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ON THE EXISTENCE OF SOLUTIONS OF OPERATOR-DIFFERENTIAL EQUATIONS

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Abstract: In this paper there is investigated the existence of solutions of operator-differential equations.

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1. Introduction

Let R^n be the n-dimensional vector space with a norm $\|.\|$, $R = (-\infty, \infty)$, $R_+ = [0, \infty)$. Let $C_{loc}(R_+, R^n)$ denote the space of continuous functions $u: R_+ \to R^n$ with the topology of locally uniform convergence on R_+ and let $L_{loc}(R_+, R^n)$ be the space of locally Lebesque integrable functions $u: R_+ \to R^n$ with the topology of convergence in the mean on every compact subinterval of R_+. Let $T: C_{loc}(R_+, R^n) \to L_{loc}(R_+, R^n)$ be a continuous operator of volterra type. Let $A: R_+ \to R^{n \times n}$ be a locally integrable matrix function.

Throughout the paper the vertical bars $|.|$ denote vectors or matrices formed from absolute values of their components. Further we put $u \leq v$ ($U \leq V$) if for the corresponding components the inequality $u_i \leq v_i$ ($U_{ij} \leq V_{ij}$) is valid and in this sence we also understand the monotonicity of vector or matrix functions.

We consider an operator-differential equation of the form

$$y'(t) = A(t)y(t) + T(y)(t) \qquad (1.1)$$

and the corresponding unperturbed linear equation

$$x'(t) = A(t)x(t) \,. \qquad (1.2)$$

By a solution of (1.1) we understand any function $y: [0, t^*) \to R^n$ which is locally absolutely continuous on $[0, t^*)$ and satisfies (1.1) almost everywhere on $[0, t^*)$ and which is maximally extended to the right.

Let $X(t,s)$ be the Cauchy matrix for the equation (1.2) such that $X(t,t)$ is the identity matrix.

It is well-known that (1.1) is almost everywhere on the existence interval $[0, t^*)$ equivalent to the integral equation

$$y(t) = x(t) + \int_0^t X(t,s)T(y)(s)ds \,, \quad t \in [0, t^*) \,, \qquad (1.3)$$

where x is a solution of (1.2).

Define on the space $C_{loc}(R_+, R^n)$ the successive approximations

$$\{u_k\}_{k=1}^{\infty} \qquad (1.4)$$

by

$$\begin{aligned} u_0(t) &= x(t) \\ u_k(t) &= x(t) + \int_0^t X(t,s)T(u_{k-1})(s)ds \,, \quad k = 1,2,\ldots, \\ & \qquad t \in R_+ \,. \end{aligned} \qquad (1.5)$$

In this paper we provide sufficient conditions for the existence solutions of (1.1) on R_+. These results generalize the results of [1], [2] and [3].

2. Results.

Theorem 2.1. Let the following assumptions hold:

1. $X(t,s) \geq 0$, $s \leq t$, $s,t \in R_+$,
2. the operator T is monotone and nonnegative on $C_{loc}(R_+,R^n)$,
3. for every constant vector $a > 0$ and every $t \in R_+$ the following inequality is fulfilled

$$\int_0^t X(t,s)T(a)ds \leq \frac{a}{2} . \tag{2.1}$$

Then for every bounded solution x of (1.2) there exists a solution y of (1.1) on R_+ which is a locally uniform limit of the nondecreasing sequence (1.5) on R_+.

P r o o f . Let x be bounded solution of (1.2) on R_+. Denote $b = \sup_{t \in R_+} |x(t)|$. From (1.5) it follows that the functions $u_k(t)$ are defined and continuous on R_+ for every $k = 0,1,2,\ldots$. With respect to the assumptions of Theorem 2.1 and (2.1), from (1.5) by using the principle of mathematical induction, we obtain

$$-b \leq u_{k-1}(t) \leq u_k(t) \leq 2b , \quad k = 1,2,\ldots, \quad t \in R_+ . \tag{2.2}$$

Further the sequence (1.4) is nondecreasing and bounded on R_+. Therefore there exists $\lim_{k \to \infty} u_k(t) = u(t)$ for which

$$|u(t)| \leq 2b , \quad t \in R_+ .$$

With functions $u_k(t)$ fulfilling (2.2) the functions

$$X(t,s)T(u_k)(s) \tag{2.3}$$

for any fixed $t \in R_+$, are uniformly bounded for $0 \leq s \leq t$. By Lebesque´s dominanted convergence theorem it follows that

$$u(t) = x(t) + \int_0^t X(t,s)T(u)(s)ds , \quad t \in R_+ .$$

From the last equality we have that the function u is continuous and there exists such a function y that $u(t) = y(t)$, $t \in R_+$, and the inequality

$$|y(t)| \leq 2b$$

is true.

