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TO THE THEORY OF GLOBAL TRANSFORMATION OF THE SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS OF FINITE TYPE, SPECIAL

EVA TESAŘÍKOVÁ

Introduction

In Borůvka´s monograph [1], the question of global transformation of second order linear differential equations in Jacobian form

$$y'' = q(t)\, y \qquad (q)$$

$$Y'' = Q(t)\, Y \qquad (Q)$$

was treated at length on the basis of a theory of general dispersions for the equations being on both sides oscillatory, but the equations of a finite type are on the periphery of author´s interest, there.

In this paper is formed a theory of general dispersions for equations (q), (Q) in Jacobian form (where $q(t) \in C^0(j)$ and $Q(t) \in C^0(J)$) which are 1-special of a finite type $m \geq 2$ in their definition intervals $j = (a,b)$ and $J = (A,B)$, respectively. With regard to this property, the denoting $(q^{(1)})$, $(Q^{(1)})$ will be used in the following text. Except from the monograph [1] the text proceeds from three publicated articles [2], [3], [4] where a theory of central dispersions of particular kinds for equations of a finite type - special was treated on the basis of definitions of special central dispersions. Besides, the special central dispersions of the 1st kind relative to the equation of the type $(q^{(1)})$ form a finite cyclic group of m-order relative to an operation of composition in the whole definition interval except from points of 1-fundamental sequence.

General dispersions of differential equation of type $(q^{(1)})$ and their relation to transformation problem

Consider now differential equations $(q^{(1)})$, $(Q^{(1)})$ of the same finite type m, 1-special in the intervals j, J, respectively. Let us denote the space of all solutions of equation $(q^{(1)})$ and $(Q^{(1)})$ in corresponding definition intervals by r and R (in this order), while (u,v), (U,V) will be arbitrary bases of r, R, and w, W the Wronskians of these bases. Furthermore, consider a linear mapping p of the space r onto the space R determined by pair of bases (u,v). (U,V), i.e. mapping p where (U,V) = (pu,pv). The proportion of Wronskians w:W of these bases is called the characteristic χ_p of the mapping p. In connection with this mapping let us recapitulate same facts given in § 19 of [1].

Mapping p is uniquely ordered by pair of bases (u,v), (U,V), it is schlicht and has the property such that maps lineary independent elements of the space R. Moreover, for every element $y \in r$ defined in the form $y = (c_1u + c_2v)$ where c_1, c_2 are arbitrary constants it holds that py = Y if and only if $Y = (c_1U + c_2V)$. To every mapping p determined in this way there exists an inverse mapping p^{-1} whose characteristic is given by relationship $\chi_{p^{-1}} = (\chi_p)^{-1}$.

A mapping cp, which maps the basis (u,v) onto the basis $(c_1U + c_2V)$ where $c \neq 0$ is an arbitrary constant is called a variation of the mapping p. The elements $cpy \in R$ are linearly dependent for different c. A characteristic of the variation cp of the mapping p is given by relationship $\chi_{cp} = c^{-2}\chi_p$.

With respect to the expression of the elements of the bases (u,v) and (U,V) in the forms

$$u = \varepsilon\sqrt{w}\,\frac{\sin\alpha}{\sqrt{|\alpha'|}}\ ,\qquad v = \varepsilon\sqrt{w}\,\frac{\cos\alpha}{\sqrt{|\alpha'|}}\ , \tag{1}$$

and

$$U = E\sqrt{W}\,\frac{\sin A}{\sqrt{|A'|}}\ ,\qquad V = E\sqrt{W}\,\frac{\cos A}{\sqrt{|A'|}}\ , \tag{2}$$

by means of arbitrary first phase α of the basis (u,v) and by means of arbitrary first phase A of the basis (U,V), respectively, in which ε , E take the values +1 or -1 according as the phases are proper or not relative to the bases (u,v), (U,V), and with respect to the expression the arbitrary elements $y \in r$ and $Y \in R$ in the form

$$y = k_1\,\frac{\sin(\alpha + k_2)}{\sqrt{|\alpha'|}}\ ,\qquad Y = \frac{\varepsilon E}{\sqrt{|\chi_p|}}\cdot\frac{k_1.\sin(A + k_2)}{\sqrt{|A'|}}\ , \tag{3}$$

in which k_1, k_2 are the constants for $k_1 \neq 0$, $k_2 \in \langle 0, 2\pi \rangle$ it follows that the linear mapping p of the space r onto the space R is determined not only by a concrete ordered pair of bases (u,v), (U,V) but also by an ordered pair of phases of these bases which we call phase basis corresponding with the mapping p.

For every choice of basis (U,V), the second basis of ordered pair relative to concrete mapping p is determined uniquely in the form (pu,pv). Thus for every choice of phase α of $(q^{(1)})$, the second term A of the phase basis relative mapping p is determined except for integer multiplies of π. The characteristic χ_p is independent on choice of basis.

On the contrary, an arbitrary pair of phases (α , A) of differential equations $(q^{(1)})$, $(Q^{(1)})$ forms the phase basis of infinitely many linear mapping cp, $c \neq 0$, of the space r onto R. Then, the mapping p is uniquely determined by the phase basis $(\alpha, \mathfrak{a})$, except for its variations.

Furthermore, let's point out a concept of normalized mapping. A linear mapping p of the space r onto the space R is called normalized with respect to the pair of points (z,Z), $z \in j$, $Z \in J$, if the implication $[y(z) = 0] \Rightarrow [py(Z) = 0]$ holds for every $y \in r$. This property is common to all variations cp of the mapping p. Relating to the phase basis $(\alpha, \mathfrak{a})$ the following theorem holds.

Theorem 1.

