Irena Rachůnková

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AN EXISTENCE THEOREM OF THE LERAY-SCHAUDER TYPE FOR FOUR-POINT BOUNDARY VALUE PROBLEMS

IRENA RACHŮNKOVÁ

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Abstract: An existence theorem of the Leray-Schauder type for the problem $z'' = g(t,z,z')$, $z(c) - z(a) = A$, $z(b) - z(d) = B$, where $a,b,c,d,A,B \in (-\infty,\infty)$, $a < c \leq d < b$, is proved.

Key words: Four-point BVPs at resonance, Carathéodory conditions, Fredholm mapping of index zero, L-compact mapping, the Brouwer degree.

MS Classification: 34B10, 34B15

1. The existence theorems of the Leray-Schauder type have been proved for the Picard and periodic problems for example in [1,2]. Here, we shall prove such a theorem for the following four-point problem at resonance

$$z'' = g(t,z,z') \tag{1.1}$$

$$z(c) - z(a) = A, \quad z(b) - z(d) = B, \tag{1.2}$$

where $a,b,c,d,A,B \in (-\infty,\infty)$ $(=R)$, $a < c \leq d < b$, and g satisfies the local Carathéodory conditions on $[a,b] \times R^2$.

The questions of existence and uniqueness of the solutions of problem (1.1), (1.2) were studied in [3-5] and various effective conditions were found. The proofs were based on the Schauder fixed-point theorem and a priori estimates there.

The Leray-Schauder type theorem which is proved here enables to obtain further effective existence conditions. Using this theorem we do not need to prove a priori estimates for the solutions of (1.1), (1.2).

Let $g_0(t) = c_1 t^2 + c_2 t$ for each $t \in [a,b]$, where

$c_1 = [B/(b - d) - A/(c - a)]/(b - c + d - a)$,

$c_2 = [A(b+d)/(c-a) - B(c+a)/(b-d)]/(b-c + d-a)$.

Putting

$f(t,x,y) = g(t,x + g_0(t), y + g_0'(t)) - 2c_1$,

$u(t) = z(t) - g_0(t)$,

we get from (1.1), (1.2) the problem

$$u'' = f(t,u,u'), \qquad (1.3)$$

$$u(c) - u(a) = 0, \quad u(b) - u(d) = 0 . \qquad (1.4)$$

So, from now on, we can consider problem (1.3), (1.4).

2. Notations, definitions and auxiliary results

We shall use the terminology from [1,2]. Let X,Y be real vector normed spaces and $\mathrm{dom}L \subset X$ a vector subspace. In what follows

$L: \mathrm{dom}L \to Y$

will be a linear mapping and

$N: X \to Y$

will be a mapping not necessarily linear.

Definition 1. L will be called a Fredholm mapping of index zero iff

(i) $\dim \mathrm{Ker}\, L = \mathrm{codim}\, \mathrm{Im}\, L < +\infty$;

(ii) Im L is closed in Y.

It follows from the definition above and from basic results of linear functional analysis that there exist continuous projectors

$$P: X \to X \quad \text{and} \quad Q: Y \to Y$$

such that

$$\text{Im } P = \text{Ker } L \quad \text{and} \quad \text{Ker } Q = \text{Im } L,$$

so that $X = \text{Ker } L \oplus \text{Ker } P$, $Y = \text{Im } L \oplus \text{Im } Q$ as topological direct sums.

Consequently, the restriction L_p of L to dom $L \cap \text{Ker } P$ is one-to-one and onto Im L, so that its (algebraic) inverse

$$K_p: \text{Im } L \to \text{dom } L \cap \text{Ker } P$$

is defined.

Definition 2. Let L be a Fredholm mapping of index zero and let $\Omega \subset X$ be an open bounded set. A continuous mapping N will be called L-compact on $\bar{\Omega}$ iff the mappings $QN: \bar{\Omega} \to Y$ and $K_p(I-Q)N: \bar{\Omega} \to X$ are compact, i.e. continuous on $\bar{\Omega}$ and such that $QN(\bar{\Omega})$ and $K_p(I-Q)N(\bar{\Omega})$ are relatively compact sets.

One can show that Definition 2 does not depend upon the choice of the continuous projectors P and Q, which justifies the terminology. See [1, p.13].

