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ON THE UNIFORM BOUNDEDNESS OF ELEMENTS OF THE TYPE exp (ih) IN HERMITIAN lmc-ALGEBRAS

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Abstract.

It has been shown, s.f. Š t ě r b o v á, D. (1978), that a Banach star algebra is Hermitian if and only if the spectra of elements of the form exp(ih), where h is selfadjoint, are uniformly bounded. The present paper will show that this is true also for every complete lmc-star algebra with a not necessarily continuous involution.

1. Notations and preliminaries.

All linear spaces are over the complex field C. The reader is assumed to be familiar with the basic concepts concerning the topological algebras, namely the Banach algebras, locally multiplicatively convex (lmc) algebras, with the notion of representation, and so on. See for B o n s a l l, F., D u n c a n, J. (1973), M i ch a e l, E.

(1952), N a j m a r k, A. (1968), Ž e l a z k o, W. (1971).
Let us recall now some notations and preliminary facts.
Let A be a complete lmc-algebra and let $\{q_\alpha\}_{\alpha\in\Sigma}$ be the directed set of submultiplicative seminorms on A separating points at A, such that $A = \varprojlim A_\alpha$, where A_α denotes the completion of the normed algebra $\{A/\mathrm{Ker}\, q_\alpha, q_\alpha\}$. By π_α we denote the natural homomorphism mapping A onto A_α. Throughout, the spectrum of an element $x\in A$ will be denoted by $\sigma(x,A)$ pointing out that it is taken with respect to A. Obviously, $x\in A$ is regular if and only if for each $\alpha\in\Sigma$ the $\pi_\alpha(x)$ is regular in the algebra A_α yielding the equality $\sigma(x,A) = \bigcup_{\alpha\in\Sigma}\sigma(x_\alpha,A_\alpha)$. If there is no risk of confusion we use the notation $\pi_\alpha(x) = x_\alpha$. The set of all multiplicative linear (multiplicative continuous linear, multiplicative linear continuous with respect to q_α) functionals will be denoted by $\mathcal{M}(A)$, $(\mathcal{M}^c(A), \mathcal{M}^c_\alpha(A))$ respectively. Obviously for each $\alpha\in\Sigma$ $\mathcal{M}^c_\alpha(A)\subset\mathcal{M}^c(A)\subset\mathcal{M}(A)$ and $\mathcal{M}^c(A) =$
$= \bigcup_{\alpha\in\Sigma}\mathcal{M}^c_\alpha(A)$.

Definition 1.1.: Let A be a commutative lmc-algebra. The Gelfand representation of A is the mapping $G: A\to C(\mathcal{M}^c(A))$, $G(x)(f) = f(x)$ for each $f\in\mathcal{M}^c(A)$ and each $x\in A$. Recall, that $C(\mathcal{M}^c(A))$ means as usual the algebra of all complex continuous functions on $\mathcal{M}^c(A)$ with respect to the topology defined by all neighbourhoods $\mathcal{U}(f_o\varepsilon, x_1,\dots x_n) = \{f\in\mathcal{M}^c(A): |f(x_i) -$
$- f_o(x_i)| < \varepsilon,\ i = 1,2,\dots n$ where $f_o\in\mathcal{M}^c(A)$, $\varepsilon > 0$.

For each $\alpha\in\Sigma$ we evidently may identify the space $\mathcal{M}^c_\alpha(A)$ with $\mathcal{M}(A_\alpha)$ by the homeomorphism F defined as follows: $F(f)(x) =$
$= f(x_\alpha)$, $x\in A$, $f\in\mathcal{M}(A_\alpha)$. For the spectral radius $|x|^A_\sigma =$
$= \sup|\pi_\alpha(x)|_\sigma^{A_\alpha}$.

Definition 1.2.: The element $x\in A$ is said to be Hermitian if $x = x^*$. The set of all Hermitian elements of A will be denoted by H(A). The element $x\in A$ is said to be normal if $xx^* = x^*x$. The set of all normal elements will be denoted by

N(A). The set $M \subset A$ is said to be normal if all elements of the set $M \cup M^*$ are pairwise commuting.

Definition 1.3.: The star algebra A is said to be Hermitian if the spectrum $\sigma(x)$ is real for any $x \in H(A)$.
Recall now some useful propositions.

Propositions 1.4.: Let A be a complete commutative lmc-star algebra with the unit element e, $\{q_\alpha\}_{\alpha \in \Sigma}$ the corresponding directed set of submultiplicative seminorms defining the topology on A. Then for each $x \in A$: $\sigma_A(x) = G(x)(\mathcal{M}^c A) = \bigcup_{\alpha \in \Sigma} G_\alpha(\pi_\alpha(x))((A_\alpha)) = \bigcup_{\alpha \in \Sigma} \mathcal{M}(A_\alpha)(x_\alpha)$, where G_α denotes the Gelfand representation mapping of A_α.

P r o o f : This fact is wellknown for each Banach algebra. The rest of the verification follows immediately on using the equality $\sigma_A(x) = \bigcup_{\alpha \in \Sigma} \sigma_{A_\alpha}(\pi_\alpha(x))$. Q.E.D.

