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SOME PROPERTIES OF INTERPOLATING QUADRATIC SPLINE

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1. INTRODUCTION

Interpolating cubic splines with breakpoints coinciding with points of interpolation are usually used in applications. In the theory we can find generalizations to splines of odd degrees (see [1], [9]). The purpose of small interest in quadratic splines (and splines of even degrees) on such a set of points can be found in [2]:

- such a spline needn't exist in some cases;
- even in case, when such a spline can be constructed, it has some unpleasant properties (unsymmetry of defining conditions, strong global influence of interpolatory data and boundary conditions).

We can obtain a quadratic spline with better properties - even better than by cubic splines in some sense - by separating breakpoints and points of interpolation of the spline. Such formulation and solution of the problem of existence and uniqueness can be found in [4], [17]. The possibility of choosing the breakpoints of the spline is frequently used in the constructions of shape-preserving splines in the last time (see [13] - [16]).

Let us have the sets of breakpoints x_i, $i = 0(1)n+1$ and points of interpolation t_i, $i = 0(1)n$ for a quadratic spline $S_{21}(x) \equiv S(x)$ on the interval $[a,b]$ with

$$(\Delta x \Delta t): \qquad x_0 \leqq a = t_0 < x_1 < t_1 < x_2 < \dots < x_n < t_n = b \leqq x_{n+1} \ .$$

<u>We define</u> a function $S_{21}(x) = S(x)$ to be quadratic interpolating spline on $(\Delta x \Delta t)$ to the given data g_i, $i = 0(1)n$, if

1^o $S(x) \in C^{(1)}[a,b]$;

2^o $S(x)$ is a quadratic polynomial on each interval $[x_i, x_{i+1}]$, $i = 0(1)n$;

3^o $S(t_i) = g_i$, $i = 0(1)n$.

Using the parameters $m_i = S'(x_i)$, $i = 0(1)n+1$ for the representation of such a spline, we have (see [5], [6])

$$(Sm) \qquad S(x) = g_i + (q-d_i)[m_i + (m_{i+1}-m_i)(q+d_i)/(2h_i)] \qquad \text{for } x \in [x_i, x_{i+1}]$$

with $h_i = x_{i+1}-x_i$, $q = x-x_i$, $d_i = t_i-x_i$.
The continuity conditions for $S(x)$ lead to the following system of linear equations

$$(m) \qquad a_i\, m_{i-1} + b_i\, m_i + c_i\, m_{i+1} = f_i \ , \qquad i = 1(1)n \ ,$$

where
$$a_i = (h_{i-1} - d_{i-1})^2 > 0 \ ,$$
$$b_i = d_i(2h_i - d_i)h_{i-1}/h_i + (h_{i-1}^2 - d_{i-1}^2) > 0 \ ,$$
$$c_i = d_i^2\, h_{i-1}/h_i > 0 \ ,$$
$$f_i = 2h_{i-1}(g_i - g_{i-1}) \ ,$$

for the parameters m_i used in representation (Sm) (see [5],

[6]). Two conditions needed for the unique determination of m_i are usually given through some type of boundary conditions. The algorithms for computing these parameters under various boundary conditions (first or second derivative of $S(x)$, periodicity), the questions of existence are studied in [5] - [7].

In the following we shall study the local (or localizing) properties of $S(x)$ (local influence of the data g_i, boundary conditions), shape preserving properties (preserving monotonicity or concavity of the data g_i by $S(x)$) and finally we shall give the algorithm for finding the spline with minimal curvature. It is also the purpose of this article to compare these features with analogical properties of the cubic splines. We find that the quadratic splines on $(\Delta x \Delta t)$ - on the contrary to quadratic splines on the set (x_i) only - have quite nice properties, in some sense better than the cubic splines.

