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ON A BOUNDARY VALUE PROBLEM FOR $x''' = f(t, x, x', x'')$

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1. In the twenty past years great attention has been devoted to the study of two-point or three-point boundary value problems (hereafter only BVPs) for the above equation. As far as we know, about thirty corresponding titles [1 - 24], [26 - 32] have appeared up to the present time.

Among them we regard the result obtained by L.Jackson and K.Schrader in [17, 18] to be of extraordinary importance, because they gave an affirmative answer to the old problem whether the uniqueness of solutions of all two-point or three-point BVPs for our equation implies the existence under some natural additional restrictions.

Further result of particular importance is due to D.Barr and T.Sherman who have shown how solutions x(t) of our equation, satisfying the boundary conditions in two points, namely

$$x(a) = A, \; x(b) = B, \; x'(b) = B'$$

and $x(b) = B$, $x'(b) = B'$, $x(c) = C$,

can be "matched" to yield a unique solution satisfying the boundary conditions at thre points, namely

$$x(a) = A,\ x(b) = B,\ x(c) = C.$$

Thus many earlier or more recent papers dealing with two-point BVPs could be applied in this way (cf. e.g. [1 - 6], [12], [26]). Let us note that in the last quoted papers improved error bounds for the Picard iterations (whence the employed technique) have been successively given.

Taking into account other interesting papers, let us mention [9], where the coincidence degree technique has been used for solving a periodic BVP and [32], where an asymptotic BVP has been of interest.

2. In [7] we have proved by a manner similar to that of [9] that the following BVP:

$$x''' = f(t,x,x',x''),\quad f \in C^1(\langle 0,\Theta\rangle \times R^3), \tag{1}$$

$$x(\Theta) - x(0) = A_0,\ x'(\Theta) - x'(0) = A_1.\ x''(\Theta) - x''(0) = A_2\ , \tag{2}$$

where Θ, A_0, A_1, A_2 are suitable reals, admits a solution, provided the function f is bounded for all its arguments, but not necessarily x and

$$\left| f(t,x,y,z)\right| \leq M|x| + \text{const.} \tag{3}$$

is satisfied everywhere for a small enough constant M together with

$$\liminf_{|x|\to\infty} f(t,x,y,z)\,\mathrm{sgn}\, x \geq \left|\frac{A_2}{\Theta}\right| \ \left(\text{or } \limsup_{|x|\to\infty} f(t,x,y,z)\,\mathrm{sgn}\, x \leq -\left|\frac{A_2}{\Theta}\right|\right). \tag{4}$$

Here we would like to show that the same conditions (i.e. (3), (4)) imply for $\Theta = 2a$ also the solvability of the following incomplete BVP, namely (1) and

$$x'(0) = x'(a) = x'(2a), \tag{5}$$

even if (4) is replaced by

$$\liminf_{|x|\to\infty} f(t,x,y,z)\,\mathrm{sgn}\, x > 0 \quad (\text{or } \limsup_{|x|\to\infty} f(t,x,y,z)\,\mathrm{sgn}\, x < 0) \tag{4_0}$$

for $t \in \langle 0,2a\rangle$, $(y,z) \in R^2$.

In fact we will prove the same for (1) and (6), where

$$x(0) = x(a), \qquad x'(0) = x'(a) = x'(2a) \tag{6}$$

3. For this purpose let us define the modified Levinson operator, where $\mu, \nu \in \langle 0,1\rangle$ are parameters and $X(0) = (x(0), x'(0), x''(0))$ are Cauchy's initial values, in the following way:

$$T_{\mu,\nu}X(0) = \begin{cases} [a(x(a)-x(0)), x'(a)-x'(0), x'(2a)-2x'(a)+ \\ + x'(0)]\, a^{-2} & \text{for } \mu = \nu = 1, \\ [\mu a(x(\nu a)-x(0)), \mu(x'(\nu a)-x'(0)), (x'((\mu+\nu)a)- \\ -x'(\nu a)-(x'(\mu a)-x'(0))](\mu\nu)^{-1}a^{-2} & \\ & \text{for } \mu,\nu \in (0,1\rangle, \\ [x(\nu a)-x(0), x'(\nu a)-x'(0), x''(\nu a) - \\ -x''(0)]\,(\nu a)^{-1} & \text{for } \mu = 0, \nu \in (0,1\rangle, \\ [x'(0), x''(0), f(0,x'(0),x''(0))] & \\ & \text{for } \mu = \nu = 0 . \end{cases}$$

<u>Theorem 1</u>. The problem of (1), (6) is solvable, provided

(i) $\frac{f(0,x(0),0,0)}{|f(0,x(0),0,0)|} \neq \frac{f(0,-x(0),0,0)}{|f(0,-x(0),0,0)|}$ $(f(0,x(0),0,0) \neq 0)$

holds for $|x(0)| \geqq R_0$, where R_0 is a sufficiently large positive constant and

(ii) $T_{\mu,1}X(0) \neq 0$, $T_{0,\nu}X(0) \neq 0$ for $\|X(0)\| \geqq R \geqq R_0$ (great enough R),

independently of $\mu, \nu \in (0,1\rangle$.

P̲r̲o̲o̲f̲. It is clear that our problem is solvable iff $T_{1,1}X(0) = 0$. Since we will here employ the topological degree arguments, the fundamental requirement for ensuring this reads

$$T_{1,1}X(0) \neq 0 \tag{7}$$

on the sphere $\|X(0)\| = R > 0$. But assuming (ii), condition (7) can be replaced by

$$T_{0,0}X(0) \neq 0 \quad \text{for } \|X(0)\| = R \tag{8}$$

by virtue of the well-known invariance under homotopy [25]. Furthermore, since the degree of an odd operator is not equal to zero on the sphere according to the classical Borsuk's antipodal theorem [25], namely

$$d\left[T_{0,0}X(0)-T_{0,0}(-X(0)), \|X(0)\| \leqq R, 0\right] \neq 0 \quad \text{for } \|X(0)\| = R,$$

condition (8) can be replaced by

$$T_{0,0}X(0) - (1-\lambda)T_{0,0}(-X(0)) \neq 0 \qquad \text{for } \lambda \in (0,1\rangle ,$$

which is certainly implied by (i) for $f(0,x(0),x'(0),x''(0)) \neq 0$. This completes the proof.

Lemma 1. If all solutions $x(t)$ of (1), satisfying the following boundary conditions

$$x(\nu a)=x(0)\ x'(\nu a)=x'(0), x''(\nu a)=x''(0) \quad \text{for all } \nu \in (0,1\rangle, \qquad (9)$$

$$x(a)=x(0), x'((\mu+1)a)=x'(\mu a), x'(a) = = x'(0) \qquad \text{for all } \mu \in (0,1\rangle \qquad (10)$$

are uniformly a priori bounded with their derivatives $x'(t)$ in (10) and $x'(t), x''(t)$ in (9), then condition (ii) is fulfilled.

Proof. It can be readily checked that

$$T_{\mu,1}X(0) \neq 0 \text{ if } x(a) \neq x(0) \text{ or } x'(a) \neq x'(0) \text{ or } x'((\mu+1)a) \neq x'(\mu a) \text{ for all } \mu \in (0,1\rangle\ ,$$

$$T_{0,\nu}X(0) \neq 0 \text{ if } x(\nu a) \neq x(0) \text{ or } x'(\nu a) \neq x'(0) \text{ or } x''(\nu a) \neq x''(0) \text{ for all } \nu \in (0,1\rangle\ .$$

Therefore assuming a priori estimates as above, these inequalities are satisfied successively, which was to be proved.

Lemma 2. The a priori estimates of Lemma 1 exist, provided (4_0) and (3) with M small enough.