The function y fulfils the relation

$$y(t) = x(t) + \int_0^t X(t,s)T(y)(s)ds , \qquad t \in R_+ .$$

Therefore y is the solution of (1.1) almost everywhere on R_+. By Dini's theorem the sequence (1.4) is uniformly convergent to y on every compact subinterval from R_+. Thus the theorem is proved.

Theorem 2.2. Let the assumptions of Theorem 2.1 hold and let the solution of (1.1) be uniquely determined by the initial condition

$$y(0+) = y_0 . \tag{2.4}$$

Then for every solution x of (1.2) with $x(0+) = x_0$ there exists a unique solution y of initial problem (1.1), (2.4) on R_+ with $y(0+) = x(0+) = x_0$.

P r o o f . Choose a sequence of compact intervals $\{I_k\}_{k=1}^{\infty}$ so that $\bigcup_{k=1}^{\infty} I_k = R_+$ and for any $k \in N$, $I_k \subset I_{k+1}$ is true.

Consider an arbitrary interval I_k. Since the solution x of (1.2) is bounded on I_k, we can repeat the whole proof of Theorem 2.1 only with one exception: consideration will be carried out on the interval I_k and not on R_+. Thus we obtain the existence of the solution y_k of the initial problem (1.1), (2.4) on I_k. With respect to uniqueness of the problem (1.1), (2.4), $y_k(t) = y_p(t)$ for any $t \in I_k$, $p > k$. Therefore y defined on R_+ by relation

$$y(t) = y_k(t) , \qquad t \in I_k, \qquad k = 1,2,\dots ,$$

is already a solution of the initial problem (1.1), (2.4) on the whole interval.

Theorem 2.3. Let the following assumptions hold:

1. there exists locally integrable matrix functions M,N: $R_+ \longrightarrow R_+^{n\times n}$ such that
$$|X(t,s)| \leq M(t)N(s)\,, \quad t \geq s\,, \quad t,s \in R_+\,, \qquad (2.5)$$

2. there exists a function $\omega : R_+ \times R^n \longrightarrow R^n$ nondecreasing in the second argument for every fixed $t \in R_+$ and
$$N(t)|T(u)(t)| \leq \omega(t,|u(t)|) \quad \text{a.e.} \quad t \in R_+\,, \qquad (2.6)$$

3. there exists positive constant vectors q,r such that $q - r > 0$ and
$$\int_0^{\infty} \omega(t,M(t)N(0)q)dt < N(0)[q-r]\,. \qquad (2.7)$$

Then for every solution x of (1.2) with
$$|x(0+)| = |x_0| = r \qquad (2.8)$$
there exists a solutions y of (1.1) on R_+ such that
$$|y(t)| \leq M(t)N(0)q \qquad (2.9)$$
is true for any $t \in R_+$.

P r o o f . Let x be an arbitrary solution of (1.2) such that (2.8) is true. Let y be a solution of (1.1) with its existence interval $[0,t^*)$. Then from (1.3) with regard to the assumptions of Theorem 2.3 we get

$$|y(t)| \leq |x(t)| + \int_0^t |X(t,s)||T(y)(s)|ds \leq |X(t,0)x(0)| +$$
$$+ \int_0^t M(t)N(s)|T(y)(s)|ds \leq M(t)N(0)|x_0| +$$
$$+ M(t)\int_0^t N(s)|T(y)(s)|ds \leq M(t)\Big\{N(0)|x_0| +$$
$$+ \int_0^t \omega(s,|y(s)|)ds\Big\}\,, \text{ for } t \in [0,t^*)\,. \qquad (2.10)$$

Define

$$u(t) = N(0)|x_0| + \int_0^t \omega(s,|y(s)|)ds , \quad t\in[0,t^*) . \qquad (2.11)$$

Then we can write (2.10) in the form

$$|y(t)| \leq M(t)u(t) , \qquad t\in[0,t^*) . \qquad (2.12)$$

Using (2.12), we obtain from (2.11) the inequality

$$u(t) \leq N(0)|x_0| + \int_0^t \omega(s,M(s)u(s))ds , \quad t\in[0,t^*) . \qquad (2.13)$$

We will show that the inequality $u(t) \leq N(0)q$ is true for every $t\in[0,t^*)$. We prove it in inderect way.

