Svatoslav Staněk

On the Floquet theory of differential equations $y'' = Q(t)y$ with a complex coefficient of the real variable

Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica, Vol. 27 (1988), No. 1, 149--183

Persistent URL: `http://dml.cz/dmlcz/120191`

ACTA UNIVERSITATIS PALACKIANAE OLOMUCENSIS
FACULTAS RERUM NATURALIUM
1988 MATHEMATICA XXVII VOL. 91

Katedra matematické analýzy a numerické matematiky
přírodovědecké fakulty Univerzity Palackého v Olomouci
Vedoucí katedry: Doc.RNDr. Jindřich Palát, CSc.

ON THE FLOQUET THEORY OF DIFFERENTIAL EQUATIONS $y''=Q(t)y$ WITH A COMPLEX COEFFICIENT OF THE REAL VARIABLE

SVATOSLAV STANĚK

(Received January 7, 1987)

1. Problem

A differential equation

$$y'' = Q(t)y, \quad \operatorname{Im} Q(t) \not\equiv 0, \tag{Q}$$

is investigated, where Q is a continuous and π- periodic complex function on R. From the Floquet theory (see for instance [7]) it then follows that there exist independent solutions u, v of (Q) such that

either

$$u(t+\pi) = \varrho . u(t), \quad v(t+\pi) = \varrho^{-1}.v(t), \quad t \in R, \quad 0 \neq \varrho \in C \tag{1}$$

or

$$u(t+\pi) = \varrho . u(t) + v(t), \quad v(t+\pi) = \varrho . v(t), \quad t \in R, \quad \varrho^2 = 1. \tag{2}$$

Generally complex numbers ϱ, ϱ^{-1} are called characteristic (or Floquet's) multipliers of (Q).

In [2] - [6], [8], [9], [11], [12] the values of the characteristic multipliers of (q): $y'' = q(t)y$, q being a continuous π-periodic real function on R, where expressed by a phase and the (1st kind) central dispersion of (q).

The present article offers a new look at the Floquet theory of (Q) based on the phase theory point of view.

2. Basic notations, relations and preparatory lemmas

The symbol $C^n(R)$ ($\tilde{C}^n(R)$), where n=0,1,2,... , will refer to a set of real (complex) functions with continuous derivatives (on R) up to and including the order n. Trivial solutions of linear equations will not be considered.

In analogy with [13] a function $\alpha \in \tilde{C}^3(R)$ will be said to be a phase of an equation

$$y'' = P(t)y, \quad P \in \tilde{C}^0(R), \quad \mathrm{Im}\, P(t) \not\equiv 0, \tag{P}$$

exactly if there exist independent solutions u, v of this equation such that

a) $u^2(t) + v^2(t) \neq 0$ for $t \in R$,

b) $\alpha'(t) = -\dfrac{w}{u^2(t) + v^2(t)}$ for $t \in R$, where $w := uv' - u'v$.

If moreover $\mathrm{tg}\,\alpha(t_0) = \dfrac{u(t_0)}{v(t_0)}$ at a point $t_0 \in R$, where $v(t_0) \neq 0$, then α is said to be a phase of the basis (u,v) of (P). In such a case $u(t) = c\,\dfrac{\sin\alpha(t)}{\sqrt{\alpha'(t)}}$, $v(t) = c\,\dfrac{\cos\alpha(t)}{\sqrt{\alpha'(t)}}$ for $t \in R$, where $0 \neq c \in C$.

A function α is a phase of (P) exactly if it is a solution (on R) of a nonlinear 3rd order differential equation

$$-\{\alpha, t\} - \alpha'^2(t) = P(t),$$

where $\{\alpha, t\} := \frac{\alpha'''(t)}{2\alpha'(t)} - \frac{3}{4}\left(\frac{\alpha''(t)}{\alpha'(t)}\right)^2$ denotes the Schwarzian derivative of α at the point t.

If α is a phase of (P), then every solution of (P) may be written either as

$$c_1 \frac{\sin(\alpha(t) + c_2)}{\sqrt{\alpha'(t)}}, \tag{3}$$

or

$$c_3 \frac{e^{i\nu\alpha(t)}}{\sqrt{\alpha'(t)}}, \tag{4}$$

where $\nu^2 = 1$, $c_1, c_2, c_3 \in C$, $c_1 \neq 0 \neq c_3$. The converse is valid, too: For arbitrary complex numbers c_1, c_2, c_3, $c_1 \neq 0 \neq c_3$, and a number ν, $\nu^2 = 1$, the functions defined by (3) and (4) are solutions of (P). Hereby $\sqrt{\alpha'(t)}$ means a continuous and single-valued branch of the square root of the function $\alpha'(t)$.

If u is a solution of (P), $u(t) \neq 0$ for $t \in R$, then there exists a phase α of (P) and a number $c \in C$, $c \neq 0$, such that

$$u(t) = c\,\frac{e^{i\alpha(t)}}{\sqrt{\alpha'(t)}}, \quad t \in R.$$

All the above properties have been presented and proved in [13].

<u>Lemma 1</u>. Let α be a phase of (P). Then

$$(P(t)) = -\{\alpha, t\} - \alpha'^2(t) = \left(i\alpha'(t) - \frac{\alpha''(t)}{2\alpha'(t)}\right)' + \left(i\alpha'(t) - \frac{\alpha''(t)}{2\alpha'(t)}\right)^2, \quad t \in R.$$

<u>Proof</u>. Setting $u(t) := \frac{e^{i\alpha(t)}}{\sqrt{\alpha'(t)}}$ $(\neq 0)$ for $t \in R$, then u is a solution of (P). From the equalities $\frac{u'}{u} = i\alpha' - \frac{\alpha''}{2\alpha'}$

and $(-\{\alpha,t\} - \alpha'^2(t) =)\ P(t) = (\frac{u'(t)}{u(t)})' + (\frac{u'(t)}{u(t)})^2$ then there follows the assertion of Lemma 1.

In analogy with [14] a function X will be said to be a (complete) <u>transformator of</u> (P) if

(i) $X \in C^3(R)$, $X'(t) \neq 0$ for $t \in R$, $X(R) = R$;

(ii) for every solution y of (P) the function $\frac{y[X(t)]}{\sqrt{|X'(t)|}}$ is again a solution of this equation.

The set of increasing transformators of (P) constitutes a group L_P^+ relative to the composition of functions. We will say that L_P^+ is a <u>planar group</u>, if to every $(t_0, x_0) \in R \times R$ there exists exactly one function $X \in L_P^+$ such that $X(t_0) = x_0$.

A transformator X of (P), $X'(t) > 0$ for $t \in R$, will be called a <u>central transformator of</u> (P) if

$$\frac{y[X(t)]}{\sqrt{X'(t)}} = \nu . y(t) \quad \text{for } t \in R,$$

where $\nu^2 = 1$, for every solution y of (P). The set of all central transformators of (P) constitutes a group relative to the composition of functions, which we will write as L_P^c; $L_P^c \subset L_P^+$ (see [14]).

<u>Lemma 2</u>. Let α be a phase of (P). Then P is a π-periodic function exactly if the function $\alpha(t+\pi)$ is a phase of (P), too.

<u>Proof</u>. ($\Longrightarrow$) Suppose P is a π-periodic function and set $\beta(t) := \alpha(t+\pi)$, $t \in R$. Then

$$-\{\beta,t\} - \beta'^2(t) = -\{\alpha, t+\pi\} - \alpha'^2(t+\pi) =$$

$$= P(t+\pi) = P(t),$$

so that

$$-\{\beta,t\} - \beta'^2(t) = P(t), \qquad t \in R, \tag{5}$$

whence it follows that β is a phase of (P).

($\Longleftarrow$) Suppose β (defined analogous to the first part of the proof) is a phase of (P). Then (5) is true and consequently

$$-\left\{\alpha, t+\pi\right\} - \alpha'^2(t+\pi) = P(t), \qquad t \in R.$$

It follows from this and from the equality $-\left\{\alpha, t\right\} - \alpha'^2(t) = P(t)$, $t \in R$, that $P(t+\pi) = P(t)$ for $t \in R$.

<u>Lemma 3</u>. Let $a \in R$, $\mathrm{Re}\, P(t) + a.\mathrm{Im}\, P(t) \geqq q(t)$ for $t \in R$, where $q \in C^0(R)$ and $(q): y'' = q(t)y$ be not oscillatory (i.e. any solution of (q) has at most a finite number of zeros on R). Then any solution of (P) has at most a finite number of zeros on R.

<u>Proof</u>. Suppose, there exists a solution z of (P) with an infinite number of zeros, and ∞ is their cluster point. Let u be a solution of (q), $u(t) > 0$ for $t \geqq b$ and $z(t_1) = z(t_2) = 0$ for $b \leqq t_1 < t_2$, $z(t) \neq 0$ for $t \in (t_1, t_2)$. Since

$$(z'(t)\bar{z}(t))' = P(t)|z(t)|^2 + |z'(t)|^2 ,$$

then

$$\int_{t_1}^{t_2} \left\{ |z'(s)|^2 + (\mathrm{Re}\, P(s) + i.\mathrm{Im}\, P(s))\, |z(s)|^2 \right\} ds = 0.$$

It then follows

$$\int_{t_1}^{t_2} \left\{ |z'(s)|^2 + \mathrm{Re}\, P(s)\, |z(s)|^2 \right\} ds = 0,$$

$$\int_{t_1}^{t_2} \mathrm{Im}\, P(s)\, |z(s)|^2\, ds = 0$$

so that

$$\int_{t_1}^{t_2}\left\{|z'(s)|^2 + q(s)|z(s)|^2\right\}ds \leqq 0.$$

Since $|z(t)|'^2 \leqq |z'(t)|^2$ for $t \in (t_1, t_2)$, we obtain

$$\int_{t_1}^{t_2}\left\{ r'^2(s) + q(s)r^2(s)\right\} ds \leqq 0,$$

where $r(t) := |z(t)|$, $t \in R$. Then, by Lemma 1.3 ([15] p.3), the solution u has a zero on (t_1, t_2), which is a contradiction.

