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ON A TRANSFORMATION OF SOLUTIONS OF THE DIFFERENTIAL EQUATION $y'' = \underline{Q} = (t)y$ WITH A COMPLEX COEFFICIENT $\underline{Q}$ OF A REAL VARIABLE

SVATOSLAV STANĚK

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Dedicated to Professor M.Laitoch on his 65th birthday

1. Introduction

Academician O.Borůvka ([2]) has considered a transformation of solutions of the differential equation of the type

$$y'' = q(t)y \qquad (q)$$

with a continuous real coefficient q on R into the set of solutions of the above equation, connected with the concept of the (1st kind) dispersion of (q).

The present paper now indicates how one can investigate this transformation for differential equations of the type

$$y'' = Q(t)y, \ \mathrm{Im}\, Q(t) \not\equiv 0 \qquad (Q)$$

with a continuous complex coefficient Q on R, using thereby most recent results ([1], [3] - [5], [10] - [12]) releted to the algebraic structure of the intersection of dispersion groups two equations of the type ($\underline{q}$).

2. Basic concepts and notation

The symbol $C^n(R)$ ($\tilde{C}^n(R)$) will denote the set of real functions (the set of complex functions), having continuous derivatives up to and including the order n (n = 0,1,2,...) on R.

A function $\alpha \in C^0(R)$ is understood to be a (first) phase of the equation

$$y'' = q(t)y, \quad q \in C^0(R), \tag{q}$$

if there exist independent solutions u, v such that

$$\operatorname{tg} \alpha(t) = \frac{u(t)}{v(t)} \quad \text{for } t \in R - \{t;\ v(t) = 0\}.$$

Every phase α of (q) possesses the following properties:

$$\alpha \in C^3(R), \quad \alpha'(t) \neq 0 \text{ for } t \in R .$$

The phases of the equation $y'' = -y$, constitutes a so-called fundamental group relative to the rule of composition of functions, which will be writtem as E.

A function α is a phase of (q) exactly if it is a solution (on R) of the differential equation

$$-\{\alpha, t\} - \alpha'^2(t) = q(t) ,$$

where $\{\alpha, t\} := \dfrac{\alpha'''(t)}{2\alpha'(t)} - \dfrac{3}{4}\left(\dfrac{\alpha''(t)}{\alpha'(t)}\right)^2$ is Schwarzian derivative of the function α at the point t.

Let α be a phase of (q). Then $E\alpha := \{\varepsilon\alpha;\ \varepsilon \in E\}$ is the set of phases of (q).

A function X, X(R)= R be called a (complete) dispersion of (q) if it is a solution (on R) of the differential equation

$$-\{X, t\} + X'^2 . q(X) = q(t).$$

The dispersion X of (q) has the following characteristic property:
for every solution y of (q), the function $\dfrac{y[X(t)]}{\sqrt{|X'(t)|}}$ is also a solution of (q) (on R).

The set of dispersions (the set of increasing dispersions) of (q) constitutes a group relative to the rule of composition of functions, which will be written as L_q (L_q^+).

Let α be a phase of (q). If X is a dispersion of (q) then there exists an $\mathcal{E} \in E$ such that $X(t) = \alpha^{-1} \circ \mathcal{E} \circ \alpha(t)$ for $t \in R$ and vice versa, for every $\mathcal{E} \in E$, $\mathcal{E}(j) = j$, where $j := \alpha(R)$, the function $X(t) := \alpha^{-1} \circ \mathcal{E} \circ \alpha(t)$, $t \in R$, is a dispersion of (q).

The above properties and definitions are given for instance in [2].

Say, in accordance with [13] that the function α is a phase of the equation

$$y'' = Q(t)y, \quad Q \in C^0(R), \quad \operatorname{Im} Q(t) \not\equiv 0, \tag{Q}$$

if it is a solution (on R) of the differential equation

$$-\{\alpha, t\} - \alpha'^2(t) = Q(t).$$

If α is a phase of (Q), then $\alpha'(t) \neq 0$ for $t \in R$ and $\frac{\sin \alpha(t)}{\sqrt{\alpha'(t)}}$, $\frac{\cos \alpha(t)}{\sqrt{\alpha'(t)}}$ are its independent solutions. Here $\sqrt{\alpha'(t)}$ denotes a continuous unique branch of the square root of the function α'.