If $t = 0$, then $u(0) \leq N(0)q$. With respect to the continuity of u there exists a $\delta > 0$ such that for $t\in[0,\delta)$, $u(t) \leq N(0)q$, holds. Let $t_0\in[0,t^*)$ be the first point on the right from 0 such that $u(t_0) = N(0)q$. Then for $t\in[0,t_0)$ from (2.12) we get

$$|y(t)| \leq M(t)u(t) \leq M(t)N(0)q .$$

Thus, from (2.13) we have

$$N(0)q = u(t_0) \leq N(0)|x_0| + \int_0^t \omega(s,M(s)u(s))ds \leq N(0)|x_0| +$$

$$+ \int_0^t \omega(s,M(s)N(0)q)ds < N(0)|x_0| + N(0)\left[q - |x_0|\right] =$$

$$= N(0)q .$$

This is a contradiction. In this way we have proved that $u(t) \leq$ $\leq N(0)q$ for $t\in[0,t^*)$. With respect to (2.12) we obtain (2.9) on the interval $[0,t^*)$. Since (2.9) means the boundedness of a solution of (1.1) on $[0,t^*)$, we have that $t^* = +\infty$. Thus the proof is complete.

Next we shall consider a functional differential equation

$$y'(t) = A(t)y(t) + f(t,\sigma(y;h(t))) \qquad (2.14)$$

where A has the same meaning as above, $f: R_+ \times R^n \to R^n$ fulfils local Carathéodory conditions and σ is an operator defined by

$$\sigma(u;t) = \begin{cases} u(t) & \text{for } t \in R_+ \\ 0 & \text{for } t < 0 , \end{cases}$$

where $h: R_+ \to R$ is a continuous function such that $h(t) \leq t$ for $t \in R_+$.

Corollary 2.1. Let the following assumptions hold:

1. $X(t,s) \geq 0$, $s \leq t$, $s,t \in R_+$,
2. $f \geq 0$ for every $(t,u) \in R_+ \times R^n$ and f is nondecreasing in the second argument for every fixed $t \in R_+$,
3. for every constant vector $a > 0$ and every $t \in R_+$ the inequality

$$\int_0^t X(t,s)f(s,a)ds \leq \frac{a}{2}$$

holds.

Then for every bounded solution x of (1.2) there exists a solution y of (2.14) on R_+ which is a locally uniform limit of the nondecreasing sequence

$$u_0(t) = x(t)$$

$$u_k(t) = x(t) + \int_0^t X(t,s)f(s,\sigma(u_{k-1};h(s)))ds,$$

$$k = 1,2,\ldots, \quad t \in R_+ .$$

Corollary 2.2. Let the hypotheses of Theorem 2.3 be satisfied, except of (2.6). Let instead of (2.6)

$$N(t)f(t,u) \leq \omega(t,|u|) \quad \text{for a.e.} \quad t \in R_+$$

hold. Then for every x of (1.2) with $|x(0)| = r < q$ there exists a solution y of (2.14) on R_+ such that $|y(t)| \leq M(t)N(0)q$ is true for each $t \in R_+$.

REFERENCES

[1] F u t á k, J.: Existence and boundedness of solutions of the n-th order nonlinear differential equation with delay. Práce a štúdie VŠDS, Žilina, 4 (1981), 7-20.

[2] F u t á k, J.: On the asymptotic behaviour of the solutions of operator-differential equations. Fasciculi Math., Posnaniae (to appear).

[3] R o s a, V.: Existence theorems for initial value problems with nonlinear differential system with delay. Acta Math. Univ.Comen., XL-XLI (1982), 51-58.

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