2. LOCAL INFLUENCE OF BOUNDARY CONDITIONS

For the sake of simplicity we shall study the simplest case of $(\Delta x \Delta t)$ with $h_i = h$, $t_i = x_i + h/2$, $i = 0(1)n$. The system of equations (m) reads now (multiplied by four)

$$m_{i-1} + 6m_i + m_{i+1} = f_i \quad \text{with } f_i = 8(g_i - g_{i-1})/h . \qquad (1)$$

We complete this system by adding boundary conditions $m_0 = S'(x_0)$, $m_{n+1} = S'(x_{n+1})$ with m_0, m_{n+1} given; then (1) results in the system of n linear equations for n parameters m_i, $i = 1(1)n$. The matrix A_n of this system is tridiagonal -

$$A_n = \begin{bmatrix} 6 & 1 & & & \\ 1 & 6 & 1 & & \\ & \cdot & \cdot & \cdot & \\ & & \cdot & \cdot & \cdot \\ & & & \cdot & \cdot & \cdot \\ & & & 1 & 6 & 1 \\ & & & & 1 & 6 \end{bmatrix} .$$

For the determinant $D_n = \det(A_n)$ we have $D_1 = 6$, $D_2 = 35$ and the recurrence relation $D_{n+2} = 6\, D_{n+1} - D_n$.

Solving this difference equation with initial conditions given we obtain

$$D_n = c_1(3 + 2\sqrt{2})^n + c_2(3 - 2\sqrt{2})^n \, , \qquad c_1 \sim 0{,}9 \; ; \quad c_2 \sim -\,0{,}01 \; . \tag{2}$$

So - for n large enough - we have $D_n \sim 6^n$.

With $(\Delta x \Delta t)$, g_i given, denote

$S(x)$ - the spline with boundary conditions m_0, m_{n+1} and parameters $[m_i] = m$, $i = 1(1)n$

$\overline{S}(x)$ - the spline with boundary conditions $\overline{m}_0$, $\overline{m}_{n+1}$ and parameters $[\overline{m}_i] = \overline{m}$.

The parameters m_i, $\overline{m}_i$, $i = 1(1)n$ of such splines $S(x)$, $\overline{S}(x)$ are the solutions of systems with the same matrix A_n

$$A_n\, m = f, \quad \text{where } f = [f_1 - m_0,\ f_1, \ldots, f_{n-1},\ f_n - m_{n+1}]^T \tag{3}$$

$$A_n\, \overline{m} = \overline{f}, \quad \text{with } \overline{f} = [f_1 - \overline{m}_0,\ f_1, \ldots, f_{n-1},\ f_n - \overline{m}_{n+1}]^T \; . \tag{$\overline{3}$}$$

For the spline $\tilde{S}(x) = \overline{S}(x) - S(x)$ we have

$$\tilde{S}(t_i) = 0, \quad \tilde{S}'(x_0) = \overline{m}_0 - m_0, \quad \tilde{S}'(x_{n+1}) = \overline{m}_{n+1} - m_{n+1} \; .$$

We can count its parameters $\tilde{m}_i = \tilde{S}'(x_i)$ solving the system

$$A_n\, \tilde{m} = f \quad \text{with } \tilde{f} = [\tilde{m}_0, 0, \ldots, 0, \tilde{m}_{n+1}]^T \, , \qquad \tilde{m}_0 = m_0 - \overline{m}_0 \, , \quad \tilde{m}_{n+1} = m_{n+1} - \overline{m}_{n+1} \; . \tag{4}$$

Using Cramer's rule and expansion of D_n, we have

$$\begin{aligned}
\tilde{m}_1 &= [\tilde{m}_0 D_{n-1} + (-1)^{n-1}\tilde{m}_{n+1}]/D_n \sim \tfrac{1}{6}\,\tilde{m}_0 + (-1)^{n-1}\tilde{m}_{n+1}/6^n \\
\tilde{m}_2 &= [-\tilde{m}_0 D_{n-2} + (-1)^{n-2}\tilde{m}_{n+1}\cdot 6]/D_n \sim -\tilde{m}_0/6^2 + (-1)^{n-2}\tilde{m}_{n+1}/6^{n-1} \\
&\;\vdots \\
n_n &= [(-1)^{n-1}\tilde{m}_0 + \tilde{m}_{n+1} D_{n-1}]/D_n \sim (-1)^{n-1}\tilde{m}_0/6^n + \tilde{m}_{n+1}/6 \; .
\end{aligned} \tag{5}$$

We can see from there, that the differences $\tilde{m}_0$, $\tilde{m}_{n+1}$ in the first derivative on the boundary influence the values of parameters $\tilde{m}_i$ substantially in the neighbourhood of boundary only (and even here with the small coefficient 1/6). We can make the same conclusions analysing analogically the system for the parameters $M_i = S''(t_i)$ in the related representation of $S(x)$, given in [5], [6] (the matrix of such a system is identical in the case of equidistant set $(\Delta x \Delta t)$ considered).

