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A NOTE TO THE LAPLACE TRANSFORMATION

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The Laplace method in partial differential equations is a problem of the last century. Combinating this method with a linear transformation enables us to solve a greater class of partial second order differential equations.

We will study the following equation

$$a_1(t)\frac{\partial^2 u}{\partial t^2} + a_2(t)\frac{\partial u}{\partial t} + a_3(t)u = b_1(y)\frac{\partial^2 u}{\partial y^2} + b_2(y)\frac{\partial u}{\partial y} + b_3(y) + q(t,y) \tag{1}$$

for $t \in J_1$, where $J_1 = \langle 0, \infty)$, $y \in J_2$, where $J_2 = \langle y_1, y_2 \rangle$, y_1 and y_2 are real numbers, $y_1 < y_2$ with boundary conditions

$$u(t,y_1) = g_1(t), \quad u(t,y_2) = g_2(t) \tag{2}$$

$$u(o,y) = b_1(y), \quad \frac{\partial}{\partial t} u(o,y) = f_2(y), \tag{3}$$

there $a_1(t) \in C^2(J_1)$, $a_2(t) \in C^1(J_1)$, $g_1(t)$, $g_2(t) \in C(J_1)$, $b_1(y)$, $b_2(y)$, $b_3(y) \in C(J_2)$, $g(t,y) \in C(J_1 \times J_2)$ and $a_1(t) > 0$ for $t \in J_1$. Let us denote

$$x(t) = \int_0^t \frac{dz}{\sqrt{a_1(z)}} \tag{4}$$

and $t(x)$ as the inversion function of $x(t)$, and further

$$P(x) = \int_0^x \frac{1}{2\sqrt{a_1(t)}} \left(a_2(t) - \frac{1}{2} a_1'(t)\right) dx \tag{5}$$

$$A_2(x) = a_2(t(x)) \frac{1}{t'(x)} - a_1(t(x)) \frac{t''(x)}{t'^3(x)} \tag{6}$$

$$A_3(x) = \dot{a}_3(t(x)) \tag{7}$$

Suppose $\lim_{t \to \infty} x(t) = \infty$ and there exist Laplace images of functions

$$e^{P(x)} q(t(x),y), \quad e^{P(x)} g_1(t(x)), \quad e^{P(x)} g_2(t(x))$$

for every $y \in J_2$. Let us denote them $Q(\lambda,y)$, $G_1(\lambda,y_1)$ and $G_2(\lambda,y_2)$. Denote next that

$$S(\lambda,y) = f_1(y)(P'(0) - \lambda - A_2(0)) - f_2(y) - Q(\lambda,y).$$

On the other hand we will be interested in the equation

$$(c_0+\lambda^2)\, v(\lambda,y) + S(\lambda,y) = b_1(y)\frac{\partial^2 v}{\partial y^2} + b_2(y)\frac{\partial v}{\partial y} + b_3(y) v \tag{8}$$

with boundary conditions

$$v(\lambda, y_1) = G_1(\lambda, y_1), \quad v(\lambda, y_2) = G_2(\lambda, y_2) \tag{9}$$

where $\lambda \in (0, \infty)$, $y \in J_2$ and c_o is a real number. Let

$$a_3(t) = c_0 + \frac{1}{2}\sqrt{a_1(t)}\left(\frac{a_2(t) - \frac{1}{2}a_1'(t)}{\sqrt{a_1(t)}}\right)' +$$

$$+ \frac{1}{4a_1(t)}\left(a_2(t) - \frac{1}{2}a_1'(t)\right)^2 \tag{10}$$

hold.

Definition: We call $u(t,y)$ a solution of equation (1), where $t \in J_1$, $y \in J_2$ if $u(t,y)$ satisfies equation (1) everywhere and besides $u(t,y) \in C^2(J_1 \times J_2)$.

Theorem: Let $v(\lambda, y)$ be

1) the Laplace image of the function $e^{P(x)}u(t(x),y)$,

2) the solution of the differential equation (8) with boundary conditions (9).