3. Main results

In what follows we will investigate equations of the type

$$y'' = Q(t)y, \quad Q \in \tilde{C}^0(R), \ \operatorname{Im} Q(t) \not\equiv 0, \ Q(t+\pi) = Q(t) \quad \text{for } t \in R. \tag{Q}$$

Lemma 4. There exists a phase α of (Q) such that the function $i\alpha' - \frac{\alpha''}{2\alpha'}$ is π-periodic exactly if for a solution u of (Q)

$$u(t+\pi) = \rho . u(t), \ u(t) \neq 0 \text{ for } t \in R \tag{6}$$

is valid, where $0 \neq \rho \in C$.

Proof. ($\Longrightarrow$) Suppose there exists a phase α of (Q) such that the function $i\alpha' - \frac{\alpha''}{2\alpha'}$ is π-periodic. If we set $u(t) := \frac{e^{i\alpha(t)}}{\sqrt{\alpha'(t)}}$ $(\neq 0)$, $t \in R$, then u is a solution of (Q) and

$$\frac{u'}{u} = i\alpha' - \frac{\alpha''}{2\alpha'} \quad (:= p),$$

so that $\frac{u'}{u}$ is a π-periodic function. Further $u(t) = u(0)$.

$\cdot \exp(\int_0^t p(s)ds)$ which yields

$$u(t+\pi) = \varrho \cdot u(t), \text{ where } \varrho = \exp(\int_0^{\pi} p(s)ds).$$

($\Longleftarrow$) Let (6) hold for a solution u of (Q), where $0 \neq \varrho \in C$. Since $u(t) \neq 0$ for $t \in R$, there exists a phase α of (Q) and a $c \in C$ such that $u(t) = c \dfrac{e^{i\alpha(t)}}{\sqrt{\alpha'(t)}}$. On account of the fact that $\frac{u'}{u}$ is a π-periodic function and $\frac{u'}{u} =$ $= i\alpha' - \frac{\alpha''}{2\alpha'}$ it is clear that $i\alpha' - \frac{\alpha''}{2\alpha'}$ is also a function with a period π.

Remark 1. If (6) holds for a solution u of (Q), where $0 \neq \varrho \in C$, then ϱ is a characteristic multiplier of (Q).

Remark 2. If β is a phase of (P) and $i\beta' - \frac{\beta''}{2\beta'}$ is a π-periodic function, then the coefficient P of (P) is also a π-periodic function, as it readily follows from Lemma 1.

Remark 3. In the terminology of transformators equation (P) $t+\pi \in L_P^+$ exactly if P is a π-periodic function.

Corollary 1. Suppose there exists a phase α of (Q) such that $i\alpha' - \frac{\alpha''}{2\alpha'}$ is a π-periodic function. Then

$$\frac{\sqrt{\alpha'(0)}}{\sqrt{\alpha'(\pi)}} \exp\left\{i(\alpha(\pi) - \alpha(0))\right\}, \quad \frac{\sqrt{\alpha'(\pi)}}{\sqrt{\alpha'(0)}} \exp\left\{i(\alpha(0) - \alpha(\pi))\right\}$$

are characteristic multipliers of (Q).

Proof. It follows from Remark 2 that the coefficient Q of (Q) is a π-periodic function. Besides we obtain from the proof ($\Longrightarrow$) of Lemma 4 and Remark 1 that $\exp(\int_0^{\pi} p(s)ds)$

$\exp(-\int_0^{\pi} p(s)ds)$, where $p := i\alpha' - \frac{\alpha''}{2\alpha'}$, are characteristic multipliers of (Q). From this and from the equality

$$\int_0^{\pi} p(s)ds = i(\alpha(\pi) - \alpha(0)) + \ln \frac{\sqrt{\alpha'(0)}}{\sqrt{\alpha'(\pi)}}$$

immediately follows the assertion of Corollary 1.

Lemma 5. Suppose there exists a phase α of (Q) such that $i\alpha'(t) - \frac{\alpha''(t)}{2\alpha'(t)}$ $(=: p(t),\ t \in R)$ is a π-periodic function. Then for a phase β of (Q)

$$i\beta'(t) - \frac{\beta''(t)}{2\beta'(t)} = p(t) \text{ for } t \in R \tag{7}$$

is fulfilled exactly if there exist $k, k_1 \in C$, $ke^{2i\alpha(t)} \neq 1$ for $t \in R$ such that

$$\beta(t) = \alpha(t) + \frac{i}{2}\ln(1 - ke^{2i\alpha(t)}) + k_1, \quad t \in R. \tag{8}$$

Proof. ($\Longrightarrow$) Suppose β is such a phase of (Q) that

$$(p(t) =)\ i\alpha'(t) - \frac{\alpha''(t)}{2\alpha'(t)} = i\beta'(t) - \frac{\beta''(t)}{2\beta'(t)}, \quad t \in R. \tag{9}$$

Then from Theorem 4 [13] there follows the equality $\beta(t) = c[\alpha(t)]$,

$$c'(z) = \frac{1}{(c_1 \cos z + c_2 \sin z)^2 + (c_3 \cos z + c_4 \sin z)^2} \tag{10}$$

for all $z \in C$, where $(c_1 \cos z + c_2 \sin z)^2 + (c_3 \cos z + c_4 \sin z)^2 \neq 0$ and $c_1, c_2, c_3, c_4 \in C$, $c_2c_3 - c_1c_4 = 1$. Then $\beta'(t) = c'[\alpha(t)] \cdot \alpha'(t)$, $\beta''(t) = c''[\alpha(t)] \cdot \alpha'^2(t) + c'[\alpha(t)] \cdot \alpha''(t)$ and on substituting in (9) we get

$$i = i.c'[\alpha(t)] - \frac{c''[\alpha(t)]}{2c'[\alpha(t)]}$$

All solutions of the above equation are of the form $c'[\alpha(t)] = \frac{1}{1-ke^{2i\alpha(t)}}$, where $k \in C$ is an arbitrary number such that $ke^{2i\alpha(t)} \neq 1$ for $t \in R$. There is an infinite number of such k and if we proceed in the same manner as in [13] we may prove the Lebesque measure (the complex number is taken as a point in Gauss plane) of the set of such numbers k is equals to infinity. Here $c'(z)$ has the form (10). In the case of $k \neq 1$ it suffices to put $c_1 = 0$, $c_2 = \frac{1}{\sqrt{1-k}}$, $c_3 = \sqrt{1-k}$, $c_4 = -\frac{ik}{\sqrt{1-k}}$ while in the case of $k = 1$ we put $c_1 = -\frac{\sqrt{2}}{2}$, $c_2 = 0$, $c_3 = -i\frac{\sqrt{2}}{2}$, $c_4 = \sqrt{2}$. Hence $\beta'(t) = \frac{\alpha'(t)}{1-k\,e^{2i\alpha(t)}}$ and integrating the latter equality from 0 to t gives

$$\beta(t) = \beta(0) + \int_0^t \frac{\alpha'(s)ds}{1-k\,e^{2i\alpha(s)}} = \beta(0) + \alpha(t) +$$

$$+ \frac{i}{2}\ln(1-ke^{2i\alpha(t)}) - \alpha(0) - \frac{i}{2}\ln(1-ke^{2i\alpha(0)}) =$$

$$= \alpha(t) + \frac{i}{2}\ln(1-ke^{2i\alpha(t)}) + k_1,$$

where $k_1 := \beta(0) - \alpha(0) - \frac{i}{2}\ln(1-e^{2i\alpha(0)})$.

($\Longleftarrow$) Suppose β is the function defined by (8), where $k, k_1 \in C$, $ke^{2i\alpha(t)} \neq 1$ for $t \in R$. By a direct computation it may be verified that β is a phase of (Q) and (7) is true.

Lemma 6. Let all solutions of (Q) not be $\mathcal{T}$-periodic or $\mathcal{T}$-halfperiodic and let there exist a phase α of (Q) such that the function $i\alpha' - \frac{\alpha''}{2\alpha'}$ ($=: p_1$) is $\mathcal{T}$-periodic. Then there exists at most one π-periodic function p_2, $p_1 \neq p_2$, such that $p_2 = i\beta' - \frac{\beta''}{2\beta'}$ for a phase β of (Q).

Proof. Following Remark 2, it suffices to prove that the Riccati equation

$$u' + u^2 = Q(t) \tag{11}$$

has at most two different π-periodic solutions (defined on R) under the assumption that all solutions of (Q) are not π-periodic or π-halfperiodic. First, the function p_1 is a π-periodic solution of (11). We assume that there exist further two π-periodic solutions p_2, p_3 of (11), $p_1 \neq p_2$, $p_1 \neq p_3$, $p_2 \neq p_3$. Integreting the equalities

$$\frac{(p_3 - p_2)'}{p_3 - p_2} - \frac{(p_3 - p_1)'}{p_3 - p_1} = p_1 - p_2,$$

$$\frac{(p_2 - p_3)'}{p_2 - p_3} - \frac{(p_2 - p_1)'}{p_2 - p_1} = p_1 - p_3,$$

$$\frac{(p_3 - p_1)'}{p_2 - p_1} = - p_3 - p_1,$$

from 0 to π yields

$$\int_0^{\pi}(p_1(t)-p_2(t))dt = 2im\pi, \qquad \int_0^{\pi}(p_1(t)-p_3(t))dt = 2in\pi,$$

$$\int_0^{\pi}(p_1(t)+p_3(t))dt = 2ir\pi,$$

where m, n, s are integers, whence

$$\int_0^{\pi}p_1(t)dt=i(n+r)\pi, \quad \int_0^{\pi}p_2(t)dt=i(n+r-2m)\pi, \quad \int_0^{\pi}p_3(t)dt=i(r-n)\pi.$$

Since $p_1(t) = \frac{y_1'(t)}{y_1(t)}$, $p_2(t) = \frac{y_2'(t)}{y_2(t)}$, where y_1, y_2 are suitable independent solutions of (Q), $y_1(t) \neq 0$, $y_2(t) \neq 0$ for $t \in R$, there exist $k_1, k_2 \in C$ such that $y_i(t) = k_i \exp(\int_0^t p_i(s)ds)$, $i = 1,2$, $t \in R$. Naturally, then

$$y_i(t+\pi)=k_i\exp(\int_0^t p_i(s)ds)\exp(\int_0^{\pi} p_i(s)ds) = (-1)^{n+r}y_i(t)$$

($i=1,2$, $t \in R$), hence all solutions of (Q) are π-periodic or π-halfperiodic, which is a contradiction.

