),

$\|y\| \leq \frac{1}{2}$, $\|y'\| \leq 1$ and any such solution y satisfies
$\lim_{|t| \to \infty} y(t) = 0$.

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