The linear mapping of the space r onto R is normalized with respect to the pair of points (z,Z), $z \in j$, $Z \in J$ if and only if the values of the phases of the corresponding phase basis $(\alpha, \mathfrak{a})$ differ by an integral multiple of π, i.e. $\alpha(z) - \mathfrak{a}(Z) = n\pi$, where n is an integer, holds.

Proof: Let p is a linear mapping of r onto R. From $[y(z) = 0] \Longrightarrow [Y(Z) - py(Z) = 0]$ and from the expressions y(z), Y(Z) in the form (3) it follows that $\alpha(z) + k_2 = \mathfrak{a}(Z) + k_2 + n\pi$, and vice versa.

From Theorem 1 it follows immediately that for linear mapping which is normalized with respect to pair of points z, Z there always exists a phase basis $(\alpha, \hat{\mathfrak{a}})$ fulfilling the condition: $\alpha(z) = 0$, $\mathfrak{a}(Z) = 0$. We will cai it the canonical phase basis of mapping with respect to points z, Z.

So much for basic concepts of § 19 in [1] whic are concerned the theory of general dispersions of oscill ory equations (q) mentioned in introduction. In the following text, we will start with treating a theory of general dispe sions as a transformation theory of equations $(q^{(1)})$, $(Q^{(1)})$ of the same finite type $m \geq 2$, 1-special in definition intervals. First of all it is necessary to introduce some further basic concepts.

By 1-fundamental sequences $(a^{(1)})$ and $(A^{(1)})$ of equation $(q^{(1)})$ and $(Q^{(1)})$, respectively, we mean zeros of 1-fundamental solution of this equation in the following ordering

$$a = a_0 < a_1 < a_2 \dots a_{m-1} < a_m = b \qquad (a^{(1)})$$

and

$$A = A_0 < A_1 < A_2 \dots A_{m-1} < A_m = B \qquad (A^{(1)})$$

Relating to these sequences, we will call the points $t \in j$ and $T \in J$, (in this order) directly or indirectly associated, if either

$$t = a_i \quad \text{and} \quad T = A_i \quad \text{or} \quad T = A_{m-i}$$

or

$$t \in (a_{i-1}, a_i) \quad \text{and} \quad T \in (A_{i-1}, A_i) \quad \text{or} \quad T \in (A_{m-i}, A_{m-i+1}) \ ,$$

where $i = 1,2,\dots,m$, holds.

We will call the phases α, $\mathcal{A}$ which are corresponding to $(q^{(1)})$, $(Q^{(1)})$ directly or indirectly similar, if they are taking the same values in directly or indirectly associated points of sequences $(a^{(1)})$, $(A^{(1)})$, respectively.

Now consider some of 1-fundamental basis (u_1, v) and (U_1, V) of equation $(q^{(1)})$ and $(Q^{(1)})$, respectively. Let´s point out, that u_1 and U_1 are some of 1-fundamental solutions of corresponding equations, v and V are arbitrary solutions independent on u_1 and U_1, respectively. Let p be a linear mapping of the space r onto the space R defined by pair of these bases. Such a mapping is always normalized with respect to all the pairs of elements of sequences $(a^{(1)})$, $(A^{(1)})$, and conversely, every mapping p normalized with respect to all the pairs of elements of sequences $(a^{(1)})$, $(A^{(1)})$ maps every 1-fundamental basis (u_1, v) on the corresponding 1-fundamental basis (pu_1, pv). Such mapping p is called the canonical mapping of the space r onto the space R.

To every canonical mapping p there always exists the phase basis $(\alpha, \mathcal{A})$, which is canonical with respect to the chosen pair (a_i, A_i) or (a_i, A_{m-i}) of directly or indirectly associated points of sequences $(a^{(1)})$ or $(A^{(1)})$. It´s composed of directly or indirectly similar phases α, A of the phase systems of bases (u_1,v), (U_1,V). A concrete canonical mapping p isn't determined only by a chosen pair of 1-fundamental bases (u_1,v), (U_1,V), but also by an arbitrary pair of bases $(\bar{u},\bar{v})$, $(\bar{U},\bar{V})$ where $\bar{U} = p\bar{u}$, $\bar{V} = p\bar{v}$. It follows that a concrete canonical mapping p can be determined also by a phase basis $(\bar{\alpha},\bar{\mathcal{A}})$, which is composed of the phases of phase systems of bases $(\bar{u},\bar{v})$, $(\bar{U},\bar{V})$. Moreover the following theorem holds.

Theorem 2.

Some of canonical mappings of the space r onto R is determined by the phase basis $(\alpha,\mathcal{A})$, which is canonical to some directly or indirectly associated two points t_0, T_0, $t_0 \in j$, $T_0 \in J$, if and only if the α, $\mathcal{A}$ are normal phases directly or indirectly similar.

Proof: By Theorem 1 the linear mapping p or r onto R is determined by the phase basis $(\alpha,\mathcal{A})$ which is canonical if and only if $\alpha(a_i) - \mathcal{A}(A_k) = n\pi$ (n an integer) holds for an arbitrary pair of points a_i, A_k (where $i,k \in \{1,2,\ldots,m-1\}$) of 1-fundamental sequences $(a^{(1)})$, $(A^{(1)})$. If $\alpha(t_0) = 0$, $A(T_0) = 0$ for two directly or indirectly associated points t_0, T_0, then $\alpha(a_i) - \mathcal{A}(A_i) = 0$ or $\alpha(a_i) - \mathcal{A}(A_{m-i}) = 0$ for every pair of directly or indirectly associated points of sequences $(a^{(1)})$, $(A^{(1)})$ and conversely.

Henceforward the phase basis, which is corresponding with the canonical mapping p and composed of directly or indirectly similar normal phases, will be called the forming phase basis of this mapping. Now to every canonical mapping p of r onto R, we will define the function which maps zeros of an arbitrary element $y \in r$ onto corresponding zeros of element $Y = py \in R$.