Since dim Ker L = dim Im $Q < \infty$, there exists an isomorphism

$$J : \text{Im } Q \to \text{Ker } L. \tag{2.1}$$

Let us consider the mappings

$$N^* : \bar{\Omega} \times [0,1] \to Y, \qquad (x,\lambda) \mapsto N^*(x,\lambda)$$

with $N^*(.,1) = N$, and

$$N_o = JQN^*(.,0) : \text{Ker } L \to \text{Ker } L. \tag{2.2}$$

We shall need the following theorem, which is proved in [1,p.29].

Continuation theorem. Let L be a Fredholm mapping of index zero and let $\Omega \subset X$ be an open bounded set. Let N be L-compact on $\bar{\Omega} \times [0,1]$. Suppose

a) for each $\lambda \in (0,1)$, every solution x of
$$Lx = N^*(x,\lambda)$$
is such that $x \notin \partial\Omega$,

b) $QN^*(x,0) \neq 0$ for each $x \in \mathrm{Ker}\, L \cap \partial\Omega$ and

c) the Brouwer degree $d[N_0, \Omega \cap \mathrm{Ker}\, L, 0] \neq 0$.

Then the equation

$Lx = Nx$

has a least one solution in dom $L \cap \overline{\Omega}$.

In what follows

$AC^i(a,b)$ $[C^i(a,b)]$ denotes the set of all real functions having absolutely continuous [continuous] i-th derivatives on $[a,b]$, $i = 0,1$,

$L^p(a,b)$ is the set of all real functions y with $|y|^p$ Lebesgue integrable on $[a,b]$, $p \in [1,\infty)$.

We say that some property is satisfied on $D = [a,b] \times R^2$, if it is satisfied for almost each(=a.e.) $t \in [a,b]$ and for each $x,y \in R$.

We shall suppose that f satisfies the local Careathéodory conditions on D, i.e.

$f(\cdot,x,y)$: $[a,b] \to R$ is Lebesgue measurable on $[a,b]$ for each $x,y \in R$,

$f(t,\cdot,\cdot)$: $R^2 \to R$ is continuous on R^2 for a.e. $t \in [a,b]$ and

$\sup\{|f(t,x,y)| : |x| + |y| \leq \rho\} \in L^1(a,b)$ for each $\rho \in (0,+\infty)$.

We shall write $f \in Car_{loc}(D)$.

By a solution to (1.3), (1.4), we mean a function $x \in AC^1(a,b)$ verifying (1.3) for a.e. $t \in [a,b]$ and (1.4).

Let us denote

$X = C^1(a,b)$ with the C^1-norm $\|x\|_{C^1} = \max_{t\in[a,b]} \{|x(t)| + |x'(t)|\}$,

$Y = L^1(a,b)$ with the L^1-norm $\|y\|_{L^1} = \int_a^b |y(t)|dt$

and

dom $L = \{x \in AC^1(a,b),\ x$ satisfies $(1.4)\}$.

Further, let us define the mappings

$$L: \mathrm{dom}\, L \to Y,\ x \mapsto x'' \tag{2.3}$$

and

$$N: X \to Y,\quad x \mapsto f(\cdot,x(\cdot), x'(\cdot)) . \tag{2.4}$$

The problem (1.3), (1.4) is equivalent to the equation

$$Lx = Nx \,, \tag{2.5}$$

i.e. $x \in \text{dom } L$ satisfies equation (2.5) iff $x \in AC^1(a,b)$ is a solution of (1.3), (1.4).

3. Lemmas

Lemma 1. The linear mapping (2.3) is a Fredholm mapping of index zero.

P r o o f . Ker L consists of all solutions of the homogeneous problem $u'' = 0$, (1.4) and thus Ker L consists of all constant functions on $[a,b]$.and

$$\dim \operatorname{Ker} L = \dim R = 1 \,. \tag{3.1}$$

Im L is the set of all functions $y \in L^1(a,b)$ for which there exist functions $x \in \text{dom } L$ verifying the equation

$$x''(t) = y(t) \tag{3.2}$$

for a.e. $t \in [a,b]$. The solution of (3.2) has the form

$$x(t) = \alpha t + \beta + \int_a^t \int_a^s y(\tau)\, d\tau\, ds \,, \tag{3.3}$$

where $\alpha, \beta \in R$. The condition $x \in \text{dom } L$ implies $x(a) = x(c)$ and $x(b) = x(d)$ and therefore

$$\alpha = -\frac{1}{c-a} \int_a^c \int_a^s y(\tau)\, d\tau\, ds = -\frac{1}{b-d} \int_d^b \int_a^s y(\tau)\, d\tau\, ds \,. \tag{3.4}$$

Let us denote $c_0 = (b+d)/2 - (c+a)/2$ and

$$\hat{y} = \frac{1}{c_0} \left[\frac{1}{b-d} \int_d^b \int_a^s y(\tau)\, d\tau\, ds - \frac{1}{c-a} \int_a^c \int_a^s y(\tau)\, d\tau\, ds \right] . \tag{3.5}$$

We can see that

$$\operatorname{Im} L = \{ y \in L^1(a,b) \,,\ \hat{y} = 0 \} \,. \tag{3.6}$$

If $y \in Y \setminus \operatorname{Im} L$, then $\hat{y} \neq 0$ and $y - \hat{y} \in \operatorname{Im} L$. It means that

$$\dim Y/\operatorname{Im} L = \dim R = 1 \,. \tag{3.7}$$

How, we shall prove that Im L is closed in Y. Let $y_n \in \operatorname{Im} L$ $(\forall n \in N)$ and let there exists $y \in Y$ such that

$$\lim_{n\to\infty} \| y_n - y \|_{L^1} = 0 \,.$$

Then there exists a subsequence $(y_{m_n})_1^\infty$ converging to y for a.e. $t \in [a,b]$. This implies the existence of $h \in L^1(a,b)$ such that

$$|y_{m_n}(t)| < h(t) \quad \text{for a.e.} \quad t \in [a,b] .$$

So, we can use the Lebesgue convergence theorem and get

$$\hat{y} = \frac{1}{c_0}\left[\frac{1}{b-d}\int_d^b\int_a^s y(\tau)d\tau ds - \frac{1}{c-a}\int_a^c\int_a^s y(\tau)d\tau ds\right] =$$

$$= \frac{1}{c_0}\left[\frac{1}{b-d}\int_d^b\int_a^s \lim_{n\to\infty} y_{m_n}(\tau)d\tau ds - \frac{1}{c-a}\int_a^c\int_a^s \lim_{n\to\infty} y_{m_n}(\tau)d\tau ds\right] =$$

$$= \lim_{n\to\infty} \hat{y}_{m_n} = 0 .$$

We have proved $\hat{y} = 0$, i.e. $y \in \operatorname{Im} L$, which completes the proof.

Let us put

$$P: X \to X , \quad x \mapsto x(a)$$
$$Q: Y \to Y , \quad y \mapsto \hat{y} . \tag{3.8}$$

<u>Lemma 2</u>. The mappings P,Q defined in (3.8) are continuous projectors.

P r o o f . From (3.8) it follows that $P^2 = P$, $Q^2 = Q$ and P, Q are linear. We can also see that P, Q are continuous because

$$\|Px_1 - Px_2\|_{C^1} \le \|x_1 - x_2\|_{C^1} \text{ for any } x_1, x_2 \in X \text{ and}$$

$$\|Qy_1 - Qy_2\|_{L^1} \le 2(b-a)|c_0|^{-1}\|y_1 - y_2\|_{L^1} \text{ for any } y_1, y_2 \in Y .$$

<u>Lemma 3</u>. Let $\Omega \subset X$ be an open bounded set. Let Q and N be the mappings (3.8) and (2.4), respectively. The the mapping $QN: \overline{\Omega} \to Y$ is compact.

P r o o f . Since Ω is bounded, there exists $M_1 \in (0,+\infty)$ such that

$$\|x\|_{C^1} < M_1 ,$$

for any $x \in \overline{\Omega}$ and since $f \in Car_{loc}(D)$, there exists $h \in L^1(a,b)$ such that

$$|f(t,x(t),x'(t))| \leq h(t) \quad \text{for a.e.} \quad t \in [a,b] \tag{3.9}$$

and for any $x \in \overline{\Omega}$.

Let us choose arbitrary functions $x_n \in \Omega$ $(n \in N)$ and let there exists $x_0 \in \overline{\Omega}$ such that

$$\lim_{n\to\infty} \|x_n - x_0\|_{C^1} = 0 .$$

Then

$$\lim_{n\to\infty} f(t,x_n(t),x_n'(t)) = f(t,x_0(t),x_0'(t)) \tag{3.10}$$
$$\text{for a.e. } t \in [a,b] .$$

In view of (3.9) and (3.10), using the Lebesgue convergence theorem, we get

$$\lim_{n\to\infty} \int_a^b [f(t,x_n(t),x_n'(t)) - f(t,x_0(t),x_0'(t))]dt =$$

$$= \int_a^b [\lim_{n\to\infty} f(t,x_n(t),x_n'(t)) - f(t,x_0(t),x_0'(t))]dt = 0 ,$$

i.e. $\lim_{n\to\infty} \|Nx_n - Nx_0\|_{L^1} = 0$, which means that N is continuous. From this, by Lemma 2, it follows that QN is continuous.

According to (3.7), dim ImQ = 1 and therefore $QN(\overline{\Omega})$ is relatively compact iff it is bounded in Y. Let us choose an arbitrary $x \in \overline{\Omega}$. Then, by (3.9),

$$\|QNx\|_{L^1} \leq M_2 , \tag{3.11}$$

where $M_2 = \dfrac{2(b-a)}{|c_0|}\displaystyle\int_a^b h(t)dt$. Lemma is proved.

<u>Lemma 4</u>. Let $\Omega \subset X$ be an open bounded set. Let L and N be the mappings (2.3) and (2.4), respectively. Then N is L-compact on $\overline{\Omega}$.

P r o o f . Let P,Q be the mappings (3.8). We shall verify the conditions of Definition 2. In Lemma 1 we have proved that L is a Fredholm mapping of index zero and from Lemma 3 we get the compacteness of QN. Now, we shall find the mapping

$$K_p : \mathrm{Im}L \rightarrow \mathrm{dom}L \cap \mathrm{Ker}P$$

which is the generalized inverse to the restriction $L_p = L/\mathrm{Ker}P$. Clearly $\mathrm{Ker}P = \{x \in X : x(a) = 0\}$. Let us consider the equation (3.2) with the boundary condition

$$x(a) = x(c) = 0 , \qquad x(d) = x(b) . \tag{3.12}$$

Problem (3.2), (3.12) is equivalent to the equation

$$L_p x = y \ , \text{ where } y \in \mathrm{Im}L .$$

According to (3.3), (3.4), (3.5), (3.6), we can get the solution of (3.2), (3.12) in the form

$$x(t) = -\frac{t-a}{c-a}\int_a^c\int_a^s y(\tau)d\tau ds + \int_a^t\int_a^s y(\tau)d\tau ds .$$

Therefore

$$K_p : y \mapsto -\frac{t-a}{c-a}\int_a^c\int_a^s y(\tau)d\tau ds + \int_a^t\int_a^s y(\tau)d\tau ds . \tag{3.13}$$

Since $\|K_p y - K_p z\|_{C^1} = 2(b-a+1)\|y-z\|_{L^1}$ for any $y,z \in \mathrm{Im}L$, K_p is continuous. Since Q and N are continuous (see Lemma 2 and the proof of Lemma 3), the mapping $K_p(I-Q)N$ is continuous as well.

Now, let us show that the functions of $K_p(I-Q)N(\overline{\Omega})$ are equi-bounded in X. Let v be an arbitrary function of $K_p(I-Q)N(\overline{\Omega})$. Then there exists $x \in \overline{\Omega}$ such that $v = K_p(I-Q)Nx$, i.e.

$$v(t) = -\frac{t-a}{c-a}\int_a^c\int_a^s (f(\tau,x(\tau),x'(\tau)) - \hat{f})d\tau ds + \int_a^t\int_a^s (f(\tau,x(\tau),x'(\tau)) - \hat{f})d\tau ds , \tag{3.14}$$

where

$$\hat{f} = \frac{1}{c_0}\left[\frac{1}{b-d}\int_d^b\int_a^s f(\tau,x(\tau),x'(\tau))d\tau ds - \right.$$

$$\left. - \frac{1}{c-a}\int_a^c\int_a^s f(\tau,x(\tau),x'(\tau))d\tau ds\right] .$$

Consequently, in view of (3.9), (3.11), we get

$$\|v\|_{C^1} \leq M_3 , \qquad (3.15)$$

where $M_3 = 2(b-a+1)(\int_a^b h(s)ds + M_2)$.

Further let us show that the functions of $K_p(I-Q)N(\overline{\Omega})$ are equi-continuous in X. Let v have the form (3.14) and $t, s \in [a,b]$. Then

$$|v(t) - v(s)| \leq M_4|t-s| , \qquad (3.16)$$

where $M_4 = 2\int_a^b (h(\tau) + \hat{f})d\tau$, and similarly

$$|v'(t) - v'(s)| \leq |\int_t^s h(\tau)d\tau| + |t-s|M_2/(b-a) . \qquad (3.17)$$

Since $h \in L^1(a,b)$, then for any $\varepsilon > 0$ there exists $\delta_1 > 0$ such that

$$|s-t| < \delta_1 \Rightarrow |\int_t^s h(\tau)d\tau| < \frac{\varepsilon}{3} . \qquad (3.18)$$

Let us choose an arbitrary $\varepsilon \in (0,+\infty)$ and

$$\delta \leq \min\{\delta_1, \varepsilon/3M_4 , \varepsilon(b-a)/3M_2\} .$$

According to (3.16) - (3.18), we get

$$|s-t| < \delta \Rightarrow \|v(s) - v(t)\|_{C^1} < \varepsilon \qquad (3.19)$$

for each $v \in K_p(I-Q)N(\overline{\Omega})$.

From (3.15), (3.19) and the Arzelà-Ascoli theorem it follows that the set $K_p(I-Q)N(\bar{\Omega})$ is relatively compact. This completes the proof.

Lemma 5. Let $\Omega \subset X$ be an open bounded set and let $f^* \in Car_{loc}([a,b] \times R^2 \times [0,1])$. Then the mapping

$$N^*: \bar{\Omega} \times [0,1] \to Y, \quad (x,\lambda) \mapsto f(\cdot,x(\cdot),x'(\cdot),\lambda)$$

is L-compact on $\bar{\Omega} \times [0,1]$.

P r o o f . Lemma 5 can be proved in a similar way as Lemma 4. On the space $X \times [0,1]$ we work with the norm

$$\|(x,\lambda)\| = \|x\|_{C^1} + |\lambda| \quad \text{for } (x,\lambda) \in X \in [0,1] .$$

4. The main result

Let us choose the function

$$f^* \in Car_{loc}([a,b] \times R^2 \times [0,1])$$

such that

$$f^*(t,x,y,1) = f(t,x,y) \quad \text{on} \quad D,$$

and consider the set of the equations

$$u'' = \lambda f^*(t,u,u\ \ ,\lambda), \qquad \lambda \in [0,1]. \qquad (4.1\lambda)$$

Let us put

$$f_0(x) = \frac{1}{b-d}\int_d^b\int_a^s f^*(t,x,0,0)dtds - \frac{1}{c-a}\int_a^c\int_a^s f^*(t,x,0,0)dtds . \quad (4.2)$$

Existence theorem of the Leray-Schauder type. Let there exists an open bounded set $\Omega \subset X$ such that

(a) for any $\lambda \in (0,1)$, every solution u of the problem (4.1λ), (1.4) satisfies $u \notin \partial\Omega$;

(b) for any root $x_0 \in R$ of the equation $f_0(x) = 0$, the condition $x_0 \notin \partial\Omega$ is fulfilled, where x_0 is considered as a constant function $u(t) = x_0$, on $[a,b]$;

(c) the Brouwer degree $d[f_0, \Delta, 0] \neq 0$, where $\Delta \subset R$ is the set of such constants c , that the constant functions $u(t) = c$ belong to Ω .

Then problem (1.3), (1.4) has a least one solution in $\overline{\Omega}$.

P r o o f . From Lemma 1 and Lemma 5 it follows that L and N^* satisfies the conditions of the Continuation theorem. According to (3.8) and (4.2) we have $QN^*(x,0) = f_0(x)$ and in view of (2.1), (2.2) and (3.1), $N_0 = kf_0$, where $k \in R$, $k \neq 0$. Therefore the conditions (a), (b), (c) of the Continuation theorem are satisfied as well, which completes the proof.

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Department of Math.Analysis
Palacký University
Vídeňská 15, 771 46 Olomouc
Czechoslovakia

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