Proof. Denote

$$f^*(t,x,y,z) = \begin{cases} f(t,x,y,z) & \text{for } |x| \leq S \\ f(t, S\,\mathrm{sgn}\, x, y, z) & \text{for } |x| \geq S\ , \end{cases}$$

where S is a suitable constant specified bellow and consider instead of (1) the equation

$$x''' = f^*(t,x,x',x'')\ . \qquad (11)$$

Since such a constant F^* must exist that $|f^*(t,x,y,z)| \leq F^*$ for all $t \in \langle 0,2a\rangle$, $(x,y,z) \in R^3$, we have also $|x'''(t)| \leq F^*$. Furthermore, since such points $t_1, t_2 \in \langle 0,2a\rangle$ exist with respect to (9), (10) that $x'(t_1) = 0 = x''(t_2)$, the following inequalities are satisfied:

$$|x''(t)| \leq \left|\int_{t_2}^{t} |x'''(s)|\,ds\right| \leq 2aF^* , \tag{12}$$

$$|x'(t)| \leq \left|\int_{t_1}^{t} |x''(s)|\,ds\right| \leq 4a^2F^* . \tag{13}$$

Condition (4_0) implies such an R_0 (cf. (i)) that $f(t,x,y,z)\operatorname{sgn} x > 0$ or $f(t,x,y,z)\operatorname{sgn} x < 0$ holds for $|x| > R_0$ and $t \in \langle 0,2a\rangle$, $(y,z) \in R^2$ and consequently $x'''(t) > 0$ or $x'''(t) < 0$, from which follows the convexity or concavity of $x'(t)$ for $|x(t)| > R_0$, respectively. Hence (9) or (10) cannot be satisfied in this respect.

Thus $\min_{t\in\langle 0,2a\rangle} |x(t)| = |x(t_0)| \leq R_0$ in some $t_0 \in \langle 0,2a\rangle$ and we get

$$|x(t)| \leq |x(t_0)| + \left|\int_{t_0}^{t} |x'(s)|\,ds\right| \leq R_0 + 8a^3F^*$$

with respect to (13).

Obviously, the existence of such constants S,F is guaranteed by (3) that

$$R_0 + 8a^3 \max_{|x|\leq S} |f(t,x,y,z)|^* \leq R_0 + 8a^3F < S ,$$

and hence we have not only $|x(t)| \leq S$, but also (cf. (12), (13))

$$|x'(t)| + |x''(t)| \leq (1+2a)2aF .$$

The same is certainly true for solutions x(t) of (1). This completes the proof.

<u>Theorem 2</u>. There exists a solution of BVP (1), (6), provided (4_0) and (3) with M small enough.

<u>Proof</u> - follows immediately from Lemmas 1,2, because condition (i) of Theorem 1 is satisfied trivially by (4_0).

4. Remark. Although the incomplete BVP (1), (5) can be considered only as a special case of those studied in the papers [5-7], [12], [18] and some others (see also the references included), our result cannot be deduced from any obtained there, in general. However, several results are comparable in certain aspects at least.

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O JISTÉ OKRAJOVÉ ÚLOZE PRO $x''' = f(t,x,x',x'')$

Souhrn

Užitím teorie topologického stupně zobrazení jsou nalezeny efektivní podmínky řešitelnosti tříbodové periodické okrajové úlohy pro obecnou nelineární diferenciální rovnici třetího řádu. Jsou uvedeny dosud dosažené základní výsledky o okrajových úlohách pro studovanou rovnici.

ОБ ОДНОЙ КРАЕВОЙ ЗАДАЧЕ ДЛЯ $x''' = f(t,x,x',x'')$

Резюме

На основе теории топологической степени отображения получены эффективные условия разрешимости трехточечной периодической краевой задачи для одного нелинейного дифференциального уравнения третьего порядка. Представлены также основные результаты решения краевых задач, достигнутые в настоящее время для данного уравнения.

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