Definition 1

The function X(t), which associated every point $t \in j$ at the directly or indirectly associted zero $T \in J$ of image $Y = py \in R$ of the solution $y \in r$ such that $y(t) = 0$, will be called the general dispersion of equations $(q^{(1)})$, $(Q^{(1)})$

corresponding to canonical mapping p of the space r onto R, for $\chi_p > 0$ or $\chi_p < 0$, respectively.

If $\chi_p > 0$, then X(t) is a direct general dispersion.

If $\chi_p < 0$, then X(t) is an indirect general dispersion.

From theorems about ordering of zeros of solutions relative linear dofferential equations of type (q) of [1] it follows that a domain of definition of the general dispersion is a whole interval j = (a,b), the range of values is the whole interval J = (A,B).

With respect to the coincidence of zeros of images $\varphi py \in R$ relative to the element $y \in r$ for various variations φp of mapping p, from Definition 1 it is easy to see, that general dispersions X(t) of equations $(q^{(1)})$, $(Q^{(1)})$ relative to various variations of the same mapping p are equal identically in the whole definition interval.

The Definition 1 is a certain analogy of general dispersions of oscillatory equations of § 20 [1], which is adapted to conditions of equations of type $(q^{(1)})$. From the following we will see, that there is a possibility of derivating analogical properties of functions introduced in such a way. These adaptions are formed on the basic of theorems defining a relation of general dispersions and phase bases of corresponding canonical mapping p.

Theorem 3

Let p be a canonical mapping of r onto R, $(\alpha, \mathcal{A})$ be its forming phase basis. Thus a general dispersion X(t) corresponding to this mapping fullfils the functional equation

$$\alpha(t) = \mathcal{A}(X(t)) \quad (4)$$

on the whole interval j.

Proof: If expressing element y and its image Y = py in zero x in the form (3) we will come to equations

$$\alpha(x) + k_2 = n\pi$$

$$\mathcal{A}[X(x)] + k_2 = N\pi$$

where $k_2 \in \langle 0, 2\pi)$ is a constant, n, N are integers. Consequently $\alpha(x) - \mathcal{A}[X(x)] = m\pi$, $m \in Z$. With respect to direct or indirect similarity of phases, the relation $-\pi < \alpha(x) - \mathcal{A}[X(x)] < \pi$ holds too and with the above relations completes the proof of (4).

From Theorem 3 we may derive the following properties of general dispersions.

<u>Theorem 4</u>

A general dispersion X(t) of equations $(q^{(1)})$, $(Q^{(1)})$, corresponding to canonical mapping p of the space r onto the space R, passes in the relation with an arbitrary forming basis $(\alpha, \mathcal{A})$ of this mapping the following properties:

1) For every $t \in j$ the general dispersion X is inequelly determined by the relation

$$X(t) = \mathcal{A}^{-1}(\alpha(t)) \tag{5}$$

Proof: This statement immediately follows from the relation (4) of Theorem 3.

2) A direct general dispersion X is an increasing function in interval j within limits from A to B, with derivative $X' > 0$; An indirect general dispersion X is decreasing function in interval j within limits from B to A, with derivative $X' < 0$.

Proof: The statement follows immediately from the relationship (5) and from properties of phases $\alpha, \mathcal{A}$. It follows that

$$\text{sign } X' = \text{sign } \alpha' . \text{ sign } \mathcal{A}' = \text{sign}(-w).\text{sign}(-W) = \\ = \text{sign } \chi_p .$$

3) Function X^{-1} inversed to function X is a general dispersion of differential equations $(Q^{(1)})$, $(q^{(1)})$, which is corresponding to linear mapping p^{-1} of R onto r.

Proof: From (4) follows the form of inversed function X^{-1}

$$X^{-1}(T) = \alpha^{-1}[\mathcal{A}(T)] . \tag{6}$$

But it is an expression of general dispersion corresponding to canonical mapping p^{-1} of R onto r by means of forming phase basis (a, α).

4) A general dispersion X is in j three times continuously differentiable function and in two homologic point $t \in j$, $X(t) \in J$ fulfil the relationship as follows

$$X'(t) = \frac{\alpha'(t)}{a'(X)}, \quad X''(t) = \frac{1}{a'^3(X)}[\alpha''(t)\, a'^2(X) - \alpha'^2(t)\, \ddot{a}(X)] \tag{7}$$

$$\dot{a}(X) = \frac{\alpha'(t)}{X'(t)}, \quad \ddot{a}(X) = \frac{1}{X'^3(t)}[\alpha''(t)X'(t) - X''(t)\,\alpha'(t)] \tag{8}$$

Proof: We will get the above relationship (7), (8) by means of immediate double derivation of (4). The phases α, a are three times continously differentiable on the corresponding intervals j, J, and $\alpha'(t) \neq 0$, $a'(X) \neq 0$ hold for every $t \in j$, $X(t) \in J$.

5) For every $t \in j$, $t \neq a_{m-k}$, $k = 0,1,\ldots,m-1$ holds the relation

$$X[\phi_k(t)] = F_{k\varepsilon}[X(t)], \tag{9}$$

where $\varepsilon = \text{sign } X'$, $\phi_k(t)$ or $F_{k\varepsilon}(T)$ denote k-th or k-th special central dispersion of 1st kind of equation $(q^{(1)})$ or $(Q^{(1)})$.

Proof: Assume first $\mathfrak{X}_p > 0$ and thus $X' > 0$.