Proposition 1.5.: Let A be a complete, commutative Hermitian lmc-star algebra with the unit element e. Then for each $x \in A$ and for each continuous multiplicative funcional $f \in \mathcal{M}^c(A)$, $f(x^*) = \overline{f(x)}$.

P r o o f : This obviously holds for any arbitrary commutative Banach star algebra. Using the preceeding proposition we easily get for each $x \in A$:

$$\sigma_A(x) = \{ f(x),\ f \in \mathcal{M}^c(A) \} = \bigcup_{\alpha \in \Sigma} \{ f(x),\ f \in \mathcal{M}^c_\alpha(A) \} = \bigcup_{\alpha \in \Sigma} \{ f(\pi_\alpha(x)),\ f \in \mathcal{M}(A_\alpha)$$

If $h \in H(A)$, then $\sigma_A(h)$ is real thus for each $\alpha \in \Sigma$ and for each $f \in \mathcal{M}^c_\alpha(A)$ the value f(h) is real. Let further x be an arbitrary element of A. Then there exists a unique couple $h, k \in H(A)$ such that $x = h + ik$. Then there follows for each $\alpha \in \Sigma$ and for all $f \in \mathcal{M}^c_\alpha(A)$ that $f(x^*) = \overline{f(h) + if(k)} = \overline{f(x)}$. Q.E.D.

Proposition 1.6.: Let A be a complete lmc-star algebra, and $N \subset A$ be a normal subset of A. Then there exists a maximal closed commutative star subalgebra $C \subset A$ containing N as a subset and such that for each $x \in C$, $\sigma_C(x) = \sigma_A(x)$.

P r o o f : This easily follows on using Zorn's maximal principle. Q.E.D.

Definition 1.7.: Let A be a complete lmc-algebra with the unit e and a A. The symbol exp(a) denotes the element:

$$\exp(a) = e + a + \frac{a^2}{2} + \dots \frac{a^n}{n!} + \dots = \sum_{n=0}^{\infty} \frac{a^n}{n!}$$

Note 1.8.: The completeness of A yields the existence of exp(a) for an arbitrary $a \in A$. Further, for each $a \in A$ it can be easily seen that

(i) $\exp(a) = \exp(\pi_\alpha a))_{\alpha \in \Sigma}$

(ii) $\pi_\alpha(\exp(a)) = \exp(\pi_\alpha(a)) = \exp(a_\alpha)$ for an arbitrary $\alpha \in \Sigma$.

Proposition 1.9.: Let A be a complete lmc-algebra with the unit element e. Let $a, b \in A$ be such that $ab = ba$. Then the following relations are true:

(i) $\exp(a + b) = \exp(a).\exp(b)$

(ii) $(\exp(a))^{-1} = \exp(-a)$

(iii) $\exp(a) = \lim_{n \to \infty}(e + a/n)^n$

(iv) $\sigma(\exp(a)) = \exp(\sigma(a))$

P r o o f : All these facts are wellknown for Banach algebras. On using them together with the definition of the projective limit of Banach algebras there directly follows the proof of (i), (ii), (iii). Since $\sigma(x) = \bigcup_{\alpha \in \Sigma} \sigma_\alpha(x)$ there immediately follows:

$$\sigma(\exp(a)) = \bigcup_{\alpha\in\Sigma} \sigma(\pi_\alpha(\exp(a))) = \bigcup_{\alpha\in\Sigma} \sigma(\exp \pi_\alpha(a)) =$$

$$= \bigcup_{\alpha\in\Sigma} \exp(\sigma(\pi_\alpha(a))) = \bigcup_{\alpha\in\Sigma} \exp(\sigma_\alpha(a)) = \exp\sigma(a).$$

Q.E.D.

2. Spectral properties of exp(i H(A)) in Hermitian lmc-algebras.

In this part we give the main result which means the statement that the uniform boundedness of spectra for elements belonging to exp(iH(A)) is equivalent to the fact that the complete lmc-star algebra is Hermitian. As we do not suppose the continuity of the involution the proofs of the next lemmas and the theorem demand some refined methods.

Lemma 2.1.: Let A be a complete Hermitian lmc-star algebra with the unit element e and $x \in N(A)$ be a regular element of A. If $\sigma(x^* - x^{-1}) = \{0\}$, then $|\sigma(x)| = 1$.

P r o o f : Since $\sigma(x^* - x^{-1}) = \{0\}$, then by 1.4. it follows for each multiplicative functional $f \in \mathcal{M}^c(C(e,x^*,x))$ (where $C(e,x^*,x)$ is the maximal commutative subalgebra in A containing the elements e,x^*,x) that $f(x^* - x^{-1}) = 0$ and thus

$$f(x(x^* - x^{-1})) = f(x).f(x^* - x^{-1}) = f(x).0 = 0$$

Again, using 1.4. we find that

$$\sigma(x^*x - e) = \sigma(xx^* - e) = \{0\}$$

while $\sigma(x^*x) = \sigma(x\, x^*) = \{1\}$. The latter equality implies that

$$|\sigma(x^*x)| = |\{f(x^*x) : f \in \mathcal{M}^c(C(e,x^*,x))\}| = 1$$

As A is Hermitian so is $C(e,x^*,x)$ and by 1.5. for each $f \in {}^c(C(e,x^*,x))$, $f(x^*) = \overline{f(x)}$. This implies $|\sigma(x)| = 1$.