Remark. We can see that such a local property of $S(x) \equiv S_{21}(x)$ follows from the strong diagonal dominancy of the matrix A_n of the system (4). Making analogically such an analysis for the interpolating cubic spline $S_{31}(x)$ on the equidistant mesh $(\Delta x) = (x_i)$, $i = 0(1)n+1$, we get for parameters $m_i = S'_{31}(x_i)$ tridiagonal linear system with the matrix

$$B_n = \begin{bmatrix} 4 & 1 & & & \\ 1 & 4 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 4 & 1 \\ & & & 1 & 4 \end{bmatrix} .$$

We have $\det(B_n) \sim 13(2+\sqrt{3})^n \sim 13(3,7)^n$; so the rate of decreasing the influence of errors or inaccuracies in boundary conditions is smaller than by quadratic splines (the diagonal dominancy in matrix B_n is weaker than in the matrix A_n).

3. LOCAL INFLUENCE OF THE DATA g_i

In the case of $t_i = \frac{1}{2}(x_i + x_{i+1})$ we can easily show the local influence of the data g_i, $i = 0(1)n$, $g_i = g(t_i)$ with $g \in C^1[x_0, x_{n+1}]$, on the values of parameters m_j with $|j-i|$ great enough.

With m_0, m_{n+1} given, the parameters m_i $i = 1(1)n$ are computed from the system of equations (3).

The magnitude of the components of the inverse matrix $A_n^{-1} = [\bar{a}_{ij}]$ is limited by

$$|\bar{a}_{ij}| \leq \frac{1}{4} \left(\frac{1}{3}\right)^{|i-j|/2} \qquad \text{(see [18])}; \tag{6}$$

further we have $\| f \|_\infty \leq 8 \| (g_i - g_{i-1})/h \| \leq 8 \| g' \|$.

Finally, with $I_k = \{ j = i-k(1)i+k \}, k \in \{1, \ldots, n-i\}$, we obtain the inequality

$$\left| m_i - \sum_{j \in I_k} \bar{a}_{ij} f_j \right| \leq 8 . \frac{1}{4} \| g' \| \sum_{j \notin I_k} \left(\frac{1}{3}\right)^{|i-j|/2} \leq$$

$$\leq 4 . 3^{-(k-1)/2} \| g' \| \quad . \tag{7}$$

We see from there the decreasing influence of the data g_j on parameters m_i with increasing distance from the diagonal (from the breakpoint x_i).

4. SHAPE-PRESERVING PROPERTIES

Let us have the mesh $(\Delta \times \Delta t)$ of breakpoints x_i and points of interpolation t_i with the data g_i given.

We say (see [13]) that the data g_i, i=0(1)n are

- monotone, if $g_i \leq g_{i+1}$ (or $g_i \geq g_{i+1}$), i=0(1)n-1 ;
- concave, if $\delta^2 g_i \geq 0$ (or convex, if $\delta^2 g_i \leq 0$), i=1(1)n-1.

The problem we shall study now is the following: does the interpolating spline $S(x) = S_{21}(x)$ follow the monotonicity or concavity of the data g_i ?

4.1 Monotonicity preserving

Under the boundary conditions m_0, m_{n+1} prescribed, the parameters m_i, i=1(1)n satisfy the system of equations (3)

$$A_n \, m = f \quad .$$

Transforming this system equivalently by multiplying the i-th equation by two and substracting neighbouring equations multiplied by one half, we get the five-diagonal system

$$\begin{bmatrix} 11\frac{1}{2} & -1 & -\frac{1}{2} & & & & \\ & \ddots & & \ddots & & & \\ & -\frac{1}{2} & -1 & 11\frac{1}{2} & -1 & -\frac{1}{2} & \\ & & \ddots & & \ddots & & \\ & & & & -\frac{1}{2} & -1 & 11\frac{1}{2} \end{bmatrix} \begin{bmatrix} m_1 \\ \vdots \\ m_i \\ \vdots \\ m_n \end{bmatrix} \begin{bmatrix} \frac{16}{h}(g_1-g_0)-\frac{4}{h}(g_2-g_1)-2m_0 \\ \vdots \\ \frac{4}{h}[4(g_i-g_{i-1})-(g_{i+1}-g_i)-(g_{i-1}-g_{i-2})] \\ \vdots \\ \frac{16}{h}(g_n-g_{n-1})-\frac{4}{h}(g_{n-1}-g_{n-2})-2m_{n+1} \end{bmatrix} . \tag{8}$$