From functional equation (4) follows in the point $\phi_k(t)$, $t \in j$, $t \neq a_{m-k}$ the following relationship

$$\alpha[\phi_k(t)] = a[X(\phi_k(t))] \tag{10}$$

Using (4) on a modification of Abel equation (1) of [3] for special central dispersions of $(q^{(1)})$ we will come to the expression

$$a[X(\phi_k(t))] = \begin{cases} a[X(t)] + k\pi \,\text{sign}\, \alpha' & \text{for } t \in (a, a_{m-k}) \\ a[X(t)] - (m-k)\pi \,\text{sign}\, \alpha' & \text{for } t \in (a_{m-k}, b) \end{cases} \tag{11}$$

With respect to equalities

$$X(a_{m-k}) = A_{m-k} \quad ,$$

$$\lim_{t \to a^+} X(t) = A \ , \qquad \lim_{t \to b^-} X(t) = B \ ,$$

$$\operatorname{sign} \alpha' = \operatorname{sign} X' \operatorname{sign} \dot{a} = \operatorname{sign} \dot{a} \quad ,$$

and after using the modification of Abel equation for special central dispersions of equation $(Q^{(1)})$ for the expression of right side of (11) in the form

$$a[F_k(X(t))] = \begin{cases} a[X(t)] + k\pi \operatorname{sign} \dot{a} & \text{for } X(t) \in (A, A_{m-k}) \\ a[X(t)] - (m-k)\pi \operatorname{sign} \dot{a} & \text{for } X(t) \in (A_{m-k}, B) \end{cases} \quad , \tag{12}$$

we will come to the equation (9).

b) In case that $\mathfrak{X}_p < 0$ one of the phases of the forming phase basis (α, a) is increasing, the other one is decreasing. Zeros of these phases are associated points indirectly. Also in this case we come from (4) through (10) to (11). With respect to equalities

$$X(a_{m-k}) = A_k \ ;$$

$$\lim_{t \to a^+} X(t) = B \ , \qquad \lim_{t \to b^-} X(t) = A \ ,$$

$$\operatorname{sign} \alpha' = \operatorname{sign} X' \operatorname{sign} \dot{a} = -\operatorname{sign} \dot{a} \ ;$$

and by using the modification of Abel's equation for special central dispersion of $(Q^{(1)})$, the right side of the equation (11) will be expressed in this form:

$$a[F_{m-k}(X(t))] = \begin{cases} a[X(t)] - k\pi \operatorname{sign} \dot{a} & \text{for } X(t) \in (A_k, B) \\ a[X(t)] + (m-k)\pi \operatorname{sign} \dot{a} & \text{for } X(t) \in (A, A_k) \end{cases} \ . \tag{13}$$

With respect to validity of $F_{m-k}(X(t)) = F_{-k}(X(t))$ for $X(t) \in J$, $X(t) \neq A_k$ we will come to the equality (9).

6) The general dispersion X(t) is a solution of the non-linear differential equation of the 3rd order

$$- \{X,t\} + Q(X)X'^2 = q(t) \quad . \qquad (Q^{(1)}q^{(1)})$$

for every $t \in j$.

Proof: Again, we will proceed from relationship (4). When expressing the Schwarz's derivative of a composed function $a[X(t)]$ in the sence of the formula (17), 8, § 1 of [1] and by using of (7), (8) we will come to the equation

$$\{X,t\} + [\{a,X\} + a'^2(X)]X'^2 = \{\alpha,t\} + \alpha'^2(t) \ .$$

By using (16) § 5 of [1] to expres the carrier of the equation $(q^{(1)})$ and $(Q^{(1)})$ by means of 1st phase, we will get the validity of 6).

7) The function $x(T) = X^{-1}$ inverse to general dispersion X represents a solution of nonlinear third order differential equation

$$- \{x,T\} + q(x)\ \dot{x}^2 = Q(T) \ . \qquad (q^{(1)}Q^{(1)})$$

in the whole interval J.

Proof: The statement follows immediately from properties (3) and (6) of this theorem.

Theorem 5

Consider three differential equations $(q^{(1)})$ and $(Q^{(1)})$ and $(\overline{Q}^{(1)})$ of the same finite type m, 1-special on intervals $j = (a,b)$ and $J = (A,B)$ and $\overline{J} = (\overline{A},\overline{B})$, respectively. Let p be a cannonical linear mapping of the space r of all the solutions of equation $(q^{(1)})$ onto the space R of the solutions of $(Q^{(1)})$, P be a cannonical linear mapping of the space R onto the space $\overline{R}$ of all solutions of the equation $(\overline{Q}^{(1)})$. Let X(t) and $\overline{X}(t)$ be general dispersions relative to the mappings p and P, respectively. Consequently, a composed mapping Pp is a cannonical linear mapping of the space r onto $\overline{R}$ and the general dispersion $\overline{\overline{X}}$ relative to Pp is a function $\overline{X}(X(t))$.

Proof: In agreement with property 1) from Theorem 4 it follows that

$$X(t) = \mathcal{A}^{-1}[\alpha(t)] \qquad \text{for } t \in j ,$$

$$\overline{X}(t) = \overline{\mathcal{A}}^{-1}[\mathcal{A}(T)] \qquad \text{for } T \in J ,$$

and thus also

$$\overline{X}(X(t)) = \overline{\mathcal{A}}^{-1}\mathcal{A}\,\mathcal{A}^{-1}[\alpha(t)] = \overline{\mathcal{A}}^{-1}\alpha(t) \qquad \text{for } t \in j ,$$

where $(\alpha, \mathcal{A})$ is a forming phase basis of mapping p, $(\mathcal{A}, \overline{\mathcal{A}})$ is a forming phase basis of mapping P, and $(\alpha, \overline{\mathcal{A}})$ is a forming phase basis of mapping Pp. As a result holds $\overline{X}(X(t)) = \bar{\bar{X}}(t)$. $\overline{\overline{X}}$ is a direct dispersion, if both dispersions X, $\overline{X}$ are direct at the same time or indirect. Otherwice X is an indirectly dispersion, since $\mathcal{X}_{Pp} = \mathcal{X}_P \cdot \mathcal{X}_p$.