Remark 4. In assuming that all solutions of (Q) are π-periodic or π-halfperiodic, the Riccati equation (11) has infinitely many π-periodic solutions. All these solutions are of form $\frac{y'(t)}{y(t)}$, where y is a solution of (Q), $y(t) \neq 0$ for $t \in R$ (see Example 1). Here the main difference is in the number of periodic solutions of the Riccati equation in a real case, when even there the equation has at most two π-periodic solution (see [10]).

Example 1. The Riccati equation

$$u' + u^2 = -4 + 16e^{8it}$$

has π-periodic solutions, say

$$u = -2i + 4ie^{4it}\operatorname{cotg}(e^{4it}+c),$$

with $c \in C$ being an arbitrary number such that $\sin(e^{4it}+c) \neq 0$ for $t \in R$. This condition is fulfilled for $c = c_1 + ic_2$ such that $(c_1+k\pi)^2 + c_2^2 \neq 1$ for all integer k.

It becomes obvious that the investigation of π-periodicity of the function $i\alpha' - \frac{\alpha''}{2\alpha'}$, where α is a phase of (Q), is essential. The remain part of this text is deviden into three cases:

Case 1 – there exists a phase α of (Q) such that its derivative α' is a π-periodic function (and then the function $i\alpha' - \frac{\alpha''}{2\alpha'}$, too, is π-periodic);

Case 2 – there exists such a phase α of (Q) that its derivative α' is not a π-periodic function and $i\alpha' - \frac{\alpha''}{2\alpha'}$ is a π-periodic function;

Case 3 – there exists no such phase α of (Q) that $i\alpha' - \frac{\alpha''}{2\alpha'}$ is a π-periodic function.

Case 1

Theorem 1. Suppose ϱ is a characteristic multiplier of (Q), $|\varrho| \geqq 1$. Then, there exist independent solutions u, v of (Q), $u(t)v(t) \neq 0$ for $t \in R$ satisfying (1) exactly if there exists a phase α of (Q), $k_1, k_2 \in R$, $0 \leqq k_1 \leqq (1+\operatorname{sign} k_2)\pi$, $k_1 \neq 2\pi$, $k_2 \geqq 0$ and an integer n such that

$$\alpha(t+\pi) = \alpha(t) + (k_1+2n\pi) + ik_2, \quad t \in R, \tag{12}$$

$$\varrho = \nu\, e^{k_2 - ik_1} \quad \text{and} \quad \nu = \frac{\sqrt{\alpha'(t+\pi)}}{\sqrt{\alpha'(t)}} \ (= \pm 1).$$

Proof. ($\Longrightarrow$) Let ϱ be a characteristic multiplier of (Q), $|\varrho| \geqq 1$ and u, v be independent solutions of (Q) satisfying (1), $u(t)v(t) \neq 0$ for $t \in R$. Setting $U := \frac{1}{2}(u+v)$, $V := \frac{i}{2}(v-u)$ yilds that U, V are independent solutions of (Q) and $U^2(t) + V^2(t) \neq 0$ for $t \in R$. Let α be a phase of the basis (U, V) of (Q). Then there exists a $c \in C$, $c \neq 0$, such that

$$U(t) = c\,\frac{\sin\alpha(t)}{\sqrt{\alpha'(t)}}, \qquad V(t) = c\,\frac{\cos\alpha(t)}{\sqrt{\alpha'(t)}}, \qquad t \in R, \tag{13}$$

(see [13]). Since

$$c\,\frac{\sin\alpha(t)}{\sqrt{\alpha'(t)}} = \frac{1}{2}(u(t)+v(t)), \quad c\,\frac{\cos\alpha(t)}{\sqrt{\alpha'(t)}} = \frac{i}{2}(v(t)-u(t)), \quad t \in R \tag{14}$$

then

$$\frac{c^2}{\alpha'(t+\pi)} = \frac{1}{4}(\rho.u(t) + \rho^{-1}.v(t))^2 - \frac{1}{4}(\rho^{-1}.v(t) -$$

$$- \rho.u(t))^2 = u(t)v(t) = \frac{c^2}{\alpha'(t)} .$$

Naturally, then $\alpha'(t+\pi) = \alpha'(t)$ and therefore for an $a \in C$ we get

$$\alpha(t+\pi) = \alpha(t) + a, \quad t \in R. \tag{15}$$

Let $\vartheta = \frac{\sqrt{\alpha'(t+\pi)}}{\sqrt{\alpha'(t)}}$. Evidently, ϑ is either equal to 1 or equal to -1. From the definition of U,V and from (1), (13) - (15) it follows from one side

$$V(t+\pi) + i.U(t+\pi) = \vartheta c \frac{\exp i(\alpha(t)+a)}{\sqrt{\alpha'(t)}}$$

and from the other side

$$V(t+\pi) + i.U(t+\pi) = \frac{i}{2}(\rho^{-1}.v(t) - \rho.u(t)) +$$

$$+ \frac{i}{2}(\rho.u(t) + \rho^{-1}.v(t)) = i\rho^{-1}.v(t) =$$

$$= \rho^{-1}.(V(t) + i.U(t)) = c\rho^{-1} \frac{\exp i\alpha(t)}{\sqrt{\alpha'(t)}} .$$

Thus $\rho = \vartheta e^{-ia}$ and if $a = a_1+ia_2$ is $|\rho| = e^{a_2} = 1$, whence $a_2 \geqq 0$. Next let $a_1 = k_1+2n\pi$, where $0 \leqq k_1 < 2\pi$ and n is an integer. Setting $k_2 := a_2$ ($\geqq 0$), we get from (15) formula (12) and $\rho = \vartheta.\exp(-i(k_1+2n\pi) + k_2) = \vartheta.\exp(k_2-ik_1)$.

It remains to prove that in case of $a_2 = 0$, i.e. where $|\rho| = 1$, the number k_1 may be chosen to that $0 \leqq k_1 \leqq \pi$. In case of $\pi < k_1 < 2\pi$ we consider the phase $\beta := -\alpha$ in place of the phase α of (Q). Then it follows from (15)

$$\beta(t+\pi) = \beta(t) - a = \beta(t) - k_1 - 2n\pi =$$
$$= \beta(t) + (2\pi - k_1) - 2(n+1)\pi$$

and in place of the integer n in (12) we put the integer $-(n+1)$ and in place of the number k_1 we put $2\pi - k_1$. Evidently $0 < 2\pi - k_1 < \pi$.

($\Longleftarrow$) Let α be a phase of (Q), $k_1, k_2 \in R$, $0 \leqq k_1 \leqq (1+\operatorname{sign} k_2)\pi$, $k_1 \neq 2\pi$, $k_2 \geqq 0$ and n be an integer such that (12) is true. Let $\nu = \frac{\sqrt{\alpha'(t+\pi)}}{\sqrt{\alpha'(t)}}$ and set $\varrho := \nu.\exp(k_2 - ik_1)$, $U(t) := \frac{\sin\alpha(t)}{\sqrt{\alpha'(t)}}$, $V(t) := \frac{\cos\alpha(t)}{\sqrt{\alpha'(t)}}$, $u(t) := -iU(t) + V(t)$, $v(t) := iU(t) + V(t)$ for $t \in R$.

Then $|\varrho| \geqq 1$, u, v are independent solutions of (Q), $u(t)v(t) = U^2(t) + V^2(t) \neq 0$,

$$u(t+\pi) = \frac{\cos\alpha(t+\pi)}{\sqrt{\alpha'(t+\pi)}} - i\,\frac{\sin\alpha(t+\pi)}{\sqrt{\alpha'(t+\pi)}} = \frac{\exp(-i\alpha(t+\pi))}{\sqrt{\alpha'(t+\pi)}} =$$
$$= \varrho\left[\frac{\cos\alpha(t)}{\sqrt{\alpha'(t)}} - i\,\frac{\sin\alpha(t)}{\sqrt{\alpha'(t)}}\right] = \varrho.u(t),$$

$$v(t+\pi) = \frac{\cos\alpha(t+\pi)}{\sqrt{\alpha'(t+\pi)}} + i\,\frac{\sin\alpha(t+\pi)}{\sqrt{\alpha'(t+\pi)}} = \frac{\exp(i\alpha(t+\pi))}{\sqrt{\alpha'(t+\pi)}} =$$
$$= \varrho^{-1}\left[\frac{\cos\alpha(t)}{\sqrt{\alpha'(t)}} + i\,\frac{\sin\alpha(t)}{\sqrt{\alpha'(t)}}\right] = \varrho^{-1}.v(t),\quad t \in R,$$

and ϱ, ϱ^{-1} are characteristic multipliers of (Q).

<u>Corollary 2</u>. Let α be a phase of (Q). All solutions of (Q) are π-periodic or π-halfperiodic exactly if

$$\alpha(t+\pi) = \alpha(t) + k\pi,\quad t \in R, \tag{16}$$

where $k = 2n + \frac{1}{2}(1-\varepsilon)$ or $k = 2n + \frac{1}{2}(1+\varepsilon)$, $n \in Z$ and $\varepsilon = \frac{\sqrt{\alpha'(t+\pi)}}{\sqrt{\alpha'(t)}}$ (= ± 1 for $t \in R$).

Proof. ($\Longrightarrow$) Suppose all solutions of (Q) are π-periodic or π-halfperiodic. The functions $\frac{e^{i\alpha(t)}}{\sqrt{\alpha'(t)}}$, $\frac{e^{-i\alpha(t)}}{\sqrt{\alpha'(t)}}$ are independent solutions of (Q) and

$$\frac{e^{i\alpha(t+\pi)}}{\sqrt{\alpha'(t+\pi)}} = \nu\,\frac{e^{i\alpha(t)}}{\sqrt{\alpha'(t)}}\,, \qquad \frac{e^{-i\alpha(t)}}{\sqrt{\alpha'(t+\pi)}} = \nu\,\frac{e^{-i\alpha(t)}}{\sqrt{\alpha'(t)}}\,, \quad t\in R, \tag{17}$$

where $\nu^2 = 1$. Here all solutions for $\nu = 1$ ($\nu = -1$) are π-periodic (π-halfperiodic). On multiplying out both sides of (17) we get $\alpha'(t+\pi) = \alpha'(t)$, thus for any $a\in C$ we have $\alpha(t+\pi) = \alpha(t) + a$ for $t\in R$. Then from (17) there follows $e^{ia} = \nu\varepsilon$, $e^{-ia} = \nu\varepsilon$, with $\varepsilon = \frac{\sqrt{\alpha'(t+\pi)}}{\sqrt{\alpha'(t)}}$. If $a = a_1 + ia_2$ we have $a_2 = 0$, for $\nu\varepsilon = 1$ we get $\cos a_1 = 1$ and for $\nu\varepsilon = -1$ we get $\cos a_1 = -1$. In this way $a_1 = (2n + \frac{1}{2}(1-\varepsilon))\pi$ for $\nu = 1$ and $a_1 = (2n + \frac{1}{2}(1+\varepsilon))\pi$ for $\nu = -1$, where n is an appropriate integer.

($\Longleftarrow$) Suppose α is a phase of (Q) satisfying (16), where n is an integer, $\varepsilon = \frac{\sqrt{\alpha'(t+\pi)}}{\sqrt{\alpha'(t)}}$ and $k = 2n + \frac{1}{2}(1-\varepsilon)$ ($k = 2n + \frac{1}{2}(1+\varepsilon)$). Let us put $u(t) := \frac{e^{i\alpha(t)}}{\sqrt{\alpha'(t)}}$, $v(t) := \frac{e^{-i\alpha(t)}}{\sqrt{\alpha'(t)}}$ for $t\in R$. Then u, v are independent solutions of (Q), $u(t)v(t) \neq 0$, $u(t+\pi) = u(t)$, $v(t+\pi) = v(t)$ ($u(t+\pi) = -u(t)$, $v(t+\pi) = -v(t)$), $t\in R$. It immediately follows from this that all solutions of (Q) are π-periodic (π-halfperiodic).

Remark 5. In the terminology of central transformators of (Q) all solutions of (Q) are π-periodic or π-halfperiodic exactly if $t+\pi$ is a central transformator of (Q).

Remark 6. If all solutions of (Q) are π-periodic or π-halfperiodic, then the value of the number k in Corollary 2 will generally depend on the choice of the phase of (Q) - as it becomes apparent from Example 1 [14].

In the following theorem we present certain sufficient conditions for the derivative α' of a phase α of (Q) to be π-periodic.

Theorem 2. Suppose there exists a number $a \in R$ such that $\text{Re}\, Q(t) + a.\text{Im}\, Q(t) \geq q(t)$ for $t \in R$, where $q \in C^0(R)$ and $y'' = q(t)y$ is a nonoscillatory equation. Then one of the following two mutualy excluding situations arises:

(i) there exist independent solutions of (Q) such that $u(t)v(t) \neq 0$ for $t \in R$ and (1) holds, where $\varrho^2 \neq 1$;

(ii) there exist independent solutions u, v of (Q) such that $v(t) \neq 0$ for $t \in R$ and (2) holds.

Proof. From Lemma 3 there follows that every solution of (Q) has at most a finite number of zeros. Consequently, every solution u of (Q) satisfying the equality $u(t+\pi) = \varrho . u(t)$ on R, where $0 \neq \varrho \in C$, has no zeros on R, i.e. $u(t) \neq 0$ for $t \in R$. Especially from this there follows that all solutions of (Q) cannot be π-periodic or π-halfperiodic. The statement of the Theorem readily follows from the results of the Floquet theory.

Lemma 7. Suppose all solutions of (Q) are not π-periodic or π-halfperiodic. Let α, β be such phases of (Q) that

$$\alpha(t+\pi) = \alpha(t) + a, \quad t \in R, \tag{18}$$

$$\beta(t+\pi) = \beta(t) + b, \quad t \in R, \tag{19}$$

where $a, b \in C$. Then either $a = b$ (in this case $\alpha(t) - \alpha(0) = \beta(t) - \beta(0)$ for $t \in R$) or $a = -b$ (in this case $\alpha(t) - \alpha(0) = -[\beta(t) - \beta(0)]$ for $t \in R$).

Proof. We may assume without loss of generality $\alpha(0) = \beta(0)$. In the contrary case we assume instead of phases $\alpha(t)$ and $\beta(t)$, the phases $\alpha(t) - \alpha(0)$ and $\beta(t) - \beta(0)$, respectively. By Corollary 2 the numbers a,b cannot be equal to an integral multiple of π. Next, from Theorem 4 [13] there follows the existence of $k_1, k_2, k_3, k_4 \in C$, $k_1k_4 - k_2k_3 \neq 0$ such that

$$\beta'(t) = \frac{(k_2k_3 - k_1k_4)\alpha'(t)}{(k_1\cos\alpha(t) + k_2\sin\alpha(t))^2 + (k_3\cos\alpha(t) + k_4\sin\alpha(t))^2}, \quad t \in R. \qquad (20)$$

Placing t instead of $t+\pi$ in (20) then from (18) and (19) we obtain

$$\beta'(t) = $$
$$= \frac{(k_2k_3 - k_1k_4)\alpha'(t)}{(k_1\cos(\alpha(t)+a)+k_2\sin(\alpha(t)+a))^2+(k_3\cos(\alpha(t)+a))+k_4\sin(\alpha(t)+a))^2}, \quad t \in R. \qquad (21)$$

Since a is not equal to a integral multiple of π, there follows from (20) and (21) that $(k_1\cos\alpha(t) + k_2\sin\alpha(t))^2 + (k_3\cos\alpha(t) + k_4\sin\alpha(t))^2$ is a constant function on R, thus $\beta'(t) = c.\alpha'(t)$, where $c \in C$ is an appropriate number, $c \neq 0$. From the equality $(Q(t)=) -\{\alpha, t\} - \alpha'^2(t) = -\{\beta, t\} - \beta'^2(t)$ we obtain $c^2 = 1$. If $c = 1$, then $\beta'(t) = \alpha'(t)$ and therefore $\beta(t) = \alpha(t)$ for $t \in R$ and $a = b$. If $c = -1$, then $\beta'(t) = -\alpha'(t)$ and therefore $\beta(t) = -\alpha(t)$ for $t \in R$ and $a = -b$.

Corollary 3. Let all solutions of (Q) not be π-periodic or π-halfperiodic. If for any phase α of (Q) relation (12) holds, where $0 \leqq k_1 \leqq (1+\text{sign } k_2)\pi$, $k_1 \neq 2\pi$, $k_2 \geqq 0$, then the value of the integer n in this formula does not depend on the choice of the phase α of (Q) and it is defined uniquely by (Q).

Proof. Suppose α, β are the phases of (Q) such that

$$\alpha(t+\pi) = \alpha(t) + (k_1+2n\pi) + ik_2, \quad t \in R,$$

$$\beta(t+\pi) = \beta(t) + (s_1+2m\pi) + is_2, \quad t \in R,$$

where $0 \leqq k_1 \leqq (1+\text{sign } k_2)\pi$, $0 \leqq s_1 \leqq (1+\text{sign } s_2)\pi$, $k_1 \neq \neq 2\pi \neq s_1$, $k_2 \geqq 0$, $s_2 \geqq 0$ and m, n are integers. By Lemma 6 there is either $\alpha(t) - \alpha(0) = \beta(t) - \beta(0)$ or $\alpha(t) - \alpha(0) = -[\beta(t) - \beta(0)]$. If $\alpha(t) - \alpha(0) = \beta(t) - \beta(0)$, then $k_1 = s_1$, $k_2 = s_2$ and $m = n$. If $\alpha(t) - \alpha(0) = -[\beta(t) - \beta(0)]$, then $k_1 + 2n\pi + ik_2 = -(s_1 + 2m\pi + is_2)$, whence $k_2 = -s_2$ and from the assumptions to k_2, s_2 we obtain $k_2 = s_2 = 0$. Then by Theorem 1 $|\varrho| = 1$, where ϱ is one of the characteristic multipliers of (Q). Then, however, $0 \neq k_1 \neq \neq \pi \neq s_1 \neq 0$, because in the contrary case (by Corollary 2) all solutions of (Q) would be π-periodic or π-halfperiodic. Hence $k_1, s_1 \in (0, \pi)$, whence we get $0 < k_1 + s_1 < 2\pi$. From the other side it holds $k_1 + s_1 = -2(m + n)\pi$, which is a contradiction.

Theorem 3. Suppose α be a phase of (Q) and

$$\alpha(t+\pi) = \alpha(t) + a, \quad t \in R, \tag{22}$$

where $0 \neq a \in C$. Then a function β is a phase of (Q_1),

$$\beta(t+\pi) = \beta(t) + a, \quad t \in R \tag{23}$$

and

$$\frac{\sqrt{\alpha'(t+\pi)}}{\sqrt{\alpha'(t)}} = \frac{\sqrt{\beta'(t+\pi)}}{\sqrt{\beta'(t)}} \quad (= \pm 1), \tag{24}$$

exactly if

$$\beta(t) = k + d \int_0^{c(t)} e^{i\gamma(s)} \alpha'(s)ds, \quad t \in R, \tag{25}$$

where $k \in C$, $\tau \in C^2(R)$, $c \in C^3(R)$, $\tau(t+\pi) = \tau(t) + 4n\pi\ (n \in Z)$, $c(t+\pi) = c(t) + \pi$, $c'(t) > 0$ for $t \in R$ and

$$(\neq)\quad d = a\left[\int_0^{\pi} e^{i\tau(s)}\alpha'(s)ds\right]^{-1} .$$

<u>Proof</u>. ($\Longleftarrow$) Let k, d, c, τ satisfy the assumptions of Theorem 3 and the function β be defined by formula (25). Then

$$\beta(t+\pi) = k + d\int_0^{c(t)} e^{i\tau(s)}\alpha'(s)ds + d\int_{c(t)}^{c(t)+\pi} e^{i\tau(s)}\alpha'(s)ds =$$

$$= \beta(t) + a,$$

since the function $e^{i\tau(t)}\alpha'(t)$ is π-periodic and $d\int_0^{\pi} e^{i\tau(s)}\alpha'(s)ds = a$. Next $\beta \in \tilde{C}^3(R)$ and $\beta'(t) =$

$= de^{i\tau(c(t))}(\alpha(c(t)))' \neq 0$ for $t \in R$. Thus β is a phase of any (Q_1). We denote $f_{\alpha'}(t)$ ($f_{\beta'}(t)$) a continuous single-valued branch of the argument of the function α' (β') on R. Then for an integer m there is $f_{\alpha'}(t+\pi) = f_{\alpha'}(t) + 2m\pi$. From the equality $\beta' = de^{i\tau(c)}(\alpha(c))'$ there follows the existence of an integer j such that

$$f_{\beta'}(t) = \tau(c(t)) + f_{\alpha'}(c(t)) + 2j\pi + \operatorname{Arg} d,$$

whence we get

$$f_{\beta'}(t+\pi) = \tau(c(t)+\pi) + f_{\alpha'}(c(t)+\pi) + 2j\pi + \operatorname{Arg} d =$$

$$= \tau(c(t)) + f_{\alpha'}(c(t)) + 2(j+m+2n)\pi + \operatorname{Arg} d =$$

$$= f_{\beta'}(t) + 2(m+2n)\pi ,$$

i.e. (24) is true, whereby

$$\frac{\sqrt{\alpha'(t+\pi)}}{\sqrt{\alpha'(t)}} = \frac{\sqrt{\beta'(t+\pi)}}{\sqrt{\beta'(t)}} = (-1)^m .$$

($\Longrightarrow$) Let β be a phase of (Q_1) satisfying (23), where $0 \neq a \in C$. We put

$$A(t):= \int_0^t |\alpha'(s)|\,ds, \quad B(t):= \int_0^t |\beta'(s)|\,ds, \qquad t \in R.$$

Then A, B are increasing functions on R, $A, B \in C^3(R)$. Because of $|\alpha'(t+\pi)| = |\alpha'(t)|$, $|\beta'(t+\pi)| = |\beta'(t)|$ we have

$$A(t+\pi) = A(t) + a_1, \; B(t+\pi) = B(t) + b_1, \quad t \in R,$$

where $a_1 = A(\pi) > 0$, $b_1 = B(\pi) > 0$. Setting $C(t) := \frac{a_1}{b_1} B(t)$, $c(t) := A^{-1}(C(t))$, $t \in R$, yields

$$C(t+\pi) = C(t) + a_1, \quad t \in R,$$

and $c(t+\pi) = A^{-1}(C(t) + a_1) = A^{-1}(C(t)) + \pi = c(t) + \pi$, sign $c' = 1$, $c(0) = 0$.
From the equality $C(t) = A(c(t))$ it follows that $\int_0^t |\beta'(s)|\,ds = \frac{b_1}{a_1} \int_0^{c(t)} |\alpha'(s)|\,ds$ whence

$$|\beta'(t)| = \frac{b_1}{a_1} c'(t) \, |\alpha'(c(t))| , \qquad t \in R. \tag{26}$$

Let us put $\varphi(t) := \frac{\beta'(t)}{(\alpha(c(t)))'}$, $t \in R$. Then $|\varphi(t)| = \frac{b_1}{a_1}$, $\varphi(t+\pi) = \varphi(t)$, $\varphi \in \tilde{C}^2(R)$. Let f_φ denote a continuous and single-valued branch of the argument of the function φ and $f_{\alpha'}$, $f_{\beta'}$ be defined analogous to the proof ($\Longleftarrow$) above. Then for some integers k, j there holds

$$f_\varphi(t) = f_{\beta'}(t) - f_{\alpha'}(c(t)) + 2j\pi,$$

$$f_{\alpha'}(t+\pi) = f_{\alpha'}(t) + 2k\pi,$$

and from (24) there follows the existence of an integer n:

$$f_{\beta'}(t+\pi) - f_{\beta'}(t) = f_{\alpha'}(t+\pi) - f_{\alpha'}(t) + 4n\pi.$$

Furthermore

$$f_\varphi(t+\pi) = f_{\beta'}(t+\pi) - f_{\alpha'}(c(t)+\pi) + 2j\pi = f_{\beta'}(t) +$$

$$+ f_{\alpha'}(t+\pi) - f_{\alpha'}(t) + 4n\pi - f_{\alpha'}(c(t)) - 2k\pi +$$

$$+ 2j\pi = f_{\beta'}(t) - f_{\alpha'}(c(t)) + 2j\pi + 4n\pi =$$

$$= f_\varphi(t) + 4n\pi .$$

Therefore there exist an integer n and a function τ, $\tau \in C^2(R)$, $\tau(t+\pi) = \tau(t) + 4n\pi$ such that the function φ may be written as $\varphi(t) = de^{i\tau(c(t))}$, where $d := \frac{b_1}{a_1}$. From the definition of functions φ, τ and from (26) we obtain $\beta'(t) = de^{i\tau(c(t))}\cdot(\alpha(c(t)))'$. Integrating the last equality from 0 to t we get

$$\beta(t) = \beta(0) + d\int_0^t e^{i\tau(c(s))}\alpha'(c(s))c'(s)ds = \beta(0) +$$

$$+ d\int_0^{c(t)} e^{i\tau(s)}\alpha'(s)ds.$$

From this and from (23) it follows

$$\beta(0) + d\int_0^{c(t)} e^{i\tau(s)}\alpha'(s)ds + a =$$

$$= \beta(0) + d\int_0^{c(t)+\pi} e^{i\tau(s)}\alpha'(s)ds$$

and consequently

$$a = d\int_{c(t)}^{c(t)+\pi} e^{i\tau(s)}\alpha'(s)ds = d\int_0^{\pi} e^{i\tau(s)}\alpha'(s)ds.$$

If we put $d := a\left[\int_0^{\pi} e^{i\tau(s)}\alpha'(s)ds\right]^{-1}$ and k:=

$:= \beta(0) + d \int_{c(0)}^{U} e^{i\tau(s)}\alpha'(s)ds$, then the phase β may be written in the form of (25).

Remark 7. Let all solutions of (Q) not to be π-periodic or π-halfperiodic. It follows from Corollary 3 that a phase α of (Q), for which (12) holds - where $0 \leqq k_1 \leqq (1+\text{sign } k_2)\pi$, $k_1 \neq 2\pi$, $k_2 \geqq 0$ and n is an integer - is uniquely determined up to an additive constant.

Remark 7 justifies us to the following

Definition 1. Let all solutions of (Q) not be π-periodic or π-halfperiodic, n being an integer, and $\nu^2 = 1$. We say that the pair of numbers (n,ν) (it this order) is a significant pair of numbers of (Q) if there exists a phase α of (Q) such that (12) holds, where $0 \leqq k_1 \leqq (1+\text{sign } k_2)\pi$, $k_1 \neq 2\pi$, $k_2 \geqq 0$ and $\nu = \frac{\sqrt{\alpha'(\pi)}}{\sqrt{\alpha'(0)}}$ $\left(= \frac{\sqrt{\alpha'(t+\pi)}}{\sqrt{\alpha'(t)}}\right.$ for $t \in R)$.

Theorem 4. Let (n, ν) be the significant pair of numbers of (Q) and α be such a phase of (Q) that (12) holds, where $0 \leqq k_1 \leqq (1+ \text{sign } k_2)\pi$, $k_1 \neq 2\pi$, $k_2 \geqq 0$, $\nu = \frac{\sqrt{\beta'(\pi)}}{\sqrt{\beta'(0)}}$.
Then (n,ν) is the significant pair of numbers of (Q_1) and the equations (Q) and (Q_1) have equal characteristic multipliers exactly if

$$Q_1(t) = Q(c(t))c'^2(t) - \{c,t\} + (\alpha(c(t)))'^2(1-d^2e^{2i\tau(c(t))}) + \\ + \frac{c'^2(t)}{4}\left[2i\tau'(c(t))\frac{\alpha''(c(t))}{\alpha'(c(t))} - 2i\tau''(c(t)) - \tau'^2(c(t))\right], \quad t \in R \qquad (27)$$

where $\tau \in C^2(R)$, $c \in C^3(R)$, $\tau(t+\pi) = \tau(t) + 4n\pi\ (n \in Z)$, $c(t+\pi) = c(t) + \pi$, $c'(t) > 0$ for $t \in R$ and

$$d = \left(\int_0^{\pi} e^{i\tau(s)}\alpha'(s)ds\right)^{-1}.(k_1 + 2n\pi + ik_2) .$$

Proof. ($\Longrightarrow$) Let (n,ν) be significant numbers of (Q_1) and let the equations (Q) and (Q_1) have equal characteristic multipliers. From Theorem 1, Corollary 3 and from its proof then there follows the existence of such a phase β of (Q_1) that

$$\beta(t+\pi) = \beta(t) + (k_1 + 2n\pi) + ik_2, \quad t \in R,$$

and $\nu = \dfrac{\sqrt{\beta'(\pi)}}{\sqrt{\beta'(0)}}$. By Theorem 3 naturally $\beta(t) = h + {}$

$+ d \displaystyle\int_0^{c(t)} e^{i\tau(s)}\alpha'(s)ds$, where $h \in C$ and d, c, τ satisfying

the assumptions stated in the Theorem. From the equality $Q_1(t) = -\{\beta, t\} - \beta'^2(t)$ we get with some modification the form of (27) for the coefficient Q_1 of (Q_1).

($\Longleftarrow$) Let the function Q_1 be defined by (27), where d, c, τ satisfy the assumptions of the Theorem. A direct calculation shows that the function $\beta(t) := d \displaystyle\int_0^{c(t)} e^{i\tau(s)}\alpha'(s)ds$,

$t \in R$, is a phase of (Q_1). By Theorem 3 there hold (23) and (24), thus from Theorem 1 it follows that (n,ν) is the significant pair of numbers of (Q) and (Q_1) and both equations have equal characteristic multipliers.

Corollary 4. Suppose the group of increasing transformators L_Q^+ of (Q) is planar. Then there exist independent solutions u, v of (Q), $u(t)v(t) \neq 0$ for $t \in R$ satisfying (1).

Proof. By Corollary 1 [14] there exists a function $Y \in C^3(R)$, $Y(R) = R$, $Y'(t) > 0$ for $t \in R$ and a number $c \in C$, $c^2 \in C - R$ such that the function $\alpha(t) := c.Y(t)$ for $t \in R$, is a phase of (Q). Let $c = c_1 + ic_2$. Then $c_1c_2 \neq 0$ and it follows from the equality $-\{\alpha, t\} - \alpha'^2(t) = Q(t)$ that $-2c_1c_2Y'^2(t) = \operatorname{Im} Q(t)$, hence $Y'(t) = \nu\sqrt{\dfrac{-\operatorname{Im} Q(t)}{2c_1c_2}}$ for $t \in R$, where $\nu^2 = 1$. Naturally, then Y' and thus also α' are π-periodic functions and from Theorem 1 there immediately follows the assertion of the Corollary.

Remark 8. If there exist a function $Y \in C^3(R)$, $Y(R) = R$, $Y'(t) > 0$ for $t \in R$ and a number $c \in C$, $c^2 \in C - R$ such that the function $\alpha(t) := c.Y$ is a phase of (Q), i.e. the group of increasing transformators of (Q) is planar (see Corollary 1 [14]), then it follows from Corollary 1 that

$$\sqrt{\frac{Y'(0)}{Y'(\pi)}}\exp\left[ic(Y(\pi) - Y(0))\right], \quad \sqrt{\frac{Y'(\pi)}{Y'(0)}}\exp\left[ic(Y(0) - Y(\pi))\right]$$

are characteristic multipliers of (Q).

Case 2

Theorem 5. There exist independent solutions u, v of (Q) such that $u(t) \neq 0$ for $t \in R$, v has a zero at a point of R satisfying (1) exactly if there exist a phase α of (Q), an integer n, and $x \in R$ so that $\alpha'(t)$ is not a π-periodic function, the function $i\alpha'(t) - \frac{\alpha''(t)}{2\alpha'(t)}$ is π-periodic and $\alpha(x+\pi) = \alpha(x) + n\pi$.

Proof. ($\Longrightarrow$) Suppose there exist independent solutions u, v of (Q) for which (1) holds, $u(t) \neq 0$ for $t \in R$, v having a zero on R. Then, by Lemma 4, there exists a phase α of (Q) such that function $i\alpha' - \frac{\alpha''}{2\alpha'}$ is π-periodic and on acconnt of the fact that v has a zero on R, then by Theorem 1, the function α' is not π-periodic. It next follows from Theorem 8 and Theorem 5 [13] that there exist numbers $c_1, c_2 \in C$, $c_1 \neq 0$: $v(t) = c_1\frac{\sin(\alpha(t)+c_2)}{\sqrt{\alpha'(t)}}$ for $t \in R$. For an $x \in R$ let $v(x) = 0$. Then it follows from (1) that $v(x+\pi) = v(x) = 0$ i.e. there exist such integers n_1, n_2 that $\alpha(x) = -c_2 + n_1\pi$,

$\alpha(x+\pi) = -c_2 + n_2\pi$, whence $\alpha(x+\pi) = \alpha(x) + n\pi$, $n(:= n_2 - n_1)$ being an integer.

($\Longleftarrow$) Suppose there exists a phase α of (Q) such that α' is not π-periodic, the function $i\alpha' - \frac{\alpha''}{2\alpha'}$ is π-periodic and there exist an integer n and an $x \in R$: $\alpha(x+\pi) = \alpha(x) + n\pi$. From Lemma 4 there then follows the existence of a solution u of (Q), $u(t) \neq 0$ for $t \in R$, satisfying (6), where $0 \neq \varrho \in C$. If we put $v(t) := \frac{\sin(\alpha(t)-\alpha(x))}{\sqrt{\alpha'(t)}}$ for $t \in R$, then v is a solution of (Q), $v(x) = 0$. Thus u, v are independent solutions of (Q) and $v(x+\pi) = \frac{\sin(\alpha(x+\pi)-\alpha(x))}{\sqrt{\alpha'(x+\pi)}} = \frac{\sin n\pi}{\sqrt{\alpha'(x+\pi)}} = 0$. Therefore $v(x) = v(x+\pi) = 0$ and thus $v(t+\pi) = \gamma . v(t)$ for $t \in R$, where $\gamma \in C$ is a suitable number, $\gamma \neq 0$. From the Floquet theory we have $\gamma = \varrho^{-1}$. We see that the solutions u, v of (Q) satisfy (1).

Remark 9. If there exist such independent solutions u, v of (Q) that $u(t) \neq 0$ for $t \in R$, v having a zero on R and (1) is valid, then the Riccati equation (11) has exactly one π-periodic solution.

Corollary 5. Let α be such a phase of (Q) that α' is not a π-periodic function, $i\alpha' - \frac{\alpha''}{2\alpha'}$ is a π-periodic function and for an $x \in R$ we have $\alpha(x+\pi) = \alpha(x) + n\pi$, n being an integer. Then

$$(-1)^n \frac{\sqrt{\alpha'(x+\pi)}}{\sqrt{\alpha'(x)}}, \qquad (-1)^n \frac{\sqrt{\alpha'(x)}}{\sqrt{\alpha'(x+\pi)}},$$

are the values of the characteristic multipliers of (Q).

Proof. From Corollary 1 and from its proof we find that $\exp(\int_x^{x+\pi} p(s)ds)$, $\exp(-\int_x^{x+\pi} p(s)ds)$, where $p := i\alpha' - \frac{\alpha''}{2\alpha'}$, are the characteristic multipliers of (Q). Since

$$\int_x^{x+\pi} p(s)ds = i\left[\alpha(x+\pi) - \alpha(x)\right] - \frac{1}{2}\left[\ln\alpha'(x+\pi) - \right.$$

$$\left. - \ln\alpha'(x)\right] = in\pi + \ln\frac{\sqrt{\alpha'(x)}}{\sqrt{\alpha'(x+\pi)}}$$

then

$$\exp\left(\int_x^{x+\pi} p(s)ds\right) = (-1)^n \left(\frac{\sqrt{\alpha'(x)}}{\sqrt{\alpha'(x+\pi)}}\right)^{\nu}$$

where $\nu^2 = 1$.

Remark 10. The result of Corollary 5 may be proved also in other way. If we put $\beta(t) := \alpha(t) - \alpha(x)$, $v(t) := \frac{\sin\beta(t)}{\sqrt{\beta'(t)}}$, $t \in R$, then β is a phase of (Q) and v is a solution of this equation, $v(x) = v(x+\pi) = 0$. Hence, the equality $v(t+\pi) = \varrho^{-1}.v(t)$ holds for $t \in R$, where ϱ^{-1} is one of the characteristic multipliers of (Q). By differentiating the equality $\frac{\sin\beta(t+\pi)}{\sqrt{\beta'(t+\pi)}} = \varrho^{-1}\frac{\sin\beta(t)}{\sqrt{\beta'(t)}}$ and setting now in the resulting equality x instead of t, we obtain (with some modification) the equality $\frac{\beta'(x+\pi)}{\sqrt{\beta'(x+\pi)}}\cos\beta(x+\pi) = \varrho^{-1}\frac{\beta'(x)}{\sqrt{\beta'(x)}}\cos\beta(x)$, whence $\varrho = (-1)^n\frac{\sqrt{\beta'(x)}}{\sqrt{\beta'(x+\pi)}}$, thus $(-1)^n\frac{\sqrt{\alpha'(x)}}{\sqrt{\alpha'(x+\pi)}}$ and $(-1)^n\frac{\sqrt{\alpha'(x+\pi)}}{\sqrt{\alpha'(x)}}$ are the characteristic multipliers of (Q).

Remark 11. Let a phase α of (Q) satisfy the assumptions of Corollary 5. Let further $p := i\alpha' - \frac{\alpha''}{2\alpha'}$. From Lemma 5 there follows that then for every phase β of (Q) for which $i\beta' - \frac{\beta''}{2\beta'} = p$, we have $\beta(x+\pi) = \beta(x) + n\pi$.

Theorem 6. There exist independent solutions u, v of (Q), $v(t) \neq 0$ for $t \in R$ for which (2) is valid exactly if the function

$$\alpha(t) = \frac{i}{2} \ln\left[P(t) - \frac{2i\varrho t}{a\pi}\right], \quad t \in R \tag{28}$$

is a phase of (Q), where $0 \neq a \in C$, $P \in \tilde{C}^3(R)$ is a π-periodic function, $P(t) \neq \frac{2i\varrho t}{a\pi}$, $P'(t) \neq \frac{2i\varrho}{a\pi}$, $\sqrt{iP'(t+\pi) + \frac{2\varrho}{a\pi}} = \varrho\sqrt{iP'(t) + \frac{2\varrho}{a\pi}}$ for $t \in R$.

Proof. ($\Longrightarrow$) Let (2) hold, where u, v are independent solutions of (Q), $v(t) \neq 0$ for $t \in R$. Then there exists such a phase α of (Q) that

$$v(t) = \frac{e^{i\alpha(t)}}{\sqrt{\alpha'(t)}}, \qquad t \in R.$$

Every solution of (Q) may be written as $y(t) = v(t)\left[a \int_0^t \frac{ds}{v^2(s)} + b\right]$, where $a, b \in C$. An easy calculation shows that the function u satisfies (2) exactly if

$$u(t) = v(t)\left[a \int_0^t \frac{ds}{v^2(s)} + b\right],$$

where $b \in C$ is an arbitrary constant and $a\varrho \int_t^{t+\pi} \frac{ds}{v^2(s)} = 1$ (with respect to the π-periodicity of v^2 we see that $\int_t^{t+\pi} \frac{ds}{v^2(s)} =$ a constant). Then

$$\frac{1}{a} = \rho \int_t^{t+\pi} \frac{ds}{v^2(s)} = \rho \int_t^{t+\pi} \alpha'(s) e^{-2i\alpha(s)} ds =$$

$$= \frac{i\rho}{2} \left[e^{-2i\alpha(t+\pi)} - e^{-2i\alpha(t)} \right] ,$$

hence

$$e^{-2i\alpha(t+\pi)} = e^{-2i\alpha(t)} - \frac{2i\rho}{a} .$$

From the latter equality then follows the existence of a such a π-periodic function $P \in \tilde{C}^3(R)$, $P(t) \neq \frac{2i\rho t}{a\pi}$, $P'(t) \neq \frac{2i\rho}{a\pi}$ for $t \in R$ that the function $e^{-2i\alpha(t)}$ may be written as

$$e^{-2i\alpha(t)} = P(t) - \frac{2i\rho t}{a\pi} \quad \text{for } t \in R,$$

whence $\alpha(t) = \frac{i}{2} \ln(P(t) - \frac{2i\rho t}{a\pi})$. From the last relation and from the equality $v(t) = \frac{e^{i\alpha(t)}}{\sqrt{\alpha'(t)}}$ (with some modification) we obtain

$$v(t) = \sigma \frac{\sqrt{2}}{\sqrt{iP'(t) + \frac{2\rho}{a\pi}}} \quad \text{for } t \in R, \text{ where } \sigma^2 = 1.$$

It the follows from the assumption $v(t+\pi) = \rho . v(t)$ that

$$\sqrt{iP'(t+\pi) + \frac{2\rho}{a\pi}} = \rho \sqrt{iP'(t) + \frac{2\rho}{a\pi}} \quad \text{for } t \in R.$$

($\Longleftarrow$) Let the function α defined by (28) be a phase of (Q) where the function P and the number a satisfy the assumptions of the Theorem. Putting $v(t) := \frac{e^{i\alpha(t)}}{\sqrt{\alpha'(t)}}$ yields $v(t) \neq 0$, $v(t) = \sigma \frac{\sqrt{2}}{\sqrt{iP'(t) + \frac{2\rho}{a\pi}}}$, $v(t+\pi) = \rho . v(t)$ for $t \in R$, where $\sigma^2 = 1$. Let us put further $u(t) := v(t) \left[a \int_0^t \frac{ds}{v^2(s)} + \right.$

$+ \frac{ia}{2} e^{-2i\alpha(0)}\Big]$ for $t \in R$. Then u is a solution of (Q) and

$$u(t) = \frac{ia}{2} v(t) e^{-2i\alpha(t)} = \frac{\sigma a \sqrt{2}}{2} \frac{iP(t) + \frac{2\varrho t}{a\pi}}{\sqrt{iP'(t) + \frac{2\varrho}{a\pi}}} .$$ Consequently

$u(t+\pi) = \varrho . u(t) + v(t)$. So, we have proved that there exist independent solutions u, v of (Q), $v(t) \neq 0$ for $t \in R$, satisfying (2).

Corollary 6. There exist independent solutions u, v of (Q), $v(t) \neq 0$ for $t \in R$, for which (2) is valid exactly if

$$Q(t) = -\frac{1}{2} \frac{P'''(t)}{P'(t) - \frac{2i\varrho}{a\pi}} + \frac{3}{4}\left(\frac{P''(t)}{P'(t) - \frac{2i\varrho}{a\pi}}\right)^2 \quad \text{for } t \in R$$

where $0 \neq a \in C$, $P \in \tilde{C}^3(R)$ is a π-periodic function, $P(t) \neq$ $\neq \frac{2i\varrho t}{a\pi}$, $P'(t) \neq \frac{2i\varrho}{a\pi}$, $\sqrt{iP'(t+\pi) + \frac{2\varrho}{a\pi}} = \varrho \sqrt{iP'(t) + \frac{2\varrho}{a\pi}}$ for $t \in R$.

Proof. This immediately follows from the preceding Theorem and from the fact that α is a phase of (Q) exactly if it is a solution (on R) the equation $Q(t) = -\{\alpha, t\} -$ $-\alpha'^2(t)$.

Example 2. Consider the equation

$$y'' = \frac{4e^{2it}(1 - e^{2it})}{(1 + 2e^{2it})^2} y .$$

The functions $v(t) = \frac{\sqrt{\pi}}{\sqrt{1 + 2e^{2it}}}$ and $u(t) = \frac{t - ie^{2it}}{\sqrt{\pi}\sqrt{1 + 2e^{2it}}}$ are its independ solutions for which $v(t+\pi) = v(t)$, $u(t+\pi) =$ $= u(t) + v(t)$ for $t \in R$.

Case 3

<u>Theorem 7</u>. An equation (Q) has independent solutions u, v satisfying

$$u(t+\pi) = \varrho . u(t), \quad v(t+\pi) = \varrho . v(t), \quad t \in R, \ \varrho^2 \neq 1, \tag{29}$$

where u, v have zeros on R exactly if there exists such a phase α of (Q) that α' is not a π-periodic function and

$$\begin{aligned} \alpha(t_1) &= n_1\pi, & \alpha(t_2) &= \frac{\pi}{2} + n_2\pi, \\ \alpha(t_1+\pi) &= k_1\pi, & \alpha(t_2+\pi) &= \frac{\pi}{2} + k_2\pi, \end{aligned} \tag{30}$$

where $t_1, t_2 \in [0, \pi)$, $t_1 \neq t_2$ and k_1, k_2, n_1, n_2 are integers. In this case $(-1)^{k_1-n_1} \dfrac{\sqrt{\alpha'(t_1+\pi)}}{\sqrt{\alpha'(t_1)}}$, $(-1)^{k_1-n_1} \dfrac{\sqrt{\alpha'(t_1)}}{\sqrt{\alpha'(t_1+\pi)}}$

(or also $(-1)^{k_2-n_2} \dfrac{\sqrt{\alpha'(t_2+\pi)}}{\sqrt{\alpha'(t_2)}}$, $(-1)^{k_2-n_2} \dfrac{\sqrt{\alpha'(t_2)}}{\sqrt{\alpha'(t_2+\pi)}}$) are

the characteristic multipliers of (Q).

<u>Proof</u>. ($\Longrightarrow$) Let there exist independent solutions u, v of (Q) satisfying (29), both having zeros on R. Without loss of generality we may assume $u^2(t) + v^2(t) \neq 0$ for $t \in R$. From (29) there follows that u, v have zeros on $[0, \pi)$. Suppose now $u(t_1) = v(t_2) = 0$, where t_1, $t_2 \in [0, \pi)$, $t_1 \neq t_2$. Let α be a phase of the basis (u,v) of (Q). Then $u(t) = c \dfrac{\sin\alpha(t)}{\sqrt{\alpha'(t)}}$, $v(t) = c \dfrac{\cos\alpha(t)}{\sqrt{\alpha'(t)}}$ for $t \in R$, where $c \in C$, $c \neq 0$. Since $u(t_1+\pi) = u(t_1) = 0$, $v(t_2+\pi) = v(t_2) = 0$, we have $\alpha(t_1+\pi) = k_1\pi$, $\alpha(t_1) = n_1\pi$, $\alpha(t_2+\pi) = \frac{\pi}{2} + k_2\pi$, $\alpha(t_2) = \frac{\pi}{2} + n_2\pi$, where n_1, n_2, k_1, k_2 are integers. With

respect to $\varrho^2 \neq 1$, it follows from Corollary 2 that α' is not a π-periodic function.

($\Longleftarrow$) Let there exist such a phase α of (Q) that α' is not a π-periodic function and (30) is valid, where $t_1, t_2 \in [0,\pi)$, $t_1 \neq t_2$ with n_1, n_2, k_1, k_2 being integers.

Setting $u(t) := \dfrac{\sin\alpha(t)}{\sqrt{\alpha'(t)}}$, $v(t) := \dfrac{\cos\alpha(t)}{\sqrt{\alpha'(t)}}$ $(t \in R)$, then u, v are independent solutions of (Q), $u(t_1) = u(t_1+\pi) = 0$, $v(t_2) = v(t_2+\pi) = 0$. Thus (29) holds for a $\varrho \in C$, $\varrho \neq 0$ and since α' is not a π-periodic function, then - by Corollary 2 - we get $\varrho^2 \neq 1$. Writing now t_2 and t_1 for t in the equations

$$\frac{\sin\alpha(t+\pi)}{\sqrt{\alpha'(t+\pi)}} = \varrho\,\frac{\sin\alpha(t)}{\sqrt{\alpha'(t)}}\,,\qquad \frac{\cos\alpha(t+\pi)}{\sqrt{\alpha'(t+\pi)}} = \varrho^{-1}\,\frac{\cos\alpha(t)}{\sqrt{\alpha'(t)}}\,,$$

$$t \in R,$$

respectively, we obtain

$$\varrho = (-1)^{k_2-n_2}\,\frac{\sqrt{\alpha'(t_2)}}{\sqrt{\alpha'(t_2+\pi)}}\qquad \left((-1)^{k_1-n_1}\,\frac{\sqrt{\alpha'(t_1+\pi)}}{\sqrt{\alpha'(t_1)}} = \varrho\right),$$

thus $(-1)^{k_2-n_2}\dfrac{\sqrt{\alpha'(t_2)}}{\sqrt{\alpha'(t_2+\pi)}}$, $(-1)^{k_2-n_2}\dfrac{\sqrt{\alpha'(t_2+\pi)}}{\sqrt{\alpha'(t_2)}}$ (or also $(-1)^{k_1-n_1}\dfrac{\sqrt{\alpha'(t_1)}}{\sqrt{\alpha'(t_1+\pi)}}$, $(-1)^{k_1-n_1}\dfrac{\sqrt{\alpha'(t_1+\pi)}}{\sqrt{\alpha'(t_1)}}$) are the characteristic multipliers of (Q).

<ins>Theorem 8</ins>. Suppose α is a phase of (Q). This equation has independent solutions u, v satisfying (2), where v has a zero on R exactly if α' is not a π-periodic function and $\alpha(t_1+\pi) = \alpha(t_1) + k\pi$, $(-1)^k\sqrt{\alpha'(t_1+\pi)} = \varrho\sqrt{\alpha'(t_1)}$, where $t_1 \in [0,\pi)$, k being an integer.

<ins>Proof</ins>. ($\Longrightarrow$) Let (Q) have independent solutions u, v satisfying (2) where v has a zero on R. It follows from (2)

that there may be assumed without any loss on generality $v(t_1) = 0$ for $t_1 \in [0, \pi)$. If we put $\beta(t) := \alpha(t) - \alpha(t_1)$ for $t \in R$, then β is a phase of (Q), $\beta(t_1) = 0$ and $v(t) = c \frac{\sin \beta(t)}{\sqrt{\beta'(t)}}$ for $t \in R$, where $0 \neq c \in C$. Since $v(t_1 + \pi) = 0$, we have $\beta(t_1 + \pi) = k\pi$, k being an integer, hence $\beta(t_1 + \pi) - \beta(t_1) = \alpha(t_1 + \pi) - \alpha(t_1) = k\pi$.

Differentiating the equality $\frac{\sin(\alpha(t+\pi) - \alpha(t_1))}{\sqrt{\alpha'(t+\pi)}} = \varrho \frac{\sin(\alpha(t) - \alpha(t_1))}{\sqrt{\alpha'(t)}}$ and inserting t_1 in place of t in the resulting equality, we obtain $(-1)^k \sqrt{\alpha'(t_1 + \pi)} = \varrho \sqrt{\alpha'(t_1)}$. Since it follows from (2) that every solution of (Q) is not a π-periodic or π-halfperiodic, then by Corollary 2, α' is not a π-periodic function, too.

($\Longleftarrow$) Let α not to be a π-periodic function and $\alpha(t_1 + \pi) = \alpha(t_1) + k\pi$, $(-1)^k \sqrt{\alpha'(t_1 + \pi)} = \varrho \sqrt{\alpha'(t_1)}$, where $t_1 \in [0, \pi)$, k being an integer, and $\varrho^2 = 1$. Without any loss on generality there may be assumed $\alpha(t_1) = 0$. Putting $v(t) := \frac{\sin \alpha(t)}{\sqrt{\alpha'(t)}}$ for $t \in R$, then v is a solution of (Q),

$v(t_1) = v(t_1 + \pi) = 0$, thus $v(t + \pi) = \tau . v(t)$ for $t \in R$, where $\tau \in C$ is an appropriate number and from the equality $(-1)^k \sqrt{\alpha'(t_1 + \pi)} = \varrho \sqrt{\alpha'(t_1)}$ there follows $\varrho = \tau = 1$. Since α' is not a π-periodic function, it follows from Corollary 2 that every solution of (Q) is not π-periodic or π-half-periodic. Consequently it follows for (Q) from the Floquet theory that there exist such a solution u of (Q) that u, v are independent solutions of this equation and (2) holds.

<u>Remark 12</u>. If the assumptions of Theorem 7 or of Theorem 8 are satisfied, then there do not exist any π-periodic solutions of the Riccati equation (11).

REFERENCES

[1] B o r ů v k a, O.: Lineare Differential Transformations of the Second Order. The English Univ.Press, London 1971.

[2] B o r ů v k a, O.: On central dispersions of the differential equation $y'' = q(t)y$ with periodic coefficients. Lecture Notes in Mathematics, 415 (1974), 47-60.

[3] B o r ů v k a, O.: Sur les blocs des équations différentielles $y'' = q(t)y$ aux coefficients périodiques. Rend. di Mat. (2), 8 (1975), 519-532.

[4] B o r ů v k a, O.: Sur quelques compléments à la théorie de Floquet pour les équations différentielles du deuxièma ordre. Ann.Mat.Pura Appl. S.IV, CII (1975), 71-77.

[5] Б о р у в к а , О.: Теория глобальных свойств обыкновенных линейных дифференциальных уравнений второго порядка. Дифференциальные уравнения, т. ХП 1976 , 1347-1383.

[6] B o r ů v k a, O.: Sur les blocs des équations différentielles linéaires du deuxième ordre et leurs transformations. Časopis pro pěstování matematiky, 111 (1986), 78-88.

[7] M a g n u s, W. and W i n k l e r, S.: Hill's Equation. Interscience Publishers, New York, 1966.

[8] N e u m a n, F.: Note on bounded non-periodic solutions of second-order linear differential equations with periodic coefficients. Math.Nach. 39 (1969), 217-222.

[9] N e u m a n, F. and S t a n ě k, S.: On the structure of second-order periodic differential equations with given characteristic multipliers. Arch.Math. (Brno), XIII (1977), 149-157.

[10] П л и с , В.А.: Нелокальные проблемы теории колебаний. Издательство Наука, 1964.

[11] S t a n ě k, S.: A note on the disconjugate linear differential equations of the second order with periodic coefficients. Acta Univ.Palackianae Olomucensis, F.R.N., Vol.61, 1979, 93-101.

[12] S t a n ě k, S.: On limit properties of phases and of central dispersions in the oscillatory equation $y'' = q(t)y$ with a periodic coefficient. Acta Univ.Palackiane Olomucensis, F.R.N., Vol.69, 1981, 85-92.

[13] S t a n ě k, S.: A phase of the differential equation $y'' = Q(t)y$ with a complex coefficient Q of a real variable. Acta Univ.Palackianae Olomucensis, F.R.N., Mathematica XXV, vol.85, 1986, 57-73.

[14] S t a n ě k, S.: On a transformation of solutions of the differential equation $y'' = Q(t)y$ with a complex coefficient Q of a real variable. Acta Univ.Palackianae Olomucensis. Math. XXVI., Vol. 88, 1987

[15] S w a n s o n, C.A.: Comparison and Oscillation Theory of Linear Differential Equations. Academic Press, New York and London, 1968.

FLOQUETOVA TEORIE DIFERENCIÁLNÍCH ROVNIC $y'' = Q(t)y$ S KOMPLEXNÍM KOEFICIENTEM REÁLNÉ PROMĚNNÉ

Souhrn

Je vyšetřována diferenciální rovnice

$$y'' = Q(t)y, \; Q(t+\pi) = Q(t), \; \mathrm{Im}\, Q(t) \not\equiv 0 \text{ pro } t \in R, \qquad (Q)$$

kde Q je spojitá komplexní funkce na R. Z Floquetovy teorie plyne, že ke každé rovnici (Q) lze přiřadit čísla ϱ, ϱ^{-1}, která se nazývají charakteristické multiplikátory rovnice (Q). Tato čísla jsou důležitá při vyšetřování kvalitativních vlastností řešení rovnice (Q). V práci je dán nový pohled na Floquetovu teorii rovnic typu (Q) z hlediska teorie fází. Zejména je dokázáno, jak lze hodnoty charakteristických multiplikátorů vyjádřit pomocí nějaké fáze rovnice (Q).

ТЕОРИЯ ФЛОКЕ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ С КОМПЛЕКСНЫМ КОЭФФИЦИЕНТОМ ВЕЩЕСТВЕННОЙ ПЕРЕМЕННОЙ

Резюме

Изучается дифференциальное уравнение

$$y'' = Q(t)y, \; Q(t+\pi) = Q(t), \; \mathrm{Im}\, Q(t) \not\equiv 0, \; t \in R, \qquad (Q)$$

где Q непрерывная комплексная функция на R. Из теории

Флоке следует, что к каждому уравнению (Q) присоединяются числа ρ, ρ^{-1}, которые называются характеристические мультипликаторы уравнения (Q). Эти числа важные при исследовании квалитетивных свойств решений уравнения (Q).

В этой работе приводится новый взгляд на теорию Флоке уравнений типа (Q) с точки зрения теории фаз. В особенности доказывается как значения характеристик мультипликаторов уравнения (Q) представить с помощью некоторой фазы уравнения (Q).

Author's address:
RNDr. Svatoslav Staněk, CSc.
přírodovědecká fakulta
Univerzity Palackého
Gottwaldova 15
771 46 Olomouc
ČSSR /Czechoslovakia/

Acta UPO, Fac.rer.nat.,Vol.91,Mathematica XXVII,1988,149-183.