The matrix of this system is diagonally dominant and monotone (see [3]). Under conditions of positivity of the righthand side of (8),

$$(g_i-g_{g-1})/h \geqq \frac{1}{4}\left[(g_{i+1}-g_i)/h-(g_{i-1}-g_{i-2})/h\right] - \frac{1}{8}\,\overline{m}_i \ ,$$

$$\overline{m}_i = \begin{cases} m_0 & \text{for } i=2 \\ 0 & \text{for } i=3(1)n-2 \ , \\ m_{n+1} & \text{for } i=n-1 \end{cases} \tag{9}$$

$$m_0 \leqq 8(g_1-g_0)/h - 2(g_2-g_1)/h \ ,$$

$$m_{n+1} \leqq 8(g_n-g_{n-1})/h - 2(g_{n-1}-g_{n-2})/h \ ;$$

we have therefore also $m_j \geqq 0$, $j=1(1)n$.

The first derivative $S'_{21}(x)$ is a piecewise linear function on $[a,b]$ and hence from $m_j \geq 0$ also follows $S'_{21}(x) \geq 0$ for all $x \in [a,b]$. In this way we have proved the following theorem.

Theorem 1. Suppose that

- the data g_i, $i=0(1)n+1$ are increasing,
- the conditions (9) hold .

Then the interpolating spline $S_{21}(x)$ is an increasing function.

Remark. The analogical statement for decreasing data can be obtained using opposite signs of inequalities in (9).

4.2 Concavity preserving

The quantities $M_i = S''(t_i)$ of the quadratic spline $S_{21}(x)$ on the set $(\Delta \times \Delta t)$ with

$$x_{i+1} = (t_i + t_{i+1})/2 \tag{10}$$

are coupled together through the continuity conditions

$$M_{i-1} + 6\, M_i + M_{i+1} = 16[t_{i-1},\ t_i,\ t_{i+1}] \quad (=8\,\delta^2 g_i/h^2\ ) \tag{11}$$

(see [5], [6]).

Prescribing boundary conditions $M_0 = S''(t_0)$, $M_n = S''(t_n)$, we have the system of linear equations with the same matrix A_n as in 4.1. Using the same technique as before we can prove the following statement.

Theorem 2. Suppose the data g_i given for the spline $S_{21}(x)$ on the mesh $(\Delta x \Delta t)$ fulfilling (10) are concave.

Then under conditions

$$\delta^2 g_{i-1} + \delta^2 g_{i+1} < 4\,\delta^2 g_i \quad , \ i=3(1)n-3$$

$$M_0 > 0 \ , \quad M_n > 0$$

$$\delta^2 g_2 + \tfrac{1}{2}h^2 M_0 < 4\,\delta^2 g_1 < 16\,\delta^2 g_2 + \tfrac{1}{2}h^2 M_0 \tag{12}$$

$$\delta^2 g_{n-2} + \tfrac{1}{2}h^2 M_n < 4\,\delta^2 g_{n-1} < 16\,\delta^2 g_{n-2} + \tfrac{1}{2}h^2 M_n$$

the inequalities $M_i > 0$, $i=0(1)n$, hold and hence $S_{21}(x)$ is concave function.

Remark. In the case of cubic splines on equidistant set (Δx) the analogical condition of concavity preserving can be obtained (see [19]). Analogically to 4.1 we can also study the conditions, under which $g_{i+1} > g_i$ implies $m_i > 0$; but in the case of cubic splines this fact doesn't imply $S'_{31}(x) > 0$ on the interval (x_i, x_{i+1}) (- monotonicity preserving).

5. THE QUADRATIC SPLINE $S_{21}(x)$ WITH MINIMAL NORM $\| S''_{21} \|_2$

It is well-known that the cubic splines have the minimising property $\| S_{31}'' \|_2 \leq \| f'' \|_2$ on the class of functions $f \in W_2^2$ interpolating the same data and fulfilling the same boundary conditions (the "natural cubic splines" - under all types of boundary conditions; see [1], [2], [18]).

