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A NOTE TO THE FOURIER METHOD OF SOLVING PARTIAL SECOND-ORDER DIFFERENTIAL EQUATIONS

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The Fourier method is known for a long time in the partial differential equations. Combinating this method with a linear transformation enables us to solve a greater class of linear partial differential equations of the second order. This paper is a continuation of paper [3].

Let us study the equation

$$a_1(t)\frac{\partial^2 u}{\partial t^2} + a_2(t)\frac{\partial u}{\partial t} + a_3(t)u = b_1(y)\frac{\partial^2 u}{\partial y^2} + b_2(y)\frac{\partial u}{\partial y} + b_3(y)u \tag{1}$$

for $t \in J_1$, $y \in J_2$, where J_1 and J_2 are real bounded or unbounded intervals.

<u>Definition</u>: We call $u(t,y)$ a solution of equation (1) on

$(t,y) \in J_1 \times J_2$ if $u(t,y)$ satisfies equation (1) everywhere and $u(t,y) \in C^2(J_1 \times J_2)$.

Theorem 1: Let (1) be an equation defined in the intervals $t \in J_1 = \langle t_o, t_1 \rangle$, $t_o < t_1$ or $t \in J_1 = \langle t_o, \infty)$ and $y \in J_2 = \langle y_o, y_1 \rangle$, $y_o < y_1$ or $y \in J_2 = \langle y_o, \infty)$. Let $c_3 \neq 0$, c_2, c_1 be real constants such that $Q_1(Z)$ is a solution of equation

$$Q_1''(Z)Z^2 = c_1\, Z\, Q_1'(Z) + Q_1(Z)(c_2 + c_3 Z^2) \tag{2}$$

in the interval $Z \in \langle Z_1, \infty)$, $Z_1 > 0$ and let $d_3 \neq 0$, d_2, d_1 be real constants such that $Q_2(Z)$ is a solution of equation

$$Q_2''(Z)Z^2 = d_1\, Z\, Q_2'(Z) + Q_2(Z)(d_2 + d_3 Z^2) \tag{3}$$

in the interval $Z \in \langle Z_2, \infty)$, $Z_2 > 0$.

Let us suppose that $c_3 a_1(t)$, $d_3 b_1(y) > 0$, $a_1(t) \in C^2(J_1)$, $b_1(y) \in C^2(J_2)$, $a_2(t) \in C^1(J_1)$ and $b_2(y) \in C^1(J_2)$ is true for $y \in J_2$, $t \in J_1$. Denote

$$X(t) = \int_{t_o}^{t_1} \frac{ds}{\sqrt{c_3 a_1(s)}} + K_1, \qquad Y(y) = \int_{y_o}^{y} \frac{ds}{\sqrt{d_3 b_1(s)}} + K_2 .$$

$$F(t) = (X(t))^{-\frac{1}{2}c_1} (X'(t))^{-\frac{1}{2}} e^{-\frac{1}{2}\int_{t_o}^{t} \frac{a_2(s)}{a_1(s)} ds}$$

$$G(y) = (Y(y))^{-\frac{1}{2}d_1} (Y'(y))^{-\frac{1}{2}} e^{-\frac{1}{2}\int_{y_o}^{y} \frac{b_2(s)}{b_1(s)} ds}$$

where K_1, K_2 are real positive numbers. Suppose that

$$a_3(t) = \frac{-1}{c_3 (X(t))^2} \left(\frac{1}{4} c_1^2 + \frac{c_1}{2} + c_2 \right) + \frac{\frac{3}{4^2} a_1'^2(t) + \frac{1}{4} a_2^2(t)}{a_1(t)} -$$

$$- \frac{1}{4} a_1''(t) + \left(\frac{a_2(t)}{a_1(t)}\right)' \frac{a_1(t)}{2}$$

$$b_3(y) = \frac{-1}{d_3(Y(y))^2}\left(\frac{1}{4}d_1^2 + \frac{d_1}{2} + d_2\right) + \frac{\frac{3}{4^2}\, b_1'^2(y) + \frac{1}{4}b_2^2(y)}{b_1(y)} -$$

$$- \frac{1}{4}\, b_1''(y) + \left(\frac{b_2(y)}{b_1(y)}\right)' \frac{b_1(y)}{2}$$

and that the series

$$u(t,y) = \left[\sum_{\lambda\in M\subset R^+} A_\lambda Q_1(\lambda X(t))F(t)Q_2(\lambda Y(y))G(y)\right] \in C^2(J_1 \times J_2)$$

(where for every $\lambda \in M$ is A_λ a real number) is convergent and also the first and second partial derivative, term by term of this series with respect to t and y is convergent to the first and second partial derivative of the function u(t,y) with respect to t and y. Then u(t,y) is a solution of equation (1).

The proof is based on verifying equation (1), and we will leave it out.

Theorem 2: Let (1) be an equation defined in the interval $t \in J_1 = \langle t_0, t_1\rangle$, $t_0 < t_1$ or $t \in J_1 = \langle 0, \infty)$ and in the interval $y \in J_2 = \langle y_0, y_1\rangle$, $y_0 < y_1$. Let c,d be non-zero real constants such that $Q_1(Z)$ is a solution of equation

$$Q_1''(Z) = cQ_1(Z) \tag{4}$$

and $Q_2(Z)$ is a solution of equation

$$Q_2''(Z) = dQ_2(Z) \tag{5}$$

in the interval $z \in \langle 0, \infty)$. Suppose $a_i(t)$ and $b_i(y)$ $(i = 1,2)$ possess the same properties as in Theorem 1 and for every $t \in J_1$, $y \in J_2$ $ca_1(t), db_1(y) > 0$. Denote

$$X(t) = \int_{t_0}^{t} \frac{ds}{\sqrt{ca_1(s)}}\,, \qquad Y(y) = \int_{y_0}^{y} \frac{ds}{\sqrt{db_1(s)}}$$

$$F(t) = (ca_1(t))^{\frac{1}{4}} e^{-\frac{1}{2}\int_{t_o}^{t} \frac{a_2(s)}{a_1(s)} ds}$$

$$G(y) = (db_1(y))^{\frac{1}{4}} e^{-\frac{1}{2}\int_{y_o}^{y} \frac{b_2(s)}{b_1(s)} ds}$$

Suppose that

$$a_3(t) = \frac{\frac{3}{4^2} \cdot a_1'^2(t) + a_2^2(t)}{a_1(t)} - \frac{1}{4} a''(t) + \left(\frac{a_2(t)}{a_1(t)}\right)' \frac{a_1(t)}{2} \quad (6)$$

$$b_3(y) = \frac{\frac{3}{4^2} b_1'^2(y) + b_2^2(y)}{b_1(y)} - \frac{1}{4} b''(y) + \left(\frac{b_2(y)}{b_1(y)}\right)' \frac{b_1(y)}{2} \quad (7)$$

and besides that the series

$$u(t,y) = \left[\sum_{\lambda \in M \subset R^+} A_\lambda Q_1(\lambda X(t))F(t)Q_2(\lambda Y(y))G(y)\right] \in C^2(J_1 \times J_2)$$

(where for every $\lambda \in M$ A_λ is a real number) is convergent and also the first and the second partial derivative, term by term of this series with respect to t and y is convergent to the first and second partial derivative of the function $u(t,y)$ with respect to t and y. Then, $u(t,y)$ is a solution of equation (1).

This proof is also based on verifying equation (1). As in Theorem 1 we will leave it out.

Remark 1: The following method is generally known. Let (1) be the equation with the following boundary conditions

$$u(t_o,y) = f_1(y) \qquad u(t_1,y) = f_2(y)$$

$$u(t,y_o) = g_1(t) \qquad u(t,y_1) = g_2(t)$$

This problem will be divided into four partial problems:

1) $u(t_o,y) = f_1(y) \qquad u(t_1,y) = u(t,y_o) = u(t,y_1) = 0$

2) $u(t_1,y) = f_2(y) \qquad u(t_o,y) = u(t,y_o) = u(t,y_1) = 0$

3) $u(t,y_o) = g_1(t) \qquad u(t_o,y) = u(t_1,y) = u(t,y_1) = 0$

4) $u(t,y_1) = g_2(t) \qquad u(t_o,y) = u(t_1,y) = u(t,y_o) = 0$

The solution of the original problem is a sum of the solutions of those four partial problems 1) - 4).

Remark 2: For example, let us suppose that in Remark 1 the function $g_1(t)$ is expressible as a Fourier series with a weight $X'(t)$ that the ortogonal function $Q_2(Z)$ in the Fourier series is satisfying equation (3) or (5) that the function $Q_1(Z)$ is satisfying equation (2) or (4) and that for every $\lambda \in M$ we have

$$Q_2(\lambda X(t_o)) = Q_2(\lambda X(t_1)) = Q_1(\lambda Y(y_1)) = 0, \; Q_2(\lambda Y(y_o)) \neq 0$$

Then the solution of equation (1) with the boundary conditions

$$u(t_o,y) = u(t_1,y) = u(t,y_1) = 0, \qquad u(t,y_o) = g_1(t)$$

may be expressed as described in Theorem 1 or Theorem 2 under the condition that the solution u(t,y) satisfies the convergence conditions of Theorem 1 or Theorem 2.

Remark 3: Let us remark that functions sin y, cos y are suitable ortogonal functions, for example for the bounded conditions $u(t,y_o) = u(t,y_1) = 0$. On the other side, the function $e^y - e^{-y}$ is suitable for the bounded conditions $u(t,0) = \frac{I^2}{I y^2} u(t,0) = 0$. The function e^{-x} is suitable for conditions concerning the behaviour u(t,y) at infinity.

Remark 4: Conditions $u(t,y) \in C^2(t,y)$ and u(t,y) satisfying equation (1) everywhere, are very strong. We may weaken them for example in the sence that u(t,y) satisfies equation (1) everywhere excepting the set of points of the Lebesgue measure 0. Then it is necessary to change some assumptions in Theorem 1 or Theorem 2.

Example: Find a solution of equation (1) under the boundary conditions

$$u(t_o,y) = u(t_1,y) = u(t,y_o) = u(t,y_1) = 0,$$

where t_o, t_1, y_o, y_1 are real numbers, $t_o < t_1$, $y_o < y_1$, on the region $(t,y) \in J_1 \times J_2 = (\langle t_o,t_1\rangle, \langle y_o,y_1\rangle)$. Suppose that (6) and (7) hold, $a_1(t)$, $b_1(y) > 0$ for every $(t,y) \in J_1 \times J_2$ and $a_1(t) \in C^2(J_1)$, $b_1(y) \in C^2(J_2)$, $a_2(t) \in C^1(J_1)$, $b_2(y) \in C^1(J_2)$.

Solution: Write

$$c = \left(\int_{t_o}^{t_1} \frac{ds}{\sqrt{a_1(s)}}\right)^2 \frac{1}{\pi^2}, \quad d = \left(\int_{y_o}^{y_1} \frac{ds}{\sqrt{b_1(s)}}\right)^2 \frac{1}{\pi^2} .$$

In Theorem 2 we set $Q_1(Z) = Q_2(Z) = \sin Z$. Then, we have

$$u(t,y) = A \sin\left(\int_{t_o}^{t} \frac{ds}{\sqrt{c\ a_1(s)}}\right) \cdot \sin\left(\int_{y_o}^{y} \frac{ds}{\sqrt{d\ b_1(s)}}\right) \cdot$$

$$\cdot\ (a_1(t)\ b_1(y))^{\frac{1}{4}}\ e^{-\frac{1}{2}\left(\int_{t_o}^{t} \frac{a_2(s)}{a_1(s)} ds + \int_{y_o}^{y} \frac{b_2(s)}{b_1(s)} ds\right)}$$

which is the solution sought.

The paper was suggested by Professor M.Laitoch.

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POZNÁMKA K FOURIEROVĚ METODĚ PARCIÁLNÍCH DIFERENCIÁLNÍCH ROVNIC DRUHÉHO ŘÁDU

Souhrn

V článku se ukazuje možnost použití Fourierovy metody k řešení parciální diferenciální rovnice 2.řádu typu

$$a_1(t)\frac{\partial^2 u}{\partial t^2} + a_2(t)\frac{\partial u}{\partial t} + a_3(t)u = b_1(y)\frac{\partial^2 u}{\partial y^2} + b_2(y)\frac{\partial u}{\partial y} + b_3(y)u$$

Přitom se využívá Kummerovy transformace řešení obyčejné lineární diferenciální rovnice 2.řádu.

Použití metody je ukázáno na příkladě.

ЗАМЕЧАНИЕ К МЕТОДУ ФУРЬЕ ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ В ЧАСТНЫХ ПРОИЗВОДНЫХ ВТОРОГО ПОРЯДКА

Резюме

В статье показывается возможность решать дифференциаль-

ное уравнение с частными производными второго порядка типа

$$a_1(t) \frac{I^2 u}{I t^2} + a_2(t) \frac{I\ u}{I\ t} + a_3(t)u = b_1(y) \frac{I^2 u}{I y^2} + b_2(y) \frac{I\ u}{I\ y} + b_3(y)u$$

метсдом Фурье. Используется преобразование Куммера для сбыкновенного линейного дифференциального уравнения второго порядка.

Применение метода показано на примере.

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