Now, we take up the question as to how far general dispersions are characterized by having a given cannonical linear mapping p as their generator. With respect to the fact, that the function X(t) associated only direct or indirect associated zeros of its image $py \in R$ to zeros of arbitrary element $y \in r$, in the sence of definition, the general dispersion is definate uniquely by concrete cannonical mapping p. The validity of the following theorems follows from this.

Theorem 6

By means of two arbitrary directly or indirectly similar normal phases α, $\mathcal{A}$ of equations $(q^{(1)})$, $(Q^{(1)})$ there is determined only one solution of functional equation $\alpha(t) = \mathcal{A}(X(t))$. This solution is a general dispersion of presented equations relative to every cannonical mapping p for which $(\alpha, \mathcal{A})$ represents the forming phase basis.

Proof: The validity follows from the statement above. In accordence with Theorem 2, the pair of normal similar phases determinates a cannonical mapping of r onto R, for which $(\alpha, \mathcal{A})$ is a forming phase basis. This mapping is determined uniquely, except its variations. But to various c the images $cpy \in R$ of arbitrary element $y \in r$ are dependent solutions of $(Q^{(1)})$, whose corresponding zeros coincide. General dispersions, corresponding to this variations cp, $c \neq 0$ are identically equal and by Theorem 3 they represent a unique solution of equation $\alpha(t) = \mathcal{A}(X(t))$.

Theorem 7

Let $t_0 = a_i$, $X_0 = A_i$ or $t_0 = a_i$, $X_0 = A_{m-i}$ be some directly or indirectly associated points of 1-fundamental sequences $(a^{(1)})$, $(A^{(1)})$ relative equations $(q^{(1)})$, $(Q^{(1)})$, let $X_0' > 0$ or $X_0' < 0$, X_0'' are arbitrary numbers. Thus there exists, respectively, exactly one direct or indirect general dispersion $X(t)$ of equations $(q^{(1)})$, $(Q^{(1)})$ fulfiling initial conditions $X(t_0) = X_0$, $X'(t_0) = X_0'$, $X''(t_0) = X_0''$.

Proof: The existence a uniqueness of general dispersion $X(t)$ fulfilling the above conditions follows from the existence and uniqueness of the second part of forming phase basis $(\alpha, \mathcal{A})$ at the fast chosen α. At choosing the phase α of $(q^{(1)})$ which fulfils the initial conditions

$$\alpha(t_0) = 0 \ , \quad \alpha'(t_0) = 1 \ , \quad \alpha''(t_0) = 0 \ , \qquad (14)$$

from relationships (3) and (8) follow uniquely the following initial values of direct or indirect similar normal phase $\mathcal{A}$

$$\mathcal{A}(X_0) = 0 \ , \quad \mathcal{A}'(X_0) = \frac{1}{X_0'} \ , \quad \ddot{\mathcal{A}}(X_0) = \frac{X_0''}{X_0'^3} \ , \qquad (15)$$

and with their help the phase $\mathcal{A}$ of the equation $(Q^{(1)})$ is determined uniquely. Consequently the general dispersion $X(t)$ corresponding to cannonical mapping p with forming phase basis $(\alpha, \mathcal{A})$ is uniquely determined, too and it complete the proof.

Theorem 7 is in accoedance with the statement about existence of two-parametric system of similar phases vanishing in associated points of 1-fundamental sequences. At general choice of associated points t_0, X_0, the existence of the phase $\mathcal{a}$, similar to the phase α, fulfilling conditions (15) wouldn't be guaranteed, with respect to existence of only one--parametric system of corresponding similar phases.

The above mentioned results allows us to define all the regular solutions of equation $(Q^{(1)}q^{(1)})$ defined on the whole interval j, i.e. the solutions of the class $C^{(3)}(j)$ whose derivative is always non-zero.

Theorem 8

All the regular solutions X(t) of the differential equation $(Q^{(1)}q^{(1)})$ being defined on interval j and fulfilling the initial condition

$$X(a_i) = A_i \quad \text{or} \quad X(a_i) = A_{m-i} \quad ,$$

where (a_i, A_i) or (a_i, A_{m-i}), respectively, are some pairs of direct or indirect associated points of 1-fundamental sequences $(a^{(1)})$, $(A^{(1)})$ of equations $(q^{(1)})$, $(Q^{(1)})$, represent exactly all the direct or indirect general dispersions of this pair of differential equations.

Proof:

a) If X is a direct or indirect general dispersion of equations $(q^{(1)})$, $(Q^{(1)})$, corresponding to some cannonical mapping p of r onto R, thus it is fulfilling the condition $X(a_i) = A_i$ or $X(a_i) = A_{m-i}$, respectively, and by 4) and 6) of Theorem 4 it is regular solution of equation $(Q^{(1)}q^{(1)})$.