3. The algebraic structure of the group $L^+_{q_1} \cap L^+_{q_2}$

Definition 1 ([3] - [5]). *Say, a set G of functions mapping R onto R is called the planar group if:*

(i) *G is a group relative to the rule of composition of function,*

(ii) *exactly one function belongs to G passes through every point of the plane $R \times R$, i.e. there exists only one function f, $f \in G : f(x_0) = y_0$ to each point $(x_0, y_0) \in R \times R$.*

Lemma 1 ([1]). Let n be a nonnegative integer and G be a planar group of functions from the class $C^n(R)$.

Then every function from G is increasing on R and there exists $Y \in C^n(R)$, $Y(R) = R$, such that

$$G = \{ Y^{-1}[Y(t)+a] ; \quad a \in R \} .$$

If $n \geq 1$, then $Y'(t) > 0$ for $t \in R$.

Lemma 2 ([3] - [5], [10] - [12]). The group $L^+_{q_1} \cap L^+_{q_2}$, $q_1 \neq q_2$, is either a planar group or a infinite cyclic group or the trivial group.

Lemma 3. The group $L^+_{q_1} \cap L^+_{q_2}$, $q_1 \neq q_2$, is a planar group exactly if there exists a function $Y \in C^3(R)$, $Y(R) = R$, $Y'(t) > 0$ for $t \in R$ and such numbers k_1, k_2, $k_1 \neq k_2$, that

$$q_i(t) = -\{Y.t\} + k_i.Y'^2(t), \quad t \in R, \quad i = 1,2. \qquad (1)$$

In this case $L^+_{q_1} \cap L^+_{q_2} = \{ Y^{-1}[Y(t)+a] ; \ a \in R \}$.

P r o o f. ($\Longrightarrow$) Let $L^+_{q_1} \cap L^+_{q_2}$, $q_1 \neq q_2$, be a planar group. Then, by Lemma 1 there exists a function $Y \in C^3(R)$, $Y(R) = R$, $Y'(t) > 0$ for $t \in R$ such that

$$L^+_{q_1} \cap L^+_{q_2} = \{ Y^{-1}[Y(t)+a] ; \ a \in R \} .$$

Consequently, the function $X_a(t) := Y^{-1}[Y(t)+a]$, $t \in R$, is for every $a \in R$ a common solution of equations

$$-\{X,t\} + X'^2.q_i(X) = q_i(t), \quad i = 1,2. \qquad (2)$$

From the equality

$$Y[X_a(t)] = Y(t) + a$$

and from the formula

$$\{f(g),t\} = \{f,g(t)\} g'^2(t) + \{g,t\}$$

hodling for every $f,g \in C^3(R)$, $f'(t).g'(t) \neq 0$ for $t \in R$, it follows that

$$-\{Y,X_a(t)\}X_a'^2(t) - \{X_a,t\} = -\{Y,t\}, \quad t \in R. \qquad (3)$$

Setting $p(t) := -\{Y,t\} - Y'^2(t)$, $t \in R$, then from (2) (writing X_a instead of X) and from (3) we obtain

$$(p[X_a(t)] - q_i[X_a(t)])X_a'^2(t) = p(t) - q_i(t), \quad i=1,2,$$

and thus

$$X_a'^2(t).s_i[X_a(t)] = s_i(t), \quad t \in R, \quad i=1,2, \qquad (4)$$

where $s_i(t) := p(t) - q_i(t)$, $t \in R$. The equalities (4) may be written as

$$Y^{-1\,'2}(t+a).s_i[Y^{-1}(t+a)] = Y^{-1\,'2}(t).s_i[Y^{-1}(t)], \quad t \in R, \quad i=1,2. \qquad (5)$$

Putting $m_i(t) := Y^{-1\,'2}(t).s_i[Y^{-1}(t)]$, $t \in R$, $i=1,2$, then (5) may be written as

$$m_i(t+a) = m_i(t), \quad t,a \in R, \quad i=1,2.$$

Since m_i is a continuous function, $m_i(t)$ is a constant function $(:= l_i)$. Then, naturally, $s_i(t) = l_i.Y'^2(t)$ and $q_i(t) = p(t) - l_i.Y'^2(t) = -\{Y,t\} - (1+l_i)Y'^2(t)$ and we see that (1) holds, where $k_i := -1-l_i$ and because of $q_1 \neq q_2$, we have $k_1 \neq k_2$.