Suppose for every $\lambda \in (0, \infty)$, $y \in J_2$ and for $k = 1,2$ we have

$$\lim_{x\to\infty} e^{P(x)-\lambda x} \frac{\partial}{\partial x} u(t(x),y) = \lim_{x\to\infty} \frac{\partial}{\partial x}\left(e^{P(x)-\lambda x}\right)u(t(x),y) =$$

$$= \lim_{x\to\infty} A_2(x)e^{P(x)-\lambda x}\, u(t(x),y) = 0, \tag{11}$$

and

$$\frac{\partial^k}{\partial y^k}\int_0^\infty e^{P(x)-\lambda x}\, u(t(x),y)dx = \int_0^\infty \frac{\partial^k}{\partial y^k}\left(e^{P(x)-\lambda x}\, u(t(x),y)\right)dx.$$

Let us assume $u(t(x),y) \in C^2(J_1 \times J_2)$ and that there exist the Laplace images of the functions

$$e^{P(x)}\frac{\partial^2 u}{\partial x^2}, \; A_2(x)\, e^{P(x)}\frac{\partial u}{\partial x}, \; A_3(x)\, e^{P(x)}\, u .$$

If the function u(t,y) satisfies conditions (3), then it is a solution of the partial differential equation (1) with boundary conditions (2) and (3).

Proof. The proof is based on the classical method of the Laplace transformation. We have

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} \cdot \frac{1}{t'(x)}$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \cdot \frac{1}{t'^2(x)} - \frac{\partial u}{\partial x} \frac{t''(x)}{t'^3(x)} .$$

On substituting these identities in equation (1), then multiplying out by $e^{P(x)-\lambda x}$, integrating with respect to x from zero to infinity and using (4) we obtain

$$\int_0^\infty e^{P(x)-\lambda x} \frac{\partial^2 u}{\partial x^2} dx + \int_0^\infty A_2(x) e^{P(x)-\lambda x} \frac{\partial u}{\partial x} dx +$$

$$+ \int_0^\infty A_3(x) e^{P(x)-\lambda x} u \, dx = b_1(y) \int_0^\infty e^{-\lambda x+P(x)} \frac{\partial^2 u}{\partial y^2} dx +$$

$$+ b_2(y) \int_0^\infty e^{-\lambda x+P(x)} \frac{\partial u}{\partial y} dx + b_3(y) \int_0^\infty e^{-\lambda x+P(x)} u \, dx +$$

$$+ \int_0^\infty e^{-\lambda x+P(x)} q(t(x),y) \, dx . \qquad (12)$$

Integrating by parts and using assumptions (11), (3) and (4) yields

$$\int_0^\infty e^{P(x)-\lambda x} \frac{\partial^2 u}{\partial x^2} dx =$$

$$= \left[e^{P(x)-\lambda x} \frac{\partial u}{\partial x} \right]_{x=0}^{x=\infty} - \left[\frac{}{x} (e^{P(x)-\lambda x}) u \right]_{x=0}^{x=\infty} +$$

$$+ \int_0^\infty \frac{\partial^2}{\partial x^2} (e^{P(x)-\lambda x}) \; u \; dx = - f_2(y) + f_1(y)(P'(0)-\lambda) +$$

$$+ \int_0^\infty \frac{\partial^2}{\partial x^2} (e^{P(x)-\lambda x}) \; u \; dx$$

$$\int_0^\infty A_2(x) \; e^{P(x)-\lambda x} \frac{\partial u}{\partial x} dx =$$

$$= \left[A_2(x) \; e^{P(x)-\lambda x} \; u \right]_{x=0}^{x=\infty} - \int_0^\infty \frac{\partial}{\partial x} (A_2(x) e^{P(x)-\lambda x}) u \; dx =$$

$$= - A_2(0) \; f_1(y) - \int_0^\infty \frac{\partial}{\partial x} (A_2(x) \; e^{P(x)-\lambda x}) \; u \; dx \quad .$$

