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ON THE POSITION OF NODES OF ASSOCIATED EQUATIONS TO THE DIFFERENTIAL EQUATION y'' - q(t)y = r(t)

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Abstract: The present paper investigates the position of nodes of the first kind of the associated equations of constant bases to a given nonhomogeneous linear second order differential equation and this on the ground of the properties of the zeros of solutions of the respective oscillatory homogeneous equations.

Key words: N-th central dispersion of the 1st kind, node system, associated equation, separating of nodes on a curve.

MS Classification: 34C20

1. Introduction

We consider a differential equation of the second order

$$y'' - q(t)y = r(t), \qquad q,r \in C^0(j), \tag{r}$$

where $j = (a,b)(-\infty \leq a < b \leq \infty)$. The respective homogeneous differential equation of Jacobian form will be always understood to be oscillatory on j, i.e. both end points of the interval j are cluster points of any solution of equation (q). Trivial solutions of (q) will not be considered.

Let R denote the set of all real numbers, and (r), (q) stand conveniently for either a given equation or a set of solutions of that equation.

By means of the n-th central dispersion of the 1st kind $\varphi_n(t)$ of (q), introduced by O. Borůvka [1], M.Laitoch [5] defined a node system of the 1st kind belonging to (r) and to an initial condition. The node system of the 1st kind enables to modify some theorems concerning the solutions of (q) and also those of (r) (see [6], [7]).

2. Node systems of the 1st kind of (r)

Convention 1. Let $t_0 \in j, z_0, z_0' \in R$ be arbitrary numbers and $z \in (r)$ throughout be a solution, for which $z(t_0) = z_0$, $z'(t_0) = z_0'$. Let φ_n denote the n-th central dispersion of the 1st kind of (q), where $n = 0, \pm 1, \pm 2, \ldots$. Further let $S(r;t_0,z(t_0))$ or briefly S always denote a node system of the 1st kind belonging to the differential equation (r) and to the initial condition $[t_0,z_0]$, i.e. the set of all points $[\varphi_n(t_0), z(\varphi_n(t_0))]$ for n being an integer (see [5]).

Remark 1. We know from [5] that the node system of the 1st kind $S(r;t_0,z(t_0))$ is uniquely determined by anyone of its points. Here it would be well to recall the definition of the bundle of solutions of the 1st kind and the concept of the neighbouring nodes of the 1st kind:

By a bundle of solutions of the 1st kind belonging to (r) and to the initial condition $[t_0,z_0]$ (see [5]) we mean all solutions $y \in (r)$ satisfying the condition $y(t_0) = z_0$ which we write as $S(r;t_0,z(t_0))$, i.e. like the node system which all solutions are passing through. We write $y \in S(r;t_0,z(t_0))$.

Suppose $t_0, t_1 \in j$, $z \in (r)$. The points $[t_0,z(t_0)]$, $[t_1,z(t_1)] \in S(r;t_0,z(t_0))$ will be called the neighbouring nodes of the 1st kind belonging to (r) and to the initial condition $[t_0,z(t_0)]$ if the numbers t_0 and t_1 are neighbouring conjugate numbers of the 1st kind belonging to (q) (i.e. with $t_0 < t_1$ and

$t_1 = \varphi(t_0)$, where φ denotes the fundamental dispersion of the 1st kind belonging to (q)).

Theorem 1. Given a node system of the 1st kind $S(r;t_0,z(t_0))$ (belonging to (r) and to the initial condition $[t_0,z(t_0)]$). If $\bar{y}\in(r)$ is a solution not passing through these nodes, then in the set of nodes $S(r;x,z(x))$, where $x\in(t_0,\varphi(t_0))$, there exists exactly one node system of the 1st kind $\bar{S}(r;x_0,z(x_0))$ $(x_0\in(t_0,\varphi(t_0)))$ so that $\bar{y}\in\bar{S}(r;x_0,z(x_0))$.

P r o o f . This will be performed using the method of [5]. Let us consider two neighbouring nodes of S. From our assumption it then follows that $z(t_0) \neq \bar{y}(t_0)$ and $z(\varphi(t_0)) \neq \bar{y}(\varphi(t_0))$. Consequently the function $v(t):= z(t) - \bar{y}(t)$, $v\in(q)$ is such that $v(t_0) \neq 0$ and $v(\varphi(t_0)) \neq 0$. By appealing to Sturm´s theorem (see [8] p.276) the solution v possesses exactly one zero in the interval $(t_0,\varphi(t_0))$, which we write as x_0. Thus $0 = v(x_0) = z(x_0) - \bar{y}(x_0)$, whence $z(x_0) = \bar{y}(x_0)$. Therefore the point $[x_0,z(x_0)] = [x_0,\bar{y}(x_0)]$ is the only common point of solutions $z,\bar{y}$ on the interval $(t_0,\varphi(t_0))$ and the node system of the 1st kind $\bar{S}(r;x_0,z(x_0))$ $(x_0\in(t_0,\varphi(t_0)))$ (i.e. the common bundle of solutions of the 1st kind with the solution $z,\bar{y}$ too) is uniquely determined by this point.

Remark 2. From the above proof it becomes obvious that on the basis of the properties of the function φ we may also write $t_0\in(\varphi_{-1}(x_0),x_0)$.

Definition 1. Suppose that we are given the node systems of the 1st kind $S(r;t_0,z(t_0))$ and $\bar{S}(r;x_0,z(x_0))$, where $t_0 < x_0 < \varphi(t_0)$ treated in the foregoing Theorem 1. We say that the nodes of the node systems of the 1st kind S and $\bar{S}$ become separated on the curve $z(t)\in(r)$.

3. Node systems of the 1st kind of the associated equation of a constand basis

M.Laitoch [4] defined the associated equation

$$y'' = Q_1(t)y \qquad (Q_1)$$

of the basis (α, β) to a linear second order differential equation

$$y'' = q(t)y, \quad q \in C^2(j), q(t) < 0 \quad \text{for} \quad t \in j, \tag{q}$$

where $j = (a,b)$ $(-\infty \leq a < b \leq \infty)$, $\alpha, \beta \in R$, $\alpha^2 + \beta^2 > 0$. According to [4] the carrier $Q_1(t)$ relative to (Q_1) of the basis (α, β) has the form

$$Q_1 = q + \frac{\alpha\beta q'}{\alpha^2 - \beta^2 q} + \sqrt{\alpha^2 - \beta^2 q}\left(\frac{1}{\sqrt{\alpha^2 - \beta^2 q}}\right)'' . \tag{1}$$

Then between the solutions $u \in (q)$ and $U \in (Q_1)$ there exists a one-to-one mapping given by

$$U = \frac{\alpha u + \beta u'}{\sqrt{\alpha^2 - \beta^2 q}} . \tag{2}$$

Theorem 1 in [3] states that associated equation (Q_1) of the basis (α, β) to equation (q) is oscillatory exactly if (q) is oscillatory.

In [2] there is defined the associated equation

$$y'' - Q_1(t)y = R_1(t) \tag*{$[R_1]$}$$

of the basis (α, β), $\alpha^2 + \beta^2 > 0$ to the differential equation

$$y'' - q(t)y = r(t), \; q \in C^2(j), q(t) < 0 \text{ for } t \in j, \; r \in C^1(j), \tag*{[r]}$$

where the function $Q_1(t)$ is defined by formula (1) and

$$R_1 = \frac{\alpha r + \beta r'}{\sqrt{\alpha^2 - \beta^2 q}} + 2\beta\left(\frac{1}{\sqrt{\alpha^2 - \beta^2 q}}\right)' r . \tag{3}$$

Then from Lemmas 1 and 2 [2] there follows the existence of the one-to-one mapping between the solutions $y \in [r]$ and $Y \in [R_1]$ given by

$$Y = \frac{\alpha y + \beta y'}{\sqrt{\alpha^2 - \beta^2 q}} . \tag{4}$$

The aim of this paper is to investigate the position of the nodes of solutions of the associated equation [r] of the bases (α, β) or (γ, δ), and this on the basis of the properties of zeros of solutions of the corresponding oscillatory homogeneous equations.

Convention 2. Let $\alpha, \beta, \gamma, \delta \in R$, $\alpha^2 + \beta^2 > 0$, $\gamma^2 + \delta^2 > 0$ and $t_0 \in j, z_0, z_0' \in R$ be arbitrary numbers. From now on $z \in [r]$ will be assumed to denote a solution of [r] satisfying the initial conditions $z(t_0) = z_0$, $z'(t_0) = z_0'$. We set

$$Z_1 := \frac{\alpha z + \beta z'}{\sqrt{\alpha^2 - \beta^2 q}}, \qquad Z_2 := \frac{\gamma z + \delta z'}{\sqrt{\gamma^2 - \delta^2 q}}. \tag{5}$$

Obviously Z_1 and Z_2 are the solutions of the associated equations $[R_1]$ and $[R_2]$ to [r] of the bases (α, β) and (γ, δ), respectively, i.e. $Z_1 \in [R_1]$ and $Z_2 \in [R_2]$, where

$$Q_2 = q + \frac{\gamma \delta q'}{\gamma^2 - \delta^2 q} + \sqrt{\gamma^2 - \delta^2 q}\left(\frac{1}{\sqrt{\gamma^2 - \delta^2 q}}\right)'',$$

$$R_2 = \frac{\gamma r + \delta r'}{\sqrt{\gamma^2 - \delta^2 q}} + 2\delta\left(\frac{1}{\sqrt{\gamma^2 - \delta^2 q}}\right)' r .$$

Let the functions $\varphi_n(t)$, ${}^1\phi_n(t)$, ${}^2\phi_n(t)$ denote the n-th central dispersion of the 1st kind relative to the oscillatory equation (q), to its associated equation (Q_1) of the basis (α, β), to its associated equation (Q_2) of the basis (γ, δ), respectively.

Definition 2. We say that the node $[\tau, \eta]$ from the node system of the 1st kind S_1 relative to a nonhomogeneous linear second order differential equation $[r_1]$ lies between two neighbouring nodes $[t_1, y_1]$, $[t_2, y_2]$ from the node system of the 1st kind S_2 relative to $[r_2]$, if $\tau \in (t_1, t_2)$. We say also that the nodes of the node systems S_1 and S_2 become separated if there lies exactly one node from the system $S_2(S_1)$ between any two neighbouring nodes from the system $S_1(S_2)$.

Theorem 2. Let $S(r; t_0, z(t_0))$ be a node system of the 1st kind relative to [r] and $Z_1 \in [R_1]$ be defined by (5). Then there exists a node system of the 1st kind $S_1(R_1; \tau_0, Z_1(\tau_0))$, $\tau_0 \in$

$\in (t_0, \varphi(t_0))$ relative to $[R_1]$, which is the associated equation of the basis (α, β) to $[r]$, such that the nodes of the node systems S and S_1 become separated.

P r o o f .

(I) Suppose $\beta \neq 0$. We have $[t_0, z(t_0)]$, $[\varphi(t_0), z(\varphi(t_0))]$ two neighbouring nodes of the 1st kind from the node system S. Let $y \in [r]$ be such a solution that $y \in S$, $y \neq z$. Setting $u := z - y$, then u is a solution of the oscillatory equation (q) satisfying $u(\varphi_n(t_0)) = u(\varphi_{n+1}(t_0)) = 0$, $u(t) \neq 0$ for $t \in (\varphi_n(t_0), \varphi_{n+1}(t_0))$ and for every $n = 0, \pm 1, \pm 2, \ldots$. We set $U := (\alpha u + \beta u')(\alpha^2 - \beta^2 q)^{-1/2}$. Then $U \in (Q_1)$. We know from Theorem 3 [3] that the zeros of solutions of (q) and (Q_1) with $\beta \neq 0$ become separated. Thus, there exists exactly one number $\tau_0 \in (t_0, \varphi(t_0))$ so that $U(\tau_0) = 0$. From this it immediately follows that the node system of the 1st kind $S_1(R_1; \tau_0, Z_1(\tau_0))$, where Z_1 is given by (5), possesses such a property that the nodes of the system S and S_1 become separated.

(II) Suppose $\beta = 0$. In this special case we have $Q_1 = q$ by (1), $R_1 = \operatorname{sgn}\alpha . r$ by (3), $U = \operatorname{sgn}\alpha . u$ by (2), and $Z_1 = \operatorname{sgn}\alpha . z$ by (5).

(a) If $\operatorname{sgn}\alpha = 1$, then every node system of the 1st kind S_1 - possessing the properties stated in the above Theorem - may be defined as a node system of the 1st kind relative to $[r]$, whose nodes become separated with the nodes of the original node system S on the curve $(Z_1(t) =) z(t) \in [r]$. By Theorem 1 it is sufficient here to take such a solution $\bar{y} \in [r]$, where $\bar{y} \notin S$. Thus we may define

$$S_1 := \bar{S}(r; x_0, z(x_0)), \quad \text{where } (\tau_0 :=) x_0 \in (t_0, \varphi(t_0)). \tag{6}$$

(b) If $\operatorname{sgn}\alpha = -1$, then every node system of the 1st kind S_1 possessing the properties stated in the above Theorem may be defined as a node system of the 1st kind relative to $[-r]$, whose nodes become separated with the nodes of the original node system S and lie on the curve $(Z_1(t) =) - z(t) \in [-r]$. By Theorem 1 it is sufficient here to take the solution $\bar{\bar{y}}$ of equation $[-r]$ such that $\bar{\bar{y}} := -\bar{y}$, where $\bar{y} \in [r]$, $y \notin S$ is exactly that solution considered in the first part (Ia) of the proof. Hence we may define

$$S_1 := \tilde{\tilde{S}}(-r; x_0, -z(x_0)), \quad \text{where } (\tau_0 :=) x_0 \in (t_0, \varphi(t_0)) . \tag{7}$$

Definition 3. Let $S(r; t_0, z(t_0))$ be a node system of the 1st kind relative to [r]. Every node system of the 1st kind $S_1(R_1; \tau_0 . Z_1(\tau_0))$, $\tau_0 \in (t_0, \varphi(t_0))$ from the foregoing Theorem relative to $[R_1]$ will be called the associated node system of the basis (α, β) to the node system S .

Remark 3. From formulas (6) or (7) it becomes apparent what we mean by an associated node system to the given node system of the 1st kind of special bases (α, 0), if sgn α = $= \pm 1$.

Corollary 1. Let $S_1(R_1; \bar{\tau}, Z(\bar{\tau}))$ be a node system of the 1st kind relative to $[R_1]$, which is an associated equation to [r] of the basis (α, β), $Z \in [R_1]$ and $\bar{\tau} \in j$ be an arbitrary number. Let $Z = (\alpha \tilde{z} + \beta \tilde{z}')(\alpha^2 - \beta^2 q)^{-1/2}$, where $\tilde{z}$ is the appropriate solution of [r]. Then there exists a node system of the 1st kind $\tilde{S}(r; \tilde{t}, \tilde{z}(\tilde{t}))$ relative to [r], where $\tilde{t} \in ({}^1\phi_{-1}(\bar{\tau}), \tau)$ such that the nodes of the node systems S_1 and S become separated.

P r o o f . This immediately follows from the relation between the solutions $Z \in [R_1]$ and $\tilde{z} \in [r]$ (see [2]) given by formula (5) and will be carried out completely analogous to that of the foregoing Theorem.

Remark 4. We will now examine more closely the associated equations $[R_1]$ and $[R_2]$ of the bases (α, β) and (γ, δ), respectively, to equation [r] in the case when for the weight constants

$$\alpha \delta - \beta \gamma = 0$$

holds.

Then $Q_1(t) = Q_2(t)$ and there arise two fundamental alternatives.

(I) If $\beta \neq 0$ and $\delta \neq 0$,
then

$$Q_1(t) = Q_2(t) = q + \frac{\mu q}{\mu^2 - q} + \sqrt{\mu^2 - q}\left(\frac{1}{\sqrt{\mu^2 - q}}\right)'',$$

where $\mu := \alpha / \beta = \gamma / \delta$, $\mu \in R$;

(i) if $\operatorname{sgn} \delta = - \operatorname{sgn} \beta$,

then

$$R_1(t) = -R_2(t) = -\operatorname{sgn}\delta\left(\frac{\mu r + r'}{(\mu^2 - q)^{1/2}} + \frac{q' r}{(\mu^2 - q)^{3/2}}\right)$$

or

(ii) if $\operatorname{sgn} \delta = \operatorname{sgn} \beta$,

then

$$R_1(t) = R_2(t) = \frac{\mu r + r'}{(\mu^2 - q)^{1/2}} + \frac{q' r}{(\mu^2 - q)^{3/2}}$$

(II) If $\beta = \delta = 0$ ($\alpha \neq 0$, $\gamma \neq 0$),

then

$$Q_1(t) = Q_2(t) = q(t);$$

(i) $\operatorname{sgn} \gamma = - \operatorname{sgn} \alpha \Longrightarrow R_1(t) = - R_2(t) = - \operatorname{sgn} \gamma . r(t)$

(ii) $\operatorname{sgn} \gamma = \operatorname{sgn} \alpha \Longrightarrow R_1(t) = R_2(t) = r(t)$.

Definition 4. Let S_1 and S_2 be the node systems of the 1st kind relative to some nonhomogeneous linear second order differential equations $[r_1]$ and $[r_2]$, respectively. If for any node $[\tau, \eta] \in S_1$ holds that $[\tau, -\eta] \in S_2$, we say that the nodes of the node systems S_1 and S_2 are symmetric.

Theorem 3. Suppose that we are given a node system of the 1st kind $S(r; t_0, z(t_0))$ relative to $[r]$. Then there exist to S the associated node systems $S_1(R_1; \tau_0, Z_1(\tau_0))$ and $S_2(R_2; \bar{\tau}_0, Z_2(\bar{\tau}_0))$ of the bases (α, β) and (γ, δ), respectively, having the following properties:

If $\alpha\delta - \beta\gamma \neq 0$, then the nodes of the node systems S_1 and S_2 become separated.

If $\alpha\delta - \beta\gamma \neq 0$, then

1. when the conditions (Ii) or (IIi) from Remark 4 are fulfilled then the nodes of the associated node systems S_1 and S_2 are symmetric and

$$S_2 = S(-R_1; \tau_0, - Z_1(\tau_0)), \quad \bar{\tau}_0 = \tau_0;$$

2. when the conditions (Iii) or (IIii) are fulfilled,

then both associated node systems are identical and

$$S_2 = S_1 .$$

P r o o f . It follows from Definition 3 of the associated node systems S_1 and S_2 that $\tau_0, \bar{\tau}_0 \in (t_0, \varphi(t_0))$.

Suppose now that $\alpha\delta - \beta\gamma \neq 0$ and we have $[\tau_0, Z_1(\tau_0)]$, $[{}^1\phi(\tau_0), Z_1({}^1\phi(\tau_0))]$ two neighbouring nodes from S_1. Let $Y_1 \in [R_1]$ be such that $Y_1 \in S_1$, $Y_1 \neq Z_1$. Setting $Z_1 - Y_1 =: U_1$, then U_1 is such a solution of (Q_1) that $U_1({}^1\phi_n(\tau_0)) = U_1({}^1\phi_{n+1}(\tau_0)) = 0$, $U_1(t) \neq 0$ for $t \in ({}^1\phi_n(\tau_0), {}^1\phi_{n+1}(\tau_0))$ and for every $n = 0, \pm 1, \pm 2, \dots$. According to Theorem 4 [3] there exists exactly one number $\bar{\tau}_0 \in (\tau_0, {}^1\phi(\tau_0))$ and a solution $U_2 \in (Q_2)$ such that $U_2(\bar{\tau}_0) = 0$. From this there immediately follows the existence of exactly one node $[\bar{\tau}_0, Z_2(\bar{\tau}_0)] \in S_2$ lying on the curve $Z_2 \in [R_2]$ between two neighbouring nodes from S_1, that were chosen.

Let $\alpha\delta - \beta\gamma = 0$. Then, by Remark 4, $Q_1 = Q_2$. To prove the existence of nodes having the properties required, it is sufficient to find such a solution U_1 of (Q_1) - and thus also of (Q_2) - satisfying the condition $U_1(\tau_0) = 0$, where $\tau_0 \in (t_0, \varphi(t_0))$, whose zeros become separated with the zeros of any solution $u \in (q)$ satisfying the condition $u(t_0) = 0$. The solution U_1 will be obtained as follows.

In case (I) introduced in Remark 4 (where $Q_1 = Q_2 \neq q$) we may set with respect to Theorem 3 [3] $U_1 := (\alpha u + \beta u')(\alpha^2 - \beta^2 q)^{-1/2}$. Then for $U_2 := (\gamma u + \delta u')(\gamma^2 - \delta^2 q)^{-1/2}$ we have $U_2 = \pm U_1$ so that the zeros of solutions $U_2 \in (Q_1)$ are $U_1 \in (Q_1)$ are identical and $\bar{\tau}_0 = \tau_0$ holds, too.

In case (II) introduced in Remark 4 (where $Q_1 = Q_2 = q$) we may set $U_1 := v$, $U_2 := v$, where v is an arbitrary solution of (q) being linearly independent of u.

1. If the conditions (Ii) or (IIi) are fulfilled, then the right sides of the associated equations $[R_1]$ or $[R_2]$ differ in sign only, i.e. $R_2 = -R_1$ and according to (5) $Z_2 = -Z_1$.

2. If the conditions (Iii) or (IIii) are fulfilled, then $R_2 = R_1$ and $Z_2 = Z_1$. From this there immediately follows the statement of the Theorem.

Corollary 2. Given a node system of the 1st kind $S(r;t_0,z(t_0))$ and $\alpha\delta - \beta\gamma = 0$. Using the notation introduced in Theorem 3, then there exist to S associated node systems S_1 and S_2 having the following properties.

1. If (IIi): $\beta = \delta = 0$, $\operatorname{sgn}\gamma = -\operatorname{sgn}\alpha$ holds, then
   (a) with $\operatorname{sgn}\alpha = 1$ (on taking account of (6) and (7)) there holds

$$S_1 = \bar{S}(r;x_0,z(x_0)), S_2 = \bar{\bar{S}}(-r;x_0,-z(x_0)), (\tau_0 = \bar{\tau}_0 =) x_0 \in (t_0,\varphi(t_0));$$

   (b) with $\operatorname{sgn}\alpha = -1$ there holds

$$S_1 = \bar{\bar{S}}(-r;x_0,-z(x_0)), S_2 = \bar{S}(r;x_0,z(x_0)),\ x_0 \in (t_0,\varphi(t_0)) .$$

2. If (IIii): $\beta = \delta = 0$, $\operatorname{sgn}\gamma = \operatorname{sgn}\alpha$ holds, then S_1 and S_2 are identical and

$$S_1 = S_2 = \bar{S}(r;x_0,z(x_0)),\quad x_0 \in (t_0,\varphi(t_0)) .$$

Theorem 4. Suppose the nodes of the node system of the 1st kind $S(r;t_0,z(t_0))$ and $S(r;x_0,z(x_0))$ become separated on the curve $z(t) \in [r]$. Then there exist to S and $\bar{S}$ the associated node systems $S_1(R_1;\tau_0,Z_1(\tau_0))$ and $\bar{S}_2(R_2;\xi_0,Z_2(\xi_0))$ of the bases (α, β) and (γ, δ), respectively, having the following properties.

If $\alpha\delta - \beta\gamma \neq 0$, then there are at most two nodes from $\bar{S}_2(S_1)$ between any two neighbouring nodes from $S_1(\bar{S}_2)$.

If $\alpha\delta - \beta\gamma = 0$, then

1. when the conditions (Ii) or (IIi) from Remark 4 are fulfilled, then the nodes become separated so that between any two neighbouring nodes from $S_1(\bar{S}_2)$ lying on the curve $Z_1 = Z(t)$ $(Z_2 = -Z(t))$ there is exactly one node from $\bar{S}_2(S_1)$ lying on the curve $-Z(t)$ $(Z(t))$;
2. when the conditions (Iii) or (IIii) are fulfilled, then the nodes of the associated systems S_1 and $\bar{S}_2$ become separated on the curve $Z_1(t) = Z_2(t)$.

P r o o f . From the definition of the node systems S and $\bar{S}$ there follows (according to the proof of Theorem 1) the existence of such linear independent solutions $u,v \in (q)$ that their zeros become separated. Suppose $u(\varphi_n(t_0)) = v(\varphi_n(x_0)) = 0$, where

$x_0 \in (t_0, \varphi(t_0))$ for $n = 0, \pm 1, \pm 2, \ldots$. Setting $U := (\alpha u + \beta u')(\alpha^2 - \beta^2 q)^{-1/2}$, $V := (\gamma v + \delta v')(\gamma^2 - \delta^2 q)^{-1/2}$ yields $U \in (Q_1)$ and $V \in (Q_2)$.

If $\alpha\delta - \beta\gamma \neq 0$, then with respect to Theorem 6 [3] there lie between any two neighbouring zeros of solutions $U \in (Q_1)$ or $V \in (Q_2)$ at most two zeros of solutions V or U, respectively. Let us have $[\tau_0, Z_1(\tau_0)]$, $[{}^1\phi(\tau_0), Z_1({}^1\phi(\tau_0))]$, $\tau_0 \in (t_0, \varphi(t_0))$ two neighbouring nodes from S_1. Let $Y_1 \in [R_1]$ be such that $Y_1 \in S_1$, $Y_1 \neq Z_1$. If we set $Z_1 - Y_1 =: U$, then U is such a solution of (Q_1) that $U({}^1\phi_n(\tau_0)) = U({}^1\phi_{n+1}(\tau_0)) = 0$, $U(t) \neq 0$ for $t \in ({}^1\phi_n(\tau_0), {}^1\phi_{n+1}(\tau_0))$ and for any $n = 0, \pm 1, \pm 2, \ldots$. Thus, by Theorem 6 [3] there exist at most two numbers ξ_0, ${}^2\phi(\xi_0) \in (\tau_0, {}^1\phi(\tau_0))$ and a solution $V \in (Q_2)$ such that $V(\xi_0) = V({}^2\phi(\xi_0)) = 0$. From this there immediately follows the existence of at most two neighbouring nodes $[\xi_0, Z_2(\xi_0)]$, $[{}^2\phi(\xi_0), Z_2({}^2\phi(\xi_0))] \in \bar{S}_2$ lying on the curve $Z_2 \in [R_2]$ between two neighbouring nodes S, that were chosen.

Let $\alpha\delta - \beta\gamma = 0$. Then, by Remark 4, $Q_1 = Q_2$. To prove the existence of nodes having the properties required, it is sufficient to a solution $u \in (q)$, $u(t_0) = 0$ to find such a solution U_0 of (Q_1) - and thus also of (Q_2) - satisfying the condition $U_0(\tau_0) = 0$, where $\tau_0 \in (t_0, \varphi(t_0))$, whose zeros become separated with the zeros of the solution u and besides to a solution $v \in (q)$, $v(x_0) = 0$ $(x_0 \in (t_0, \varphi(t_0)))$ to find a solution V_0 of (Q_2) - and thus also of (Q_1) - satisfying the condition $V_0(\xi_0) = 0$, where $\xi_0 \in (x_0, \varphi(x_0))$, whose zeros become separated with the zeros of the solution v, whereby U_0 and V_0 are linearly independent solutions. These solutions will be obtained as follows.

In case (I) introduced in Remark 4 (where $Q_1 = Q_2 \neq q$) we may set with respect to Theorem 6 [3]

$$U_0 := (\alpha u + \beta u')(\alpha^2 - \beta^2 q)^{-1/2}, \quad V_0 := (\gamma v + \delta v')(\gamma^2 - \delta^2 q)^{-1/2} .$$

In case (II) (where $Q_1 = Q_2 = q$) we may set $U_0 := v$, $V_0 := u$. From this and on the basis of the forms of solutions $Z_1 \in [R_1]$ and $Z_2 \in [R_2]$ in (5) corresponding to cases 1 and 2 of this Theorem we obtain the statement of the Theorem.

Corollary 3. Let the nodes of the node system of the 1st kind $S(r;t_0,z(t_0))$ and $\overline{S}(r;x_0,z(x_0))$ become separated on the curve $z(t)\in[r]$ and $\alpha\delta-\beta\gamma=0$. Then, using the notation introduced in Theorem 4, there exist associated node systems S_1 and $\overline{S}_2$ having the following properties.

1. If (IIi): $\beta=\delta=0$, $\operatorname{sgn}\gamma=-\operatorname{sgn}\alpha$ holds, then

   (a) with $\operatorname{sgn}\alpha=1$ (on taking account of (6)) there holds

   $$S_1=\overline{S}(r;x_0,z(x_0)),\quad (\tau_0=)x_0\in(t_0,\varphi(t_0))$$

   and likewise

   $$\overline{S}_2[=S(-r;\varphi(t_0),-z(\varphi(t_0))]=S(-r;t_0,-z(t_0)),$$

   where $\xi_0=\varphi(t_0)\in(x_0,\varphi(x_0))$, meaning as well that the nodes from $\overline{S}_2$ and S are symmetric.

   (b) with $\operatorname{sgn}\alpha=-1$ there holds

   $$S_1=\overline{\overline{S}}(-r;x_0,-z(x_0)),\quad \overline{S}_2=S(r:t_0,z(t_0)).$$

2. If (IIii): $\beta=\delta=0$, $\operatorname{sgn}\gamma=\operatorname{sgn}\alpha$ holds, then

   $$S_1=\overline{S}(r;x_0,z(x_0)),\quad \overline{S}_2=S(r;t_0,z(t_0)).$$

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