Q.E.D.

Lemma 2.2.: Let A be the same algebra as in Lemma 2.1. Then for each $h \in H(A)$, $\sigma((\exp(ih))^* - (\exp(ih))^{-1} = \{0\}$.

P r o o f : Let us consider $C(e,h)$ the maximal commutative subalgebra of A containing elements e,h. Then clearly

$$\sigma((\exp(ih))^* - \exp(-ih)) = \{f((\exp(ih))^* - \exp(-ih)) : \\ : f \in \mathcal{M}^c(C(e,h))\} .$$

Since the algebra A is Hermitian, then also is $C(e,h)$ which means that the spectrum $\sigma_A(g) = \sigma_{C(e,g)}(g)$ is real for each $g \in H(A)$. Suppose $f \in \mathcal{M}^c(C(e,h))$. Then $f(x^*) = \overline{f(x)}$ for each $x \in C(e,h)$. Thus

$$f((\exp(ih))^*) = \overline{f(\exp(ih))} = \overline{\exp(if(h))} = \exp(-if(h)) = \\ = f(\exp(-ih))$$

(the second and the fourth equality are due to the continuity of f while the third equality follows from the fact that the value $f(h)$ is real). Now it can be easily seen that

$$\sigma((\exp(ih))^* - \exp(-ih)) = \sigma((\exp(ih))^* - \\ - (\exp(ih))^{-1} = \{0\}$$

Q.E.D.

Theorem 2.3.: Let A be a complete lmc-star algebra with the unit element e and let $\{q_\alpha\}_{\alpha \in \Sigma}$ be the corresponding directed set of seminorms on A. Then the following statements are equivalent:

(i) A is Hermitian

(ii) For each $h \in H(A)$ there is $|\sigma(\exp(ih))| = 1$

(iii) There exists a positive number $M > 0$ such that for each $h \in H(A)$ we have $|\sigma(\exp(ih))| \leqq M$

P r o o f : (i) $\rightarrow$ (ii) :

Suppose $h \in H(A)$. Using the completeness of $C(e,h)$ we easily get both exp(ih) and $(\exp(ih))^*$ belong to C(e,h). It immediately follows that they commute and hence they are normal. By 1.7. and 1.9. we have exp(ih) being regular in A and thus by the precceding two lemmas we obtain $|\sigma(\exp(ih))| = 1$.

(ii) $\longrightarrow$ (iii) : being trivial we prove the last implication (iii) $\longrightarrow$ (i) :

Suppose now conversely that $i \in \sigma(h)$ is valid for some $h \in H(A)$. Then by 1.9. $\exp \tau \in \sigma(\exp(-i\tau h))$ for any positive τ. Obviously $-\tau h \in H(A)$ and by our assumption there exists a positive M such that $|\sigma(\exp(-i\tau h))| \leqq M$ for each $\tau > 0$. For each $\tau > 0$ the above inequality immediately implies that $|\exp \tau| \leqq M$. This contradiction proves the last implication.

Q.E.D.

REFERENCES

[1] B o n s a l l, F., D u n c a n, J.: Complete Normed Algebras, Springer-Verlag (1973).

[2] M i ch a e l, E.: Locally Multiplicatively Convex Topological Algebras, Memoirs of AMS nb.11 (1952).

[3] N a h m a r k, M.A.: Normirovannyje kolca, Moskva (1968).

[4] P t á k, V.: Banach algebras with involution, Manuscripta math.6, Springer-Verlag (1972), 245-290.

[5] Š t ě r b o v á, D.: On the spectral properties of elements of the type exp(ih) in Hermitian Banach algebras, AUPO, Fac.rer.nat., Vol.57, Mathematica 17, Praha SPN (1978), 109-114.

[6] Ž e l a z k o, W.: Selected Topics in Topological Algebras, Lecture Notes Series N.31 (1971), Aarhus University.

O STEJNÉ OHRANIČENOSTI PRVKŮ TYPU exp(ih) V HERMITOVSKÉ lmc-ALGEBŘE

Souhrn

V práci je dokázáno, že úplná lokálně m-konvexní algebra s involucí je Hermiteovská, právě když jsou spektra všech prvků tvaru exp(ih) (kde h je samoadjungovaný) stejně omezená. Nepředpokládá se přitom spojitost uvažované involuce.

ОБ РАВНОМЕРНОЙ ОГРАНИЧЕННОСТИ ЭЛЕМЕНТОВ ТИПА exp/ / В ПОЛУНОРМИРОВАННЫХ КОЛЬЦАХ С ИНВОЛЮЦИЕЙ

Резюме

В настоящей статье показывается, что полное полунормированное кольцо с инволюцией является вполне симметрическим тогда и только тогда, если спектры всех элементов вида exp/ih/, где h-симметрическое, равномерно ограничены. При этом непрерывность инволюции непредполагается.

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