($\Longleftarrow$) Let q_i be defined by (1), where $k_1, k_2 \in R$, $k_1 \neq k_2$, $Y \in C^3(R)$, $Y(R) = R$ and $Y'(t) > 0$ for $t \in R$. Let us put for $a \in R$ $X_a(t) := Y^{-1}[Y(t)+a]$, $t \in R$. It then follows from the equalities

$$-\{X_a,t\} + X_a'^2(t).q_i[X_a(t)] = -\{X_a,t\} +$$

$$+ X_a'^2(t)(-\{Y,X_a(t)\} + k_i.Y'^2[X_a(t)]) = -\{Y(X_a),t\} +$$

$$+ k_i.(Y[X_a(t)])'^2 = -\{Y,t\} + k_i.Y'^2(t) = q_i(t)$$

that $X_a \in L^+_{q_1} \cap L^+_{q_2}$ and because of $q_1 \neq q_2$ and since the set of functions $\{X_a(t)\}_{a\in R}$ constitutes a planar group, we get from Lemma 2 that $L^+_{q_1}\cap L^+_{q_2}$ is a planar group.

Remark 1. Under the assumption that the equations (q_1), (q_2) are oscillatory, Lemma 3 follows from [3] - [5].

Lemma 4. The group $L^+_{q_1}\cap L^+_{q_2}$ is an infinite cyclic group exactly if there exists a function $Y\in C^3(R)$, $Y(R) = R$, $Y'(t) > 0$ for $t\in R$, and π-periodic continuous (on R) functions h_1, h_2, $h_1 \neq h_2$, from which at least one is inconstant such that

$$q_i(t) = -\{Y,t\} + Y'^2(t).h_i[Y(t)],\ t\in R,\ i=1,2. \quad (6)$$

P r o o f. ($\Longrightarrow$) Let $L^+_{q_1}\cap L^+_{q_2}$ be an infinite cyclic group and X be a generator of this group, $X(t) > t$ for $t\in R$. Next, let α_i be a phase of (q_i), i=1,2. Since X is a common dispersion of equations (q_1) and (q_2), there exist $\mathcal{E}_i\in E$ such that

$$X(t) = \alpha_i^{-1}\circ\mathcal{E}_i\circ\alpha_i(t),\ t\in R,\ i=1,2. \quad (7)$$

Let X be the fundamental central dispersion of an equation (p) and Y be an increasing phase of (p). Such an equation (p) always exists (see [2]), it is oscillatory and therefore $Y(R) = R$ and furthermore

$$Y^{-1}[Y(t) + \pi] = X(t).$$

From this and from (7) we have

$$Y\circ\alpha_i^{-1}\circ\mathcal{E}_i\circ\alpha_i\circ Y^{-1}(t) = t + \pi,\ t\in R,\ i=1,2. \quad (8)$$

Let $\gamma_i := \alpha_i\circ Y^{-1}$ be a phase of (h_i), that is $h_i(t) = -\{\gamma_i,t\} - \gamma_i'^2(t)$, $t\in R$. Then $h_i\in C^0(R)$ and we get from (8) that the function $t+\pi$ is a dispersion of (h_i), hence h_i is a π-periodic function.

Since

$$-\{Y^{-1},t\} - Y^{-1\,\prime 2}(t) = -1 - (1+p[Y^{-1}(t)])Y^{-1\,\prime 2}(t)$$

it follows that

$$h_i(t) = -\{\gamma_i, t\} - \gamma_i^{\prime 2}(t) = -\{\alpha_i(Y^{-1}), t\} -$$
$$- \alpha_i^{\prime 2}[Y^{-1}(t)].Y^{-1\,\prime 2}(t) = -\{\alpha_i, Y^{-1}(t)\}Y^{-1\,\prime 2}(t) - \{Y^{-1},t\} -$$
$$- \alpha_i^{\prime 2}[Y^{-1}(t)].Y^{-1\,\prime 2}(t) = q_i[Y^{-1}(t)].Y^{-1\,\prime 2}(t) + Y^{-1\,\prime 2}(t) -$$
$$- 1 - (1 + p[Y^{-1}(t)])Y^{-1\,\prime 2}(t),$$

hence

$$h_i(t) = -1 + (q_i[Y^{-1}(t)] - p[Y^{-1}(t)])Y^{-1\,\prime 2}(t)$$

which yields

$$q_i(t) = p(t) + Y^{\prime 2}(t) + Y^{\prime 2}(t).h_i[Y(t)] =$$
$$= -\{Y,t\} + Y^{\prime 2}(t).h_i[Y(t)] .$$

By our assumption $L^+_{q_1} \cap L^+_{q_2}$ is an infinite cyclic group. Thus, it follows from Lemma 3 that at least one of the functions h_1, h_2 is inconstant and in observing that $q_1 \neq q_2$, we obtain $h_1 \neq h_2$.