On substituting these identities in equation (12) gives

$$\int_0^\infty \Big(\frac{\partial^2}{\partial x^2} (e^{P(x)-\lambda x}) - \frac{\partial}{\partial x} (A_2(x) \; e^{P(x)-\lambda x}) +$$

$$+ A_3(x) \; e^{P(x)-\lambda x} \Big) \; u \; dx + S(\lambda, y) = b_1(y) \frac{\partial^2 v}{\partial y^2} +$$

$$+ b_2(y) \frac{\partial v}{\partial y} + b_3(y) v \tag{13}$$

We want to show equation (13) to be equivalent to (8) which is true if and only if the equations

$$P'^2(x) + P''(x) - A_2'(x) - A_2(x)\,P'(x) + A_3(x) = c_o$$

$$-\,2\,P'(x) + A_2(x) = 0$$

are fulfilled. These equations follow from (4), (5), (6), (7) and (10). Thus equation (1) is equivalent to (8) under the above assumptions. From here and from the properties of the Laplace transformation it then follows that $u(t,y)$ is a solution of equation (1) under the boundary conditions (2) and (3).

<u>Example</u>: Let us solve the equation

$$(t+1)\,\frac{\partial^2 u}{\partial t^2} + \left(-\,2\,\sqrt{t+1} + \frac{1}{2}\right)\frac{\partial u}{\partial t} =$$

$$= (y+2)\,\frac{\partial^2 u}{\partial y^2} + 3\,\frac{\partial u}{\partial y} - u - 2\sqrt{t+1}\,(y-1) + \frac{1}{2}\,(y-1) +$$

$$+ (t+1)(-\,4\sqrt{t+1} + t + y) \tag{14}$$

with boundary conditions

$$u(t,0) = t^2 + t \qquad u(t,1) = t + 1^2 \tag{15}$$

$$u(0,y) = y \qquad \frac{\partial u}{\partial t}(0,y) = y + 1 \tag{16}$$

With the help of (2) - (11) we find that

$$x(t) = 2\sqrt{t+1} - 2$$

$$t(x) = \frac{1}{4}\,x^2 + x\,, \qquad P(x) = -\,x\,, \qquad c_o = -\,1$$

$$q(t,y) = -4(t+1)^{\frac{3}{2}} - 2\sqrt{t+1}(y-1) + \frac{1}{2}\,(y-1) + (t+1)(t+1+y-1)$$

$$Q(\lambda,y) = \frac{3}{2}\,\frac{1}{(\lambda+1)^5} + \left(\frac{1}{2}y - \frac{7}{2}\right)\frac{1}{(\lambda+1)^3} \cdot 4\,\frac{1}{(\lambda+1)^2} -$$

$$-\left(\frac{1}{2}\,y + \frac{5}{2}\right)\frac{1}{\lambda+1}$$

$$S(\lambda,y) = -\lambda y - 1 - \frac{3}{2}\frac{1}{(\lambda+1)^5} - \left(\frac{1}{2}y - \frac{7}{2}\right)\frac{1}{(\lambda+1)^3} +$$
$$+ 4\frac{1}{(\lambda+1)^2} + \left(\frac{1}{2}y + \frac{5}{2}\right)\frac{1}{\lambda+1}$$

Equation (8) becomes the following form

$$(\lambda^2-1)v(\lambda,y) - \lambda y - 1 - \frac{3}{2}\frac{1}{(\lambda+1)^5} - \left(\frac{1}{2}y - \frac{7}{2}\right)\frac{1}{(\lambda+1)^3} +$$
$$+ 4\frac{1}{(\lambda+1)^2} + \left(\frac{1}{2}y + \frac{5}{2}\right)\frac{1}{\lambda+1} = (y+2)\frac{\partial^2 v}{\partial y^2} + 3\frac{\partial v}{\partial y} - v(\lambda,y)$$

with the boundary conditions

$$v(\lambda,0) = \frac{3}{2}\frac{1}{(\lambda+1)^5} + \frac{1}{(\lambda+1)^4} + \frac{5}{2}\frac{1}{(\lambda+1)^3} + \frac{1}{(\lambda+1)^2}$$
$$v(\lambda,1) = \frac{3}{2}\frac{1}{(\lambda+1)^5} + \frac{3}{(\lambda+1)^4} + \frac{3}{(\lambda+1)^3} + \frac{2}{(\lambda+1)^2} + \frac{1}{\lambda+1} .$$

Seeking for $v(\lambda,y)$ in a polynomial form, we find

$$v(\lambda,y) = \frac{3}{2}\frac{1}{(\lambda+1)^5} + \frac{3}{(\lambda+1)^4} + \left(\frac{1}{2}y + \frac{5}{2}\right)\frac{1}{(\lambda+1)^3} +$$
$$+ (y+1)\frac{1}{(\lambda+1)^2} + y\frac{1}{\lambda+1}$$

and

$$u(t,y) = (t+1)(t+y) .$$

On substituting $u(t,y)$ in (14), (15) and (16), we find that $u(t,y)$ satisfies equation (14) with the boundary conditions (15) and (16).

Remark 1: The boundary conditions (2) may be changed, but it is necessary to invert them to boundary conditions of equation (8).

Remark 2: In some cases it is suitable to interchange the Laplace transformation for a general integral from x_1 to x_2. Then it is also necessary to change some assumptions in our Theorem.

Remark 3: Putting $a_2(t) = a_3(t) = b_2(y) = b_3(y) = 0$, $a_1(t) = b_1(y) = 1$ and $q(t,y) = 0$ in equation (1) we obtain the well-known wave equation.

Remark 4: It is better sometimes to prove the existence of the solution with boundary conditions in using some other method. Then we can make use of the Laplace transformation as described above.

This paper was suggested by Professor M.Laitoch.

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POZNÁMKA K LAPLACEOVĚ TRANSFORMACI

Souhrn

Článek se zabývá řešením parciální diferenciální rovnice druhého řádu typu

$$a_1(t)\frac{\partial^2 u}{\partial t^2} + a_2(t)\frac{\partial u}{\partial t} + a_3(t)u =$$

$$= b_1(y)\frac{\partial^2 u}{\partial y^2} + b_2(y)\frac{\partial u}{\partial y} + b_3(y)u + q(t,y)$$

za předpokladu, že

$$a_3(t) = c_0 + \frac{1}{2}\sqrt{a_1(t)}\left(\frac{a_2(t) - \frac{1}{2}a_1'(t)}{\sqrt{a_1(t)}}\right) + \frac{1}{4a_1(t)}(a_2(t) -$$

$$- \frac{1}{2}a_1'(t))^2$$

Při řešení se používá metoda Laplaceovy transformace, která se kombinuje s Kummerovou transformací.

Použití metody je ukázáno na příkladě.

ЗАМЕЧАНИЕ К ПРЕОБРАЗОВАНИЮ ЛАПЛАСА

Резюме

В статье решается дифференциальное уравнение в частных производных второго порядка типа

$$a_1(t)\frac{\partial^2 u}{\partial t^2} + a_2(t)\frac{\partial u}{\partial t} + a_3(t)u =$$

$$= b_1(y)\frac{\partial^2 u}{\partial y^2} + b_2(y)\frac{\partial u}{\partial y} + b_3(y)u + q(t,y)$$

при предположении

$$a_3(t) = c_o + \frac{1}{2}\sqrt{a_1(t)}\left(\frac{a_2(t) - \frac{1}{2}a_1'(t)}{\sqrt{a_1(t)}}\right) + \frac{1}{4a_1(t)}\left(a_2(t) - \frac{1}{2}a_1'(t)\right)^2$$

Используется метод преобразования Лапласа вместе с преобразованием Куммера.

Применение метода показано на примере.

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