We couldn't find such a property under quadratic splines interpolating the data g_i. What we can do only is to find free parameters m_0, m_{n+1} in such a way as to minimize $\| S_{21}'' \|_2$ over the class of the quadratic splines interpolating the same data. This idea - for quadratic splines on the set (Δx) only - was worked out by G.Maess in [10], [11].

Let us have the set $(\Delta x \Delta t)$ with the data g_i, $i=0(1)n$ given and a quadratic spline representation

$$S(x) = g_i + (q-d_i)[m_i+(m_{i+1}-m_i)(q+d_i)/(2h_i)] \ , \qquad q = x - x_i \ , \tag{Sm}$$

where the parameters m_i, $i=1(1)n$ are computed from the system (m), and with m_0, m_{n+1} as free parameters to our disposal for minimizing $\| S'' \|_2$.

We have $\frac{d^2}{dx^2} S(x) = (m_{i+1}-m_i)/h_i$; $m_j = m_j(m_0,m_{n+1})$, $j = 1(1)n$. Denote

$$C_n = \begin{bmatrix} b_1 & c_1 & & & \\ a_2 & b_2 & c_2 & & \\ & \ddots & \ddots & \ddots & \\ & a_{n-1} & b_{n-1} & c_{n-1} & \\ & & & a_n & b_n \end{bmatrix} , \quad f = \begin{bmatrix} f_1-a_1m_0 \\ f_2 \\ \vdots \\ f_{n-1} \\ f_n-c_nm_{n+1} \end{bmatrix}$$

$$D(m_i) = \left[\frac{\partial m_j}{\partial m_i}\right], \quad i=0,n+1 \ ; \quad j=1(1)n \ ,$$

$$Q(m_0,m_1) = \| S'' \|_2^2 = \sum_{j=0}^{n} \int_{x_j}^{x_{j+1}} (S''(x))^2 dx =$$

$$= \sum_i h_i((m_{i+1}-m_i)/h_i)^2 =$$

$$= m_0^2/h_0+m_{n+1}^2/h_n+\sum_{j=1}^{n} m_j^2(\frac{1}{h_{j-1}}+\frac{1}{h_j}) \quad . \tag{13}$$

Then differentiating (m) we get

$$C_n D(m_0) = [-a_1,0,\dots,0]T \ ,$$
$$C_n D(m_{n+1}) = [0,\dots,0,-c_n]^T \quad . \tag{14}$$

The components of the solution of (14) could be written explicitly using Cramer´s rule

$$\frac{\partial m_1}{\partial m_0} = -a_1 D_1/D_n \ , \quad \frac{\partial m_2}{\partial m_0} = a_1 a_2 D_2/D_n, \dots$$
$$\dots, \quad \frac{\partial m_i}{\partial m_0} = (-1)^i a_1 \dots a_i D_i/D_n$$

$$\frac{\partial m_{n-1}}{\partial m_0} = (-1)^{n-1} a_1 a_2 \dots a_{n-1} b_n/D_n \ , \tag{15}$$

$$\frac{\partial m_n}{\partial m_0} = (-1)^n a_1 \dots a_n/D_n \ ,$$

where

$$D_i = \det \begin{bmatrix} b_{i+1} & c_{i+1} & & \\ a_{i+2} & b_{i+2} & c_{i+2} & \\ & \ddots & \ddots & \ddots \\ & & a_n & b_n \end{bmatrix}, \quad D_n = \det(C_n) \ .$$

Similarly we have

$$\frac{\partial m_1}{\partial m_{n+1}} = (-1)^n c_1 c_2 \dots c_n, \quad \frac{\partial m_2}{\partial m_{n+1}} = (-1)^{n-1} b_1 c_2 \dots c_n \ ,$$

$$\frac{\partial m_i}{\partial m_{n+1}} = (-1)^{n+1-i}\, c_i \dots c_n \bar{D}_i/D_n, \dots, \frac{\partial m_n}{\partial m_{n+1}} = -c_n\bar{D}_n/D_n ,$$

with

$$\bar{D}_i = \det \begin{bmatrix} b_1 & c_1 & & \\ a_2 & b_2 & c_2 & \\ & \ddots & \ddots & \ddots \\ & & a_{i-1} & b_{i-1} \end{bmatrix} .$$

The necessary conditions for $Q(m_0, m_{n+1})$ to have a minimum are

$$\frac{\partial Q}{\partial m_0} = \frac{2}{h_0} m_0 + 2\sum_{j=1}^{n} m_j\left(\frac{1}{h_{j-1}} + \frac{1}{h_j}\right)\frac{\partial m_j}{\partial m_0} = 0$$

$$\frac{\partial Q}{\partial m_{n+1}} = \frac{2}{h_n} m_{n+1} + 2\sum_{j=1}^{n} m_j\left(\frac{1}{h_{j-1}} + \frac{1}{h_j}\right)\frac{\partial m_j}{\partial m_{n+1}} = 0 \quad . \tag{17}$$

These two equations complete the system (m) to the system of n+2 linear equations for determining parameters m_i, $i=0(1)n+1$ (here we have to substitute for $\partial m_j/\partial m_i$ according to (15), (16)).

Denoting $a_0 = 2/h_0$, $a_{0j} = 2(h_{j-1}^{-1} + h_j^{-1})(-1)^j\, a_1 \dots$

$$\dots a_j D_j/D_n \quad , \tag{18}$$

$b_{n+1} = 2/h_n$, $b_{n+1,j} = 2(h_{j-1}^{-1}+h_j^{-1})(-1)^{n+1-j}\, c_j \dots c_n \bar{D}_h/D_n$, $j=1(1)n$,

we can write this system as

$$\begin{bmatrix} a_0 & a_{01} & a_{02} & a_{03} & \cdots & a_{0n} & 0 \\ a_1 & b_1 & c_1 & 0 & \cdots & 0 & 0 \\ \vdots & & \ddots & \ddots & \ddots & & \vdots \\ 0 & & & a_n & b_n & c_n & 0 \\ 0 & b_{n+1,1} & \cdots & & & b_{n+1,n} & b_{n+1} \end{bmatrix} \begin{bmatrix} m_0 \\ m_1 \\ \vdots \\ m_n \\ m_{n+1} \end{bmatrix} = \begin{bmatrix} 0 \\ f_1 \\ \vdots \\ f_n \\ 0 \end{bmatrix} \tag{19}$$

Solving this system, we get the parameters m_i, $i=0(1)n+1$ which define the spline $S(x)$ (via (Sm) representation) minimizing $\| S'' \|_2$.

Theorem 3. Let the mesh $(\Delta x \Delta t)$ and data g_i, $i=0(1)n$ be given. Suppose, that

$$|b_i| > |a_i| + |c_i| \qquad (a_i, b_i, c_i \text{ given in (m)}),$$

$$|a_0| > \sum_j |a_{0j}| \ , \quad |b_{n+1}| > \sum_j |B_{N+1,j}| \tag{20}$$

Then we have a unique spline $S(x)$ minimizing $\| S'' \|_2$ over all splines $S(x)$ interpolating the data g_i; the parameters m_i of such a spline can be computed from the system (19).

Remark. In case of equidistant mesh $(\Delta x \Delta t)$ we have

$$h_i = h, \quad d_i = h/2,$$

$$\frac{\partial m_i}{\partial m_0} = (-1)^i/6 \ , \quad \frac{\partial m_i}{\partial m_{n+1}} = (-1)^{n+1-i}/6^{n+1-i} \ ,$$

$$a_{0j} = (-1)^j.4/(h.6^j), \quad a_0 = 2/h > \sum_j a_{0j} = \frac{4}{7h} (1 \pm 1/6^{n+1})$$

$$b_{n+1,j} = (-1)^{n+1-j}.4(h.6^{n+1-j}), \quad b_{n+1} = 2/h > \sum_j b_{n+1,j} =$$
$$= \frac{4}{7h} (1 \pm 1/6^{n+1})$$

and the system (19) has thus unique solution.

6. EXAMPLES

Let us have the data

t_i - points of interpolation, knots of cubic splines S_{32}, S_{31};

y_i - prescribed values in t_i ($y_i = S(t_i)$);

x_i - knots of quadratic spline ($\overline{x}_i$ - knots changed);

y_i' - prescribed first derivative (for S_{32} only).
given by the following table

i	0	1	2	3	4	5	6	7	
t_i	0	1	2	3	4	6	8	9	
y_i	1.5	1	2	1.2	-0.5	-0.5	-0.5	1.5	
x_i	0	0.5	1.5	2.5	3.5	4.5	6.5	8.5	9.5
$\bar{x}_i$			—"—			4.1	7.9		—"—
y_i'	0	1	0	-1	0	0	0	0.5	

i	8	9	10	11	12	13
t_i	10	12	13	14	15	18
y_i	2	3	0.5	-2	0	1.5
x_i	10.5	12.5	13.5	14.5	16	18
$\bar{x}_i$	11.8		—"—		15.2	
y_i'	0.25	0	-1	0	1	0

We can see and compare properties of cubic and quadratic splines defined by this data on the figures 1-8:

Fig.1 - local Hermite cubic spline S_{32} given by (t_i, y_i, y_i').

Fig.2 - cubic spline given by (t_i, y_i) and boundary conditions $S_{31}'(0)$, $S_{31}'(18)$ approximated from the data.

Fig.3 - natural cubic spline S_{31} $(S_{31}''(0) = S_{31}''(18) = 0)$.

Fig.4 - quadratic spline given by (x_i, t_i, y_i); $S_{21}'(0)$, $S_{21}'(18)$ approximated from the data.

Fig.5 - quadratic spline for (x_i, t_i, y_i), $S_{21}'(0) = S_{21}'(18) = 0$.

Fig.6 - quadratic spline for (x_i', t_i, y_i), $S_{21}''(0) = S''(18) = 0$.

Fig.7 - quadratic spline with $y_{10} = 2$ changed to $y_{10} = 3.5$, 2.5, boundary conditions computed from data.

Fig.8 - quadratic spline corresponding to the function
exp(x).sin(2x), $x \in [-1.5]$;

(• - points of interpolation, o - breakpoints (knots)).

We can see the local influence of boundary conditions and local changes in values y_i on the behaviour of the quadratic spline. Let us mention also the Fig.6, where the possibilities of appropriate choice of the knots of the quadratic spline for fitting "flat parts" of the spline are demonstrated (one have to move the knots near to the ends of flat segments).

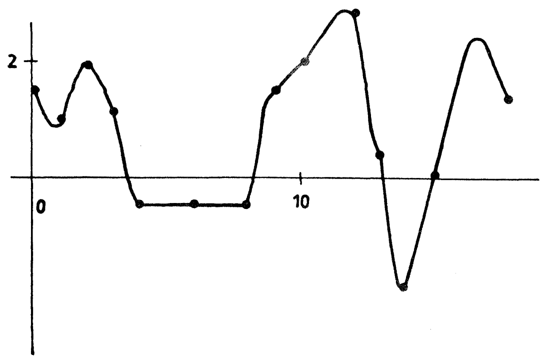

Fig.1

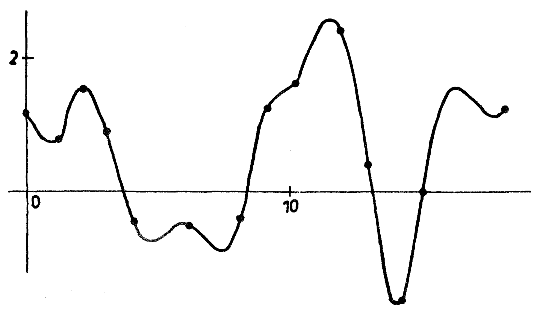

Fig.2

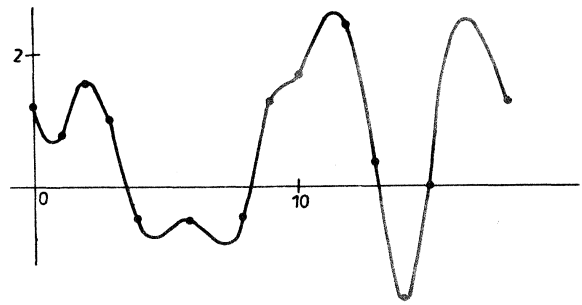

Fig.3

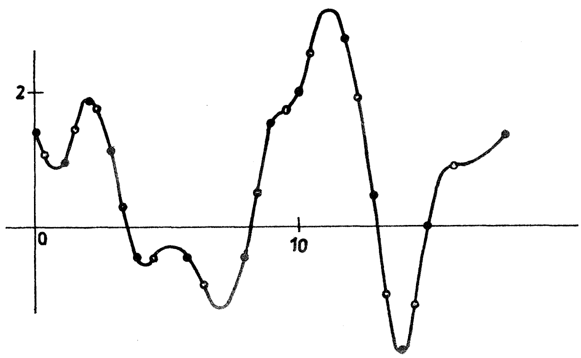

Fig.4

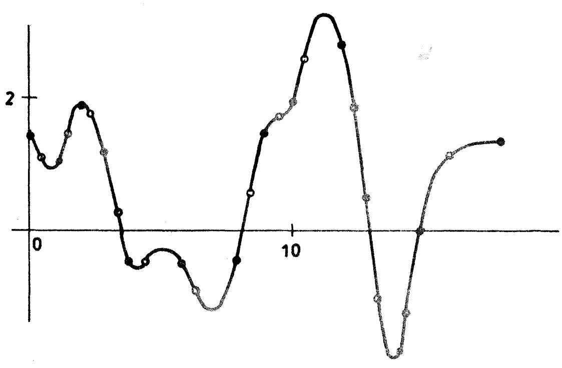

Fig.5

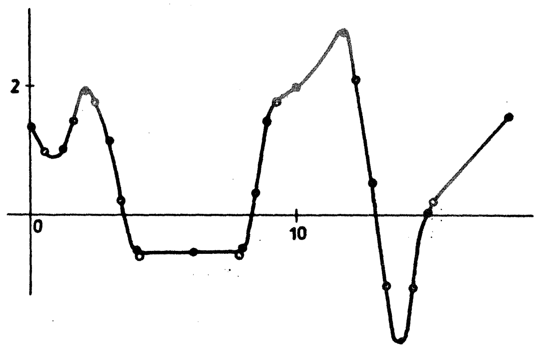

Fig.6

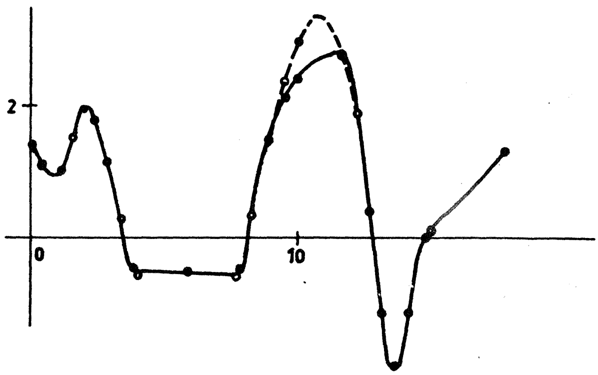

Fig.7

Fig.8

SOUHRN

NĚKTERÉ VLASTNOSTI INTERPOLUJÍCÍHO KVADRATICKÉHO SPLAJNU

JIŘÍ KOBZA

Pro kvadratický splajn s oddělenými uzly splajnu a body interpolace se dokazuje lokální vliv okrajových podmínek a změn v jednotlivých datech na průběh splajnu (tím se takový splajn liší od splajnu se splývajícími uzly splajnu a body interpolace). Dále se ukazují postačující podmínky pro to, aby kvadratický splajn zachovával monotonnost a konkávnost dat; je ukázáno, že kvadratický splajn má tyto vlastnosti ve větší míře než kubický splajn. V závěru je ukázán algoritmus výpočtu okrajových podmínek, při nichž má kvadratický interpolující splajn minimální normu druhé derivace. Problematika je ilustrována na několika příkladech.

РЕЗЮМЕ

НЕКОТОРЫЕ СВОЙСТВА КВАДРАТИЧНОГО ИНТЕРПОЛЯЦИОННОГО СПЛАЙНА

Й. КОБЗА

В работе изучаются некоторые свойства квадратичного сплайна с несовпадающими узлами сплайна с точками интерполяции. Показывается локальное влияние краевых условий и предписанных значений на поведение сплайна /в чем отличается от сплайна со совпадающими узлами и точками интерполяции/. Показаны достаточные условия сохранения монотонности и вогнутости данных квадратичным сплайном, который обладает этим свойством даже в большей мере чем кубический сплайн. Окончательно исследуется алгоритм вычисления таких краевых условий для квадратичного сплайна, которые гарантируют минимум нормы его второй производной. Результаты июстрируются на примерах.

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