($\Longleftarrow$) Let $Y \in C^3(R)$, $Y(R) = R$, $Y'(t) > 0$ for $t \in R$ and h_1, h_2 be continuous π-periodic functions from which at least one be inconstant, $h_1 \neq h_2$. Let q_i be defined by (6) and α_i be a phase of (q_i), i=1,2. Following (6) we get

$$-\{\alpha_i, t\} - \alpha_i^{\prime 2}(t) = -\{Y,t\} - Y^{\prime 2}(t).h_i[Y(t)],$$

hence

$$-\{\alpha_i, Y^{-1}(t)\} - \alpha_i^{\prime 2}[Y^{-1}(t)] = -\{Y, Y^{-1}(t)\} -$$
$$- Y^{\prime 2}[Y^{-1}(t)].h_i(t).$$

From this and from the equality $\{Y, Y^{-1}(t)\}\ Y^{-1\,\prime 2}(t) = -\{Y^{-1},t\}$

we have

$$- \{\alpha_i, Y^{-1}(t)\} Y^{-1\,'2}(t) - \alpha_i'^2 [Y^{-1}(t)] . Y^{-1\,'2}(t) - \{Y^{-1}, t\} = h_i(t)$$

and

$$- \{\alpha_i(Y^{-1}), t\} - (\alpha_i[Y^{-1}(t)])'^2 = h_i(t).$$

Then $\gamma_i := \alpha_i \circ Y^{-1}$ is a phase of (h_i) and since $t+\mathcal{T}$ is a dispersion of (h_i), there exist $\mathcal{E}_i \in E$:

$$\gamma_i^{-1} \circ \mathcal{E}_i \circ \gamma_i(t) = t + \mathcal{T}, \quad t \in R, \quad i=1,2.$$

Consequently

$$Y \circ \alpha_i^{-1} \circ \mathcal{E}_i \circ \alpha_i \circ Y^{-1}(t) = t + \mathcal{T}$$

and

$$\alpha_i^{-1} \circ \mathcal{E}_i \circ \alpha_i(t) = Y^{-1}[Y(t) + \mathcal{T}].$$

Setting $X := Y^{-1}[Y+\mathcal{T}]$, then X is a dispersion of (q_1) and (q_2), hence $L^+_{q_1} \cap L^+_{q_2}$ is not a trivial group and it follows from Lemmas 2 and 3 that this group is necessarily an infinite cyclic group.

Remark 2. Let the functions h_i $(i=1,2)$ in Lemma 4 have the least common period p, $0 < p < \mathcal{T}$ and $\mathcal{T} = jp$. Setting $\overline{Y} := j.Y$, $\overline{h}_i(t) := \frac{1}{j^2} h_i\left(\frac{t}{j}\right)$ for $t \in R$, then $\mathcal{T}$ is the least common period of functions $\overline{h}_i$ and next

$$q_i(t) = -\{Y, t\} + Y'^2(t).h_i[Y(t)] = -\{\overline{Y}, t\} + \frac{\overline{Y}'^2(t)}{j^2} h_i\left(\frac{\overline{Y}(t)}{j}\right) = -\{\overline{Y}, t\} + \overline{Y}'^2(t).\overline{h}_i[\overline{Y}(t)] .$$

Without any loss of generality it may be assumed that the functions h_i in Lemma 4 have the least common period equal to $\mathcal{T}$.

Remark 3. Let (6) be valid for functions q_i (i=1,2), where $Y \in C^3(R)$, $Y(R) = R$, $Y'(t) > 0$ for $t \in R$ and $h_1, h_2 \in C^0(R)$, $h_1 \neq h_2$. Furthermore, let $\mathcal{T}$ be the least common period of functions h_1, h_2. It then follows from the proof ($\Longleftarrow$) of Lemma 4 that

$$L^+_{q_1} \cap L^+_{q_2} = \left\{ Y^{-1}\left[Y(t) + k\mathcal{T}\right];\ k \in Z \right\},$$

where Z denotes the set of integers.

4. Transformator of the equation $y'' = Q(t)y$

Let $f \in C^2(R)$, $h \in C^3(R)$, $f(t).h'(t) \neq 0$ for $t \in R$. If the function $f(t).y\left[h(t)\right]$ is for every solution y(t) of (Q), again a solution of (Q), then necessarily $f(t) = \dfrac{c}{\sqrt{|h'(t)|}}$, where $c \in C$, as it follows from [14]. Consequently we are justified to the following

Definition 1. *Say, a function X is a (complete) transformator of (Q) if*

(i) $X \in C^3(R)$, $X'(t) \neq 0$ *for* $t \in R$, $X(R) = R$,

(ii) *for every solution* y(t) *of* (Q) *the function* $\dfrac{y[X(t)]}{\sqrt{|X'(t)|}}$ *is a solution of this equation again.*

Remark 4. In case of (q) a transformator of (Q) corresponds to a (complete) dispersion (of the 1st kind) of (q) ([2]).

Lemma 5. *A function X is a transformator of (Q) exactly if* $X(R) = R$ *and X is a solution (on R) of the system of the differential equations*

$$-\{X,t\} + X'^2.\mathrm{Re}\,Q(X) = \mathrm{Re}\,Q(t),$$
$$X'^2.\mathrm{Im}\,Q(X) = \mathrm{Im}\,Q(t).$$

P r o o f. It follows immediately from [9].

Remark 5. It becomes clear from Lemma 5 that every transformator X of (Q) is a common dispersion of equations, say (Re Q), (Re Q + Im Q), (Re Q - Im Q).

Remark 6. It follows from Remark 5 that the set of transformators (the set of increasing transformators) of (Q) constitutes a group relative to the rule of composition of functions, which will be written as L_Q (L_Q^+).

Theorem 1. *L_Q^+ is either a planar group or an infinite cyclic group or the trivial group.*

P r o o f. It follows directly from Remark 5 and Lemma 2.

Theorem 2. *Let α be a phase of (Q). If $X \in C^1(R)$, $X(R) = R$, $X'(t) \neq 0$ for $t \in R$ and the function*

$$\beta(t) := \alpha[X(t)], \quad t \in R \tag{9}$$

is a phase of (Q), then X is a transformator of (Q) and also vice versa, if X is a transformator of (Q), then the function β defined by (9) is a phase of (Q).

P r o o f. Let $X \in C^1(R)$, $X(R) = R$, $X'(t) \neq 0$ for $t \in R$ and the function β defined by (9) be a phase of (Q). Since $\alpha'(t) \neq 0$ for $t \in R$, it follows by differentiating (9) $|\beta'(t)| = \sigma |\alpha'[X(t)]| . X'(t)$, where $\sigma :=$ sign X'. This yields $X \in C^3(R)$. From the definition of the phase of (Q) we obtain

$$Q(t) = -\{\beta, t\} - \beta'^2(t) = -\{\alpha, X(t)\}X'^2(t) - \{X, t\} -$$
$$- \alpha'^2[X(t)].X'^2(t) = (-\{\alpha, X(t)\} - \alpha'^2[X(t)])X'^2(t) -$$
$$- \{X, t\} = -\{X, t\} + X'^2(t).Q[X(t)] ,$$

hence

$$-\{X, t\} + X'^2(t).Q[X(t)] = Q(t)$$

and respecting Lemma 5, X is a transformator of (Q).

Let X be a transformator of (Q) and β be defined by (9).

Then by Lemma 5

$$-\{\beta,t\} - \beta'^2(t) = -\{\alpha,X(t)\}X'^2(t) - \{X,t\} -$$
$$-\alpha'^2[X(t)].X'^2(t) = (-\{\alpha,X(t)\} - \alpha'^2[X(t)])\,X'^2(t) -$$
$$-\{X,t\} = -\{X,t\} + X'^2(t).Q[X(t)] = Q(t),$$

hence β is a phase of (Q).

<u>Theorem 3</u>. L_Q^+ *is a planar group exactly if there exists a* $Y \in C^3(R)$, $Y(R) = R$, $Y'(t) > 0$ *for* $t \in R$ *and* $s_1, s_2 \in R$, $s_2 \neq 0$, *such that*

$$\begin{aligned} \mathrm{Re}\, Q(t) &= -\{Y,t\} + s_1.Y'^2(t), \\ \mathrm{Im}\, Q(t) &= s_2.Y'^2(t), \quad t \in R . \end{aligned} \tag{10}$$

P r o o f. $L_Q^+ = L^+_{\mathrm{Re}\,Q} \cap L^+_{\mathrm{Re}\,Q\,+\,\mathrm{Im}\,Q}$, which follows from

Remark 5. Let L_Q^+ be a planar group. By Lemma 3 there exists a $Y \in C^3(R)$, $Y(R) = R$, $Y'(t) > 0$ for $t \in R$ and $k_1, k_2 \in R$, $k_1 \neq k_2$, such that

$$\mathrm{Re}\, Q(t) = -\{Y,t\} + k_1.Y'^2(t),$$
$$\mathrm{Re}\, Q(t) + \mathrm{Im}\, Q(t) = -\{Y,t\} + k_2.Y'^2(t).$$

From this immediately follows (10) if we put $s_1 := k_1$ and $s_2 := k_2 - k_1$.

Let $Y \in C^3(R)$, $Y(R) = R$, $Y'(t) > 0$ for $t \in R$ and $s_1, s_2 \in R$, $s_2 \neq 0$, and (10) be valid. Then

$$\mathrm{Re}\, Q(t) = -\{Y,t\} + s_1.Y'^2(t),$$
$$\mathrm{Re}\, Q(t) + \mathrm{Im}\, Q(t) = -\{Y,t\} + (s_1 - s_2).Y'^2(t)$$

and from Lemma 3 we get that $L^+_{\mathrm{Re}\,Q} \cap L^+_{\mathrm{Re}\,Q\,-\,\mathrm{Im}\,Q}$ is a planar group. From Remark 5 it follows that $L_Q^+ = L^+_{\mathrm{Re}\,Q} \cap L^+_{\mathrm{Re}\,Q\,-\,\mathrm{Im}\,Q}$, which reveals that L_Q^+ is also a planar group.

<u>Corollary 1</u>. L_Q^+ *is a planar group exactly if*

$$\alpha(t) = c.Y(t) \ , \ t \in R \ , \qquad (11)$$

in a phase of (Q), where

$$Y \in C^3(R), \ Y(R)=R, \ Y'(t) > 0 \ \text{for} \ t \in R, \ c^2 \in C - R. \qquad (12)$$

P r o o f. ($\Longrightarrow$) Let L_Q^+ be a planar group. Then (10) holds, where Y satisfies assumption (12), $s_1, s_2 \in R$, $s_2 \neq 0$. If we put $c := \sqrt{-s_1 - is_2}$, then $c^2 \in C - R$ and we get from the equalities

$$-\{Y,t\} + (s_1+is_2)Y'^2(t) = \operatorname{Re} Q(t) + i \operatorname{Im} Q(t) = Q(t)$$

that the function α defined by (11) is a phase of (Q).

($\Longleftarrow$) Let the function α defined by (11), where Y, c satisfy assumptions (12) be a phase of (Q). It follows from the equalities

$$Q(t) = -\{\alpha,t\} - \alpha'^2(t) = -\{Y,t\} - c^2.Y'^2(t)$$

that

$$\operatorname{Re} Q(t) = -\{Y,t\} - (c_1^2 - c_2^2)Y'^2(t) \ ,$$

$$\operatorname{Im} Q(t) = -2c_1c_2Y'^2(t),$$

where $c = c_1 + ic_2$. Since $c_1c_2 \neq 0$, it follows from Theorem 3 that L_Q^+ is a planar group.

Remark 7. If Y and c satisfy assumptions (12) and the function α defined by (11) is a phase of (Q), then evidently $L_Q^+ = \{Y^{-1}[Y(t) + a] \ ; \ a \in R\}$.

Remark 8. Let Y and c satisfy assumptions (12) and the function α defined by (11) be a phase of (Q). Let $c=c_1+ic_2$. If $c_1^2-c_2^2 > 0$, then the equation (Re Q) is oscillatory and $\sqrt{c_1^2 - c_2^2}\, Y(t)$ is its (elliptic) phase. If $c_1^2 - c_2^2 = 0$, then (Re Q) is a specially disconjugate equation and Y(t) is its parabolic phase (see [2], [7], [8]). If $c_1^2 - c_2^2 < 0$, then (Re Q) is a generally disconjugate equation and $\sqrt{c_2^2 - c_1^2}\, Y(t)$ is its hyperbolic phase (see [2], [6], [8]).

Theorem 4. L_Q^+ is an infinite cyclic group exactly if

$$\begin{aligned} \operatorname{Re} Q(t) &= -\{Y,t\} + Y'^2(t).s_1[Y(t)], \\ \operatorname{Im} Q(t) &= Y'^2(t).s_2[Y(t)], \quad t \in R, \end{aligned} \tag{13}$$

where

$$Y \in C^3(R),\ Y(R)=R,\ Y'(t) > 0 \text{ for } t \in R,\ s_1,\ s_2 \in C^0(R),\ s_2 \neq 0, \tag{14}$$

and $\mathcal{T}$ is the least common period of the functions s_1, s_2.

P r o o f. ($\Longrightarrow$) By Remark 5 we know that $L_Q^+ = L_{\operatorname{Re} Q}^+ \cap$ $\cap\, L_{\operatorname{Re} Q + \operatorname{Im} Q}^+$. If L_Q^+ is an infinite group, then from Lemma 4 and from Remark 2 follows the existence of a function Y satisfying assumptions (14) and the existence of functions h_1, $h_2 \in C^0(R)$, $h_1 \neq h_2$, having the least common period equal to $\mathcal{T}$ such that

$$\operatorname{Re} Q(t) = -\{Y,t\} + Y'^2(t).h_1[Y(t)],$$

$$\operatorname{Re} Q(t) + \operatorname{Im} Q(t) = -\{Y,t\} + Y'^2(t).h_2[Y(t)], \quad t \in R .$$

From this and if we set $s_1 := h_1$ and $s_2 := h_2 - h_1$, we obtain (13).

($\Longleftarrow$) Let (13) hold, where the functions Y, s_1, s_2 satisfy assumption (14). Then

$$\operatorname{Re} Q(t) = -\{Y,t\} + Y'^2(t).s_1[Y(t)] ,$$

$$\operatorname{Re} Q(t) - \operatorname{Im} Q(t) = -\{Y,t\} + (s_1[Y(t)] - s_2[Y(t)])Y'^2(t)$$

and it follows from Lemma 4 that $L_{\operatorname{Re} Q}^+ \cap L_{\operatorname{Re} Q - \operatorname{Im} Q}^+$ is an infinite cyclic group which, following Remark 5, is equal to L_Q^+.

Remark 9. If Y and s_1, s_2 satisfy assumption (14), then $L_Q^+ = \{ Y^{-1}[Y(t) + j\mathcal{T}];\ \ j \in Z \}$.

5. Central transformator of the equation $y'' = Q(t)y$

Lemma 6. Let X be such a transformator of (Q) that there exists to any (nontrivial) solution of (Q) such a number $\mathcal{T} \in C$ that

$$\frac{y[X(t)]}{\sqrt{|X'(t)|}} = \tau . y(t), \quad t \in R .$$

Then sign $X' = 1$, the number τ is independent of the choice of the solution y of (Q) and $\tau^2 = 1$.

P r o o f. Let u, v be independent solutions of (Q), $u(t).v(t) \neq 0$ for $t \in R$. Such solutions u, v always exists (see [13]). Then there exist numbers $\tau_1, \tau_2 \in C$:

$$\frac{u[X(t)]}{\sqrt{|X'(t)|}} = \tau_1 . u(t), \qquad \frac{v[X(t)]}{\sqrt{|X'(t)|}} = \tau_2 . v(t), \quad t \in R . \tag{15}$$

Let us put $\gamma :=$ sign X'. Since

$$\left(\frac{u[X(t)]}{\sqrt{|X'(t)|}}\right)' \frac{v[X(t)]}{\sqrt{|X'(t)|}} - \frac{u[X(t)]}{\sqrt{|X'(t)|}}\left(\frac{v[X(t)]}{\sqrt{|X'(t)|}}\right)' =$$

$$= \tau_1 \tau_2 (u'(t)v(t) - u(t)v'(t)),$$

it may be verified by an easy calculation that the expression on the right side of the last equality is equal to $\gamma(u'v-uv')$, we get

$$\gamma = \tau_1 \tau_2 . \tag{16}$$

Let $\gamma = -1$. Then $X(x) = x$ for an $x \in R$. