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ON SOME CONTINUITIES OF PHASES THEORY AND WKB METHOD FOR LINEAR DIFFERENTIAL EQUATION OF THE SECOND ORDER

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Abstract: It is found the necessary and sufficient condition for having the first phase of differential equation $y'' + q(t)y = 0$ in the form $\alpha(t) = \varphi[\int^t \sqrt{q(\tau)}d\tau]$, where $\varphi = \varphi(s) \in C^3$, $\varphi'(s) \neq 0$ is a given function. This problem is solved in a connection with the case, when the series in the exponent of the WKB solution of the given equation with a suitable coefficient q(t) reduces into a finite sum.

Key words: The first phase, the formal asymptotic solution, the WKB solution of the 2nd order linear differential equation.

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Introduction

It is known that the solution of the differential equation

$$4\,z\,z'' - 5\,z'^2 = 0 \tag{1}$$

forms the connecting article between some areas of linear differential equation of the second order

$$y'' + q(t)y = 0. \tag{2}$$

So for example in [6] there is shown that just when $q(t)$ is in (2) the positive solution (1) in the interval j, then there exists such base $u_1(t)$, $u_2(t)$ differential equation (2) in j, than the first phase ([1]) $\alpha(t)$ of this base is a primitive function to the function $\sqrt{q(t)}$ in interval j. In [7] there is proved that the differential equation with parameter λ of the form

$$y'' - \lambda^2 f(t)y = 0, \qquad (3)$$

in which the coefficient $f(t)$ is the solution of the equation (1) in j, and it shows that property that the series in exponent of their WKB solutions ([2], [7]) in interval j is reduced into a finite sum.

The solution of the equation (1) is possible to use also in asyptotic integration (for $t \to +\infty$) differential equation (2) in so called elliptic case, as it is made for example in [3], [8]. The expression $\lambda\int^t\sqrt{f(\tau)}d\tau$, respectively $\int^t\sqrt{q(\tau)}d\tau$ appears in the last mentioned cases in the main members of asymptotica.

It is possible naturally to generalize the above mentioned considerations, when we request this expression to be in the form $\varphi[\int^t\sqrt{q(\tau)}d\tau]$, where $\varphi = \varphi(s)$ is the given function which properties will be precised in the next text.

Contribution to the phases theory

Theorem 1. Let it be in the differential equation

$$y'' + q(t)y = 0 \qquad (2)$$

$q(t) > 0$ in interval j. Be $s(t) = \int^t \sqrt{q(\tau)}d\tau$ primitive function to the function $\sqrt{q(t)}$ in j with values in interval j'. Let $\varphi = \varphi(s) \in C^3$, $\dot{\varphi}(s) \neq 0$ be in j. To exist such base $u_1(t)$, $u_2(t)$ equation (2) in j, that the first phase α of this base is in the form

$$\alpha(t) = \varphi[\int^t \sqrt{q(\tau)}d\tau]$$

it is necessary and sufficient, the function $q(t)$ to satisfy in j nonlinear differential equation of the second order

$$4\, z\, z'' - 5\, z'^2 + 16[\{\varphi, s\} + \dot{\varphi}^2(s) - 1]z^3 = 0. \qquad (4)$$

In the equation (4) and it further denotes $\dot{} = \frac{d}{ds}$ and the symbol $\{\varphi, s\} = \frac{1}{2}\frac{\dddot{\varphi}}{\dot{\varphi}} - \frac{3}{4}\frac{\ddot{\varphi}^2}{\dot{\varphi}^2}$ Schwarz´s derivation of the function φ with respect to s.

P r o o f . The function $\alpha(t)$ as the first phase satisfies in j nonlinear differential equation of the third order ([1])

$$-\{\alpha, t\} - \alpha'^2(t) + q(t) = 0. \qquad (5)$$

It holds:

$$\alpha' = \dot{\varphi}\sqrt{q(t)}\ ,$$

$$\alpha'' = \ddot{\varphi}\, q(t) + \dot{\varphi}\frac{q'(t)}{2\sqrt{q(t)}}\ ,$$

$$\alpha''' = \dddot{\varphi}\, q(t)\sqrt{q(t)} + \frac{3}{2}\ddot{\varphi}\, q'(t) + \dot{\varphi}\left[\frac{1}{2}\frac{q''(t)}{\sqrt{q(t)}} - \frac{1}{4}\frac{q'^2(t)}{q(t)\sqrt{q(t)}}\right],$$

and after substitution to (5) we get identically in j

$$-q(t)\left[\frac{1}{2}\frac{\dddot{\varphi}}{\dot{\varphi}} - \frac{3}{4}\frac{\ddot{\varphi}^2}{\dot{\varphi}^2} + \dot{\varphi}^2\right] - \frac{1}{4}\frac{q''(t)}{q(t)} + \frac{5}{16}\frac{q'^2(t)}{q^2(t)} + q(t) = 0\ ,$$

i.e.

$$4\, q\, q'' - 5\, q'^2 + 16[\{\varphi, s\} + \dot{\varphi}^2(s) - 1]q^3 = 0.$$

2. Let the function q(t) satisfy in j differential equation (4). Then the function

$$\alpha(t) = \varphi\left[\int^t \sqrt{q(\tau)}d\tau\right]$$

satisfies in j nonlinear differential equation (5) and thus it is the phase of the differential equation (2) by the certain base u_1, u_2.

Remarks. 1. If $\varphi(s) \equiv s$ in j´, then the equation (4) is reduced in the equation (1). This case is studied in [6]. The functions $q(t) = (at + b)^{-4}$ satisfy to the conditions of the theorem.

2. Let $q(t) < 0$ be in j and let $s(t) = \int^t \sqrt{-q(\tau)}\,d\tau$ be the primitive function to $\sqrt{-q(t)}$ in j with values in intervaj j'. Be $\Psi = \Psi(c) \in C^3$, $\dot{\Psi}(s) \neq 0$ in j'. If we lay in the preceding theorem $\varphi = i\Psi$, then the necessary and sufficient condition (4) obtains the form

$$4zz'' - 5z'^2 + 16[-\{\Psi, s\} + \dot{\Psi}^2(s) + 1]z^3 = 0. \qquad (4')$$

Really, for $\alpha(t) = i\Psi[\int^t \sqrt{-q(\tau)}\,d\tau]$ gradually we get:

$$\alpha' = i\dot{\Psi}\sqrt{-q(t)} \quad ,$$

$$\alpha'' = -i[\ddot{\Psi}q(t) + \dot{\Psi}\frac{q'(t)}{2\sqrt{-q(t)}}] \;,$$

$$\alpha''' = -i\Big\{\dddot{\Psi}\sqrt{-q(t)}q(t) + \frac{3}{2}\ddot{\Psi}q'(t) + \dot{\Psi}\Big[\frac{q''(t)}{2\sqrt{-q(t)}} + \frac{q'^2(t)}{4\sqrt{(-q(t))^3}}\Big]\Big\} \;,$$

so after substitution to (5) it works out:

$$q[\{\Psi, s\} - \dot{\Psi}^2(s) - 1] - \frac{1}{4}\frac{q''}{q} + \frac{5}{16}\frac{q'^2}{q^2} = 0 \;,$$

what after simple modification gives (4').

<u>On a certain modification of the WKB method</u>

The substance of the classic WKB method consists in the following: there is given the differential equation

$$y'' - [\lambda^2 f(t) + g(t)]y = 0, \qquad (6)$$

in which λ is a (big) parameter, $f(t) \neq 0$ and $g(t)$ are the functions of independent variable t in interval j, which have in j derivations of the arbitrary order. By transformation

$$y = \exp[\int^t u(\tau)\,d\tau]$$

the equation (6) passes in Riccati's differential equation

$$\lambda u' + \lambda^2 u^2 - (\lambda^2 f + g) = 0. \qquad (7)$$

If we solve this equation formally by series of the form

$$u = \sum_{n=0}^{+\infty} u_n(t)\,\lambda^{-n} \quad , \tag{8}$$

for coefficients $u_n(t)$ of the series (8) work out these conditions:

$$\begin{aligned}
u_0 &= f \ , \\
u_1 &= -\frac{u_0'}{2u_0} \ , \\
u_2 &= -\frac{u_1' + u_1^2 - g}{2u_0} \ , \\
&\dots \\
u_{n+1} &= -\frac{u_n' + \sum_{n=0}^{+\infty} u_p\, u_{n+1-p}}{2\,u_0} \qquad \text{for } n \geqq 2.
\end{aligned} \tag{9}$$

Remarks. 1. Supposition about non-vanishing function f guaranties besides the sense of formulae (9) for every $t \in j$ further also that, the equation (6) hasn´t transition points in j.

2. In (7) it is among others proved that if in j identically holds $u_2(t) \equiv 0$, then $u_n(t) = 0$ in j for every $n \geqq 2$ and series (8) is reduced in a finite sum. Besides, there are found here the conditions to vanish the coefficient $u_n(t)$ for $n = 1,2,3,4$ and 5 in interval j.

Following on the preceding paragraph now let us lay in the equation (6)

$$f(t) = \dot{\varphi}^2(s)q(t), \; g(t) = [\dot{\varphi}^2(s) - 1]q(t) \ ,$$

where $s(t) = \int^t \sqrt{q(\tau)}d\tau$, so this equation will obtain the form

$$y'' - [(\lambda^2 + 1)\dot{\varphi}^2(s) - 1]q(t)y = 0. \tag{10}$$

Theorem 2. Be $\varphi = \varphi(s) \in C^3$, $\dot{\varphi}(s) \neq 0$ in interval j', $q(t) \in C^2$ positive function in interval j. Let $s(t) = \int^t \sqrt{q(\tau)}d\tau$ be primitive function to the function $\sqrt{q(t)}$ in j with values in j'.

Let further be u_n, $n = 0, 1, 2, \ldots$ denotes the coefficient of the formal series (8).

If the function $q(t)$ is a solution of nonlinear diferential equation of the second order

$$4\,z\,z'' - 5\,z'^2 + 16\left[\{\varphi, s\} + \dot{\varphi}^2(s) - 1\right]z^3 = 0 \tag{4}$$

in j, then $u_n(t) \equiv 0$ in j for every $n \geq 2$.

P r o o f . It is sufficient to prove ([7]) that in j holds $u_2(t) \equiv 0$. In correspondence with (9) we get gradually:

$$u_0 = \varepsilon\,\dot{\varphi}(s)\sqrt{q(t)}\ , \qquad \varepsilon = \pm 1\ ,$$

$$u_1 = -\frac{1}{2\dot{\varphi}\sqrt{q(t)}}\left[\ddot{\varphi}\,q(t) + \dot{\varphi}\,\frac{q'(t)}{2\sqrt{q(t)}}\right]\ ,$$

$$u_2 = -\frac{1}{2\dot{\varphi}\sqrt{q(t)}}\left\{\frac{5q'^2(t) - 4q(t)q''(t)}{16q^2(t)} - \{\varphi, s\}\,q(t) - \right.$$

$$\left. - \left[\dot{\varphi}^2(s) - 1\right]q(t)\right\} = \frac{1}{32\,\dot{\varphi}\sqrt{q^5(t)}}\left\{4q(t)q''(t) - \right.$$

$$\left. - 5q'^2(t) + 16\left[\{\varphi, s\} + \dot{\varphi}^2(s) - 1\right]q^3(t)\right\}\ ,$$

what proves the assertion of theorem.

Remarks: 1. If $\varphi(s) = s$ in j', then the equation (10) is reduced in the equation (1). In this case are formal asymptotic solutions of the equation (10) in the form

$$y_{1,2}(t,\lambda) = q^{-\frac{1}{4}}(t)\,\exp\left[\pm\lambda\int^{t}\sqrt{q(\tau)}\,d\tau\right]$$

with function $q(t) = (at + b)^{-4}$ and they are the accurate solutions of this equation. Generally, we get on the supposition, above mentioned in theorem 2, the solutions

$$y_{1,2}(t,\lambda) = \left[\dot{\varphi}^2(s)q(t)\right]^{-\frac{1}{4}}\exp\left\{\pm\lambda\,\varphi\left[\int^{t}\sqrt{q(\tau)}\,d\tau\right]\right\}\ . \tag{11}$$

2. If $q(t) < 0$, let us consider instead of (10) the equation

$$y'' - \left[(\lambda^2 - 1)\, \dot{\varphi}^2(s) + 1\right] q(t) y = 0 \ ,$$

which is possible to write in the form

$$y'' + \left[(\lambda^2 - 1)\, \dot{\varphi}^2(s) + 1\right]\left[-q(t)\right] y = 0 \ . \qquad (10')$$

In the equation (10') let us lay

$$s = \int^t \sqrt{-q(\tau)}\, d\tau \ , \qquad \varphi(s) = i\, \psi(s).$$

Then holds $\dot{\varphi}^2(s) = - \dot{\psi}^2(s)$ and the equation (10') obtains the form

$$y'' - \left[(\lambda^2 - 1)\, \dot{\psi}^2(s) - 1\right]\left[-q(t)\right] y = 0 \ . \qquad (12)$$

For its WKB coefficients we get gradually

$$u_0 = \varepsilon\, \dot{\psi}(s) \sqrt{-q(t)} \ ,$$

$$u_1 = -\frac{1}{2}\left[\frac{\ddot{\psi}}{\dot{\psi}} \sqrt{-q(t)} + \frac{q'(t)}{2q(t)}\right] ,$$

$$u_2 = -\frac{1}{2\,\dot{\psi}\sqrt{-q(t)}}\left[\left\{\psi, s\right\} q(t) + \frac{5}{16}\frac{q'^2(t)}{q^2(t)} - \frac{1}{4}\frac{q''(t)}{q(t)} - \right.$$

$$\left. - \dot{\psi}^2 q(t) - q(t)\right] = \frac{1}{32\, \dot{\psi}\, q^2(t)\sqrt{-q(t)}} \left\{4q(t)q''(t) - \right.$$

$$\left. - 5q'^2(t) - 16\left[\left\{\psi, s\right\} - \dot{\psi}^2(s) - 1\right] q^3(t)\right\} .$$

From here we can see that if $q(t)$ satisfies in j the equation

$$4\, z\, z'' - 5\, z'^2 + 16\left[-\left\{\psi, s\right\} + \dot{\psi}^2(s) + 1\right] z^3 = 0, \qquad (4')$$

then is $u_n(t) \equiv 0$ in j for every $n \geq 2$. The relevant formal solutions $y_{1,2}(t, \lambda)$ in this case will be real again and it is possible to obtain them as real, respectively, imaginary part from some of expressions

$$\left\{\dot{\psi}^2(s)\left[-q(t)\right]\right\}^{-\frac{1}{4}} \exp\left\{\pm i\lambda\,\psi\left[\int^t \sqrt{-q(\tau)}\,d\tau\right]\right\} .$$

Examples

1. If $\varphi(s) = \operatorname{arctg} s$, then $\{\varphi,s\} + \dot{\varphi}^2(s) \equiv 0$ in $j' = (-\infty, +\infty)$ and differential equation (4) has in this case the form

$$4zz'' - 5z'^2 - 16z^3 = 0$$

The functions

$$q(t) = \frac{16a^2}{\left[(at+b)^2 - 16\right]^2} , \tag{13}$$

respectively

$$q(t) = \frac{1}{(2t+c)^2} \tag{14}$$

are positive solutions of this equation ([4], 6.162); $a \neq 0$, b, c in (13), (14) are arbitrary constants.

Differential equations (2) with coefficient $q(t)$ in the form (13), (14) satisfy in convenient interval j to suppositions of theorems 1 and 2. At the same time the equation (2) with coefficient (14) is itself to itself accompanying with the base $\alpha = 0$, $\beta \neq 0$ ([5]).

2. We reach the some result as in the case 1 if we choose $\varphi(s) = \operatorname{arctg} \frac{as+b}{cs+d}$, $ad - bc > 0$. For $s \in j'$, $-\frac{d}{c} \notin j'$ we get $\{\varphi,s\} + \dot{\varphi}^2(s) = 0$ as well.

3. If $\psi(s) = \frac{1}{2} \ln \frac{1+s}{1-s}$ in interval $j' \subset (-1,1)$, then here $-\{\psi,s\} + \dot{\psi}^2(s) \equiv 0$ and the equation (4′) has in this case the form

$$4zz'' - 5z'^2 + 16z^3 = 0 .$$

Negative solutions of this equation are ([4], 6.162) functions

$$q(t) = - \frac{16a^2}{[(at + b)^2 + 16]^2} .$$

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