b) Let X is a regular solution of equation $(Q^{(1)}q^{(1)})$ defined in j, fulfilling the condition $X(a_i) = A_i$ for $X' > 0$ or $X(a_i) = A_{m-i}$ for $X' < 0$. Choosing a phase α of the phase system of some 1-fundamental basis relative to equation $(q^{(1)})$, which is vanishing in the point a_i, for example the phase determined by conditions

$$\alpha(a_i) = 0 \ , \quad \alpha'(a_i) = 1 \ , \quad \alpha''(a_i) = 0 \ .$$

With respect to it we will choose the phase $\mathcal{a}$ of $(Q^{(1)})$ fulfilling initial conditions

$$\mathfrak{a}(X_0) = 0 \ , \qquad \dot{\mathfrak{a}}(X_0) = \frac{1}{X_0'} \ , \qquad \ddot{\mathfrak{a}}(X_0) = -\frac{X_0''}{X_0'^3} \ ,$$

where X_0, X_0', X_0'' are values of function X and its first and second derivative in the point a_i. Thus, this phase $\mathfrak{a}$ is with respect to the phase α directly or indirectly similar phase of some 1-fundamental basis of equation $(Q^{(1)})$. By relationship (18) of § 5 and (17) of § 1 from [1], from following expressions of carriers of both equations on corresponding intervals in the forms

$$-\{\operatorname{tg}\alpha, t\} = q(t) \ , \qquad -\{\operatorname{tg}\mathfrak{a}, X\} = Q(X)$$

follows the relationships

$$-\{X,t\} - \{\operatorname{tg}\mathfrak{a}, X\} \cdot X'^2 = -\{\operatorname{tg}\alpha, t\} \ ,$$

$$\{\operatorname{tg}\mathfrak{a}(X), t\} = \{\operatorname{tg}\alpha, t\} \ ,$$

and thus, from the point of view of 8, § 1 of [1], also the relationship

$$\operatorname{tg}\mathfrak{a}(X) = \frac{c_{11} \ \operatorname{tg}\alpha(t) + c_{12}}{c_{21} \ \operatorname{tg}\alpha(t) + c_{22}} \ ,$$

where $c_{11},\dots,c_{22}$ are the constants. By putting the initial conditions of phases α, $\mathfrak{a}$ we get: $c_{12} = 0$, $c_{11} = c_{22}$, $c_{21} = 0$. It follows that the function X(t) is the solution of functional equation

$$\alpha(t) = \mathfrak{a}(X(t)) \ ,$$

with definition interval j = (a,b) taking the corresponding values from the interval J = (A,B), with respect to similarity of phases. Consequently, X(t) is a direct or indirect general dispersion of equations $(q^{(1)})$, $(Q^{(1)})$, relative to the cannonical mapping of the space r onto R, determined by forming phase basis $(\alpha, \dot{\mathfrak{a}})$. Thus the proof is completed.

Now, we will investigate the relation of general dispersions of equations $(q^{(1)})$, $(Q^{(1)})$ to the Kummer's transformation problem. By transformation of equation $(Q^{(1)})$ to $(q^{(1)})$ in accordance with definition of [1] we mean the two members sequence $[w,X]$ of functions $w(t)$, $X(t)$, which are defined in the open interval $i \subset j$ with properties

$$
\begin{aligned}
&1) \quad w(t) \in C^{(2)}(i)\,, \qquad X(t) \in C^{(3)}(i) \\
&2) \quad w(t)X'(t) \neq 0 \qquad \text{for } t \in i \\
&3) \quad X(i) \subset J
\end{aligned}
\tag{16}
$$

such that for every solution Y of $(Q^{(1)})$ the function $y(t)$ defined by the relationship

$$y(t) = w(t) \,.\, Y(X(t)) \tag{17}$$

is a solution of differential equation $(q^{(1)})$. If $i = j$, that the transformation is said to be global. Function $X(t)$ is called the transformation function of $(q^{(1)})$, $(Q^{(1)})$, the function $w(t)$ is called the multiplicator of transformation. From § 12 of [1] we know that every transformation function X of equations $(q^{(1)})$, $(Q^{(1)})$ is also a solution of nonlinear differential equation $(Q^{(1)}q^{(1)})$ on its own definition interval. Besides, multiplicator is uniquely determined by the transformation function X, except for multiplicative constant $k \neq 0$.

From Theorem 8 it is evident that general dispersion is in near relation to the transformation problem of considered equations.

Theorem 9

Let $X(t)$ be direct or indirect general dispersion of equations $(q^{(1)})$, $(Q^{(1)})$ corresponding to canonical mapping p of r onto R.

a) Let y be an arbitrary element of the space r, $Y = py$ is its image in the space R. Consequently, $Y(X) : \sqrt{|X'|}$ is a solution of equation $(q^{(1)})$ on the whole interval j and at the same time the relation

$$\frac{Y(X(t))}{\sqrt{|X'(t)|}} = \pm \frac{1}{\sqrt{|\mathfrak{X}_p|}}\, y(t) \,, \tag{18}$$

where + or - isn't dependent on the choice of Y, is fulfilling, too.

b) There exists such a variation $cp = p^*$ of mapping p, where

$$\frac{Y^*(X(t))}{\sqrt{|X'(t)|}} = y(t) \qquad (19)$$

holds for an arbitrary element $y \in r$ and his image $Y^* = p^* y \in R$ on the whole interval j. Besides, the characteristic of the mapping p is determined by the relationship $\chi_{p^*} = \text{sign } X'$.

c) If (U_1, V) come 1-fundamental basis of equation $(Q^{(1)})$ and W is a Wronskian of this basis, then $(U_1,(X) : \sqrt{|X'|}$, $V(X) : \sqrt{|X'|})$ is a 1-fundamental basis of $(q^{(1)})$ and for its Wronskian w the relationship $w = W \text{ sign } X'$ holds.

Proof: a) If $(\alpha, \mathcal{A})$ is a forming phase basis of mapping p, then the functional equation $\alpha(t) = \mathcal{A}(X(t))$ holds in the whole interval and for solutions $y(t)$, $Y(X(t))$ the relationship (3) holds. From it and from the expression $\alpha'(t) = \mathcal{A}'(X)X'(t)$ follows that the relationship (18) is proved.

b) At choice of mapping $p^* = cp = \varepsilon E \sqrt{|\chi_p|}p$, where $E = \pm 1$ according as the phases α, $\mathcal{A}$ are proper or unproper with respect to the bases determining the mapping p, $Y^* = Y \varepsilon E \sqrt{|\chi_p|}$ holds. General dispersions corresponding with mapping p and $cp = p^*$ are coinciding identically and the relationship (19) follows immediately by putting into the relationship (18). Besides, from $\chi_{p^*} = c^{-2} \chi_p$ follows that $|\chi_{p^*}| = 1$ and thus $\chi_{p^*} = \text{sign } \chi_p = \text{sign } X'$.

c) If (U_1, V) is a 1-fundamental basis of equation $(Q^{(1)})$ and W is its Wronskian, then there exists a canonical mapping p which maps some 1-fundamental basis (u_1, v) of $(q^{(1)})$ onto (U_1, V). With respect to the validity of (18), the solutions u_1, $U_1(X) : \sqrt{|X'|}$ or v, $V(X) : \sqrt{|X'|}$, respectively, are dependent. Consequently, $(U_1,(X) : \sqrt{|X'|}$, $V(X) : \sqrt{|X'|})$ is also a 1-fundamental basis of equation $(q^{(1)})$. The relationship $w = W \text{ sign } X'$ follows immediately from the direct commputation of w.

From the first part of Theorem 9 it follows that every direct or indirect general dispersion X of $(q^{(1)})$, $(Q^{(1)})$ is the transformation function of these equation, which is defining in the whole interval j and fulfilling the initial condition $X(a_i) = A_i$ or $X(a_i) = A_{m-i}$, respectively, for some pair of directly or indirectly associated points of $(a^{(1)})$, $(A^{(1)})$. Every transformation function of these equations has such property, that it is satisfying the equation $(Q^{(1)}q^{(1)})$. From Theorem 8 it follow immediately the following statements.

Theorem 10

Transformation functions X(t) of equations $(q^{(1)})$, $(Q^{(1)})$, defined in the whole interval j and satisfying the initial condition $X(a_i) = A_i$ for $X' > 0$ or $X(a_i) = A_{m-i}$ for $X' < 0$ where (a_i, A_i) or (a_i, A_{m-i}), respectively, is some pair of directly or indirectly associated points of 1-fundamental sequences $(a^{(1)})$, $(A^{(1)})$ are exactly all the general dispersions of equations $(q^{(1)})$, $(Q^{(1)})$.

Proof: By a) of Theorem 9 and 4) of Theorem 4, every general dispersion of considered equations satisfies the properties (16), (17). So, it is a transformation function satisfying the initial mentioned above and conversely, by 2, § 11 of [1] every transformation function of equations above is a regular solution $(Q^{(1)}q^{(1)})$. If satisfying the initial condition above, it is, by Theorem 8, a general dispersion of equations $(q^{(1)})$, $(Q^{(1)})$.

If we use yet the statement (3. § 27 of [1] about a structure of a set of global solutions of a differential equation (Qq) for equations (q), (Q) of a finite type, we may say the following corollary is true.

Corollary 1

Global transformation function X of equations $(q^{(1)})$, $(Q^{(1)})$ are exactly all the general dispersions, in the sence of Definition 1, of these equations.

Proof: The statement follows from Theorem 10 and from 3 § 27 of [1]. It is evident, now, that every complete solution of equation $(Q^{(1)}q^{(1)})$ satisfies the initial condition mentioned in Theorem 10.

In accordance with this statement, we may describe the structure of a set all general dispersions $(q^{(1)})$, $(Q^{(1)})$, corresponding with different canonical mapping of space r onto R, in the following way.

Theorem 11

A set M of all general dispersions X of equations $(q^{(1)})$, $(Q^{(1)})$, composed of two disjunct subsets M_p and M_n, respectively, of direct or indirect general dispersions, is a two-parametric system which we will call the bunch. The bunch of general dispersions is a one-parametric system of one-parametric subsystems M_σ, where σ is a real number, which are called bundels. Every bundel M_σ is composed of two disjunct subbundels M_{σ_p}, M_{σ_n} composed of only direct or only indirect general dispersions of equation mentioned above. All curves $[t,X(t)]$ for $X \in M_{\sigma_p}$ or for $X \in M_{\sigma_n}$, respectively, pass through m-1 common points $P(a_i,A_i)$ or $P(a_i,A_{m-i})$ with coordinates formed by pairs of directly or indirectly associated points of sequences $(a^{(1)})$, $(A^{(1)})$. Besides all the curves $[t,X(t)]$ for $X \in M_{\sigma_p}$ or for $X \in M_{\sigma_n}$ with fixed value σ pass through m common points $P(t_i,T_i)$ or $P(t_i,T_{m-i+1})$, where $t_i \in (a_{i-1},a_i)$ for $i = 1,2,\dots,m$ are 1-conjugate points of $(q^{(1)})$ and $T_i \in (A_{i-1},A_i)$ for $i = 1,2,\dots,m$ are 1-conjugate points of $(Q^{(1)})$, respectively.

Proof: Consider some of 1-fundamental bases (u_1,v) of the space r. To this basis there exists three-parametric system of canonical linear mappings which map this basis to some of 1-fundamental basis of the space R. Every such mapping is determined by some pair of bases (u_1,v), $(\varrho U_1, \sigma V + \bar{\sigma} U_1)$ where ϱ, σ, $\bar{\sigma}$ are real numbers, $\varrho\sigma \neq 0$, (U_1,V) is arbitrarily chosen fixed 1-fundamental basis of the space R. With respect to the fact that proportional second bases of given pair determining mapping p are corresponding with different variations of the same mapping p, the canonical independent mappings forms only two-parametric system defined by pairs of bases (u_1, v), $(\varrho U_1, V + \sigma U_1)$ and a system of appropriate general dispersions corresponds with it. Besides, all the general dispersions map points a_i of $(a^{(1)})$ at the points A_i or A_{m-i} of 1-fundamental sequence $(A^{(1)})$. Thus, corresponding curves of general dispersions pass through the points $P(a_i, A_i)$ or $P(a_i, A_{m-i})$, respectively. At concrete choice of parameter σ we talk about a bundel of dispersions. For different ϱ we obtain different elements of bundel M_σ and all of them have a property such that they map zeros in the points t_i of solution v onto directly or indirectly associated zeros in the points T_i or T_{m-i+1} of solution $V + \sigma U_1$, respectively. Hence, their curves pass through the points $P(t_i, T_i)$ or $P(t_i, T_{m-i+1})$ for $i = 1, 2, \dots, m$.

The structure of a set of general dispersions corresponds with the structure of a set of normal phases a of equation $(Q^{(1)})$, which are similar to some of chosen phase α of $(q^{(1)})$.

Corollary 2

Let A, B are an arbitrary number such that $(B-A) = m\pi$. Then every increasing or decreasing first phase $\alpha(t)$ of $(q^{(1)})$ satisfying the condition

$$\lim_{t \to a^+} \alpha(t) = A \qquad \text{or} \qquad \lim_{t \to a^+} \alpha(t) = B$$

is a direct or indirect general dispersion of equation $(q^{(1)})$, $(-1^{(1)})$, where $(-1^{(1)})$ is a differential equation $Y'' = -Y$, considered in the interval (A, B), respectively.

Proof: Every increasing or decreasing phase α of equation $(q^{(1)})$ satisfies the differential equation

$$-\{\alpha(t), t\} + \alpha'^2(t).(-1) = q(t) \quad ,$$

in the interval (a,b), with accordance with statement from 5 § 5 of [1].

The phase α fulfilling the condition

$$\lim_{t\to a^+} \alpha(t) = A \quad \text{for} \quad \alpha' > 0 \quad \text{or} \quad \lim_{t\to a^+} \alpha'(t) = B$$
$$\text{for} \quad \alpha' < 0 \quad ,$$

where A,B are arbitrary numbers such that $(B-A) = m\pi$, represents a regular solution of equation $(-1^{(1)}q^{(1)})$ where $(-1^{(1)})$ is equation $Y'' = -Y$, 1-special type m in the interval (A,B) which is $m\pi$ long and satisfying the assumptions of Theorem 8. Thus, it is direct or indirect general dispersion of $(q^{(1)})$, $(-1^{(1)})$ corresponding with some of canonical mapping of the space r onto the space of all the solution of equation $(-1^{(1)})$.

Corollary 3

To every increasing or decreasing phase α of an arbitrary equation $(q^{(1)})$ there is always such a definition interval (A,B), $m\pi$ long, of equation $(-1^{(1)})$, that the phase α is direct or indirect general dispersion of equations $(q^{(1)})$, $(-1^{(1)})$ with respect to some of canonical mapping p of the space r onto the space of all the solution of equation $(-1^{(1)})$.

Proof: For an increasing phase α we will consider the interval $(c, c+m\pi)$, where $c = \lim \alpha(t)$ for $t \longrightarrow a^+$. For a decreasing phase we may consider the interval $(d-m\pi, d)$ where $d = \lim \alpha(t)$ for $t \longrightarrow a^+$. Consequently the assumptions of Corollary 2 are fulfiled and the validity of the statement above, too.

SOUHRN

K TEORII GLOBÁLNÍCH TRANSFORMACÍ LINEÁRNÍCH DIFERENCIÁLNÍCH ROVNIC 2.ŘÁDU, KONEČNÉHO TYPU - SPECIÁLNÍCH

EVA TESAŘÍKOVÁ

V článku je rozpracována teorie transformací pro homogenní lineární diferenciální rovnice 2.řádu $y'' = q(t)y$ konečného typu $m \geq 2$ speciálních na příslušném konečném či nekonečném definičním intervalu $j = (a,b)$ za využití výsledků z teorie dispersí, formulované v literatuře [1] pro rovnice oboustranně oscilatorické ve vztahu k transformačnímu problému Kummera.

Cílem článku je zavedení pojmu obecných dispersí dvou rovnic $y'' = q(t)y$, $Y'' = Q(T)Y$, $q(t) \in C^0(j)$, $Q(T) \in C^0(J)$, 1-speciálních téhož konečného typu $m - 2$ na příslušných definičních intervalech $j = (a,b)$, $J = (A,B)$, vyšetření jejich vlastností a jejich vztahu k problému globálních transformací těchto rovnic.

РЕЗЮМЕ

К ТЕОРИИ ГЛОБАЛЬНЫХ ТРАНСФОРМАЦИЙ ЛИНЕЙНЫХ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ 2. ПОРЯДКА, КОНЕЧНОГО ТИПА - СПЕЦИАЛЬНЫХ

Э. ТЕСАРЖИКОВА

В этой статье разработана трансформационная теория для однородных линейных дифференциальных уравнений 2.порядка $y'' = q(t)y$ конечного типа $m \geq 2$, специальных на принадлежащем конечном или бесконечном интервалах опрерeления $j=(a,b)$ при помощи как можно более широкого применения результатов из теории дисперсий, основанной на литературе /1/ для уравнений с осцилирующими решениями, в связи с решением трансформационной проблемы Куммера.

Целью работы является введение понятий общих дисперсий двух уравнений $y'' = q(t)y$, $Y'' = Q(T)Y$, $q(t) \in C^{(0)}(j)$, $Q(T) \in C^{(0)}(J)$ 1-специальных того же самого типа $m - 2$ на промежутках определения $j = (a,b)$, $J = (A,B)$, исследование их свойств и их отношения к проблеме глобальных трансформаций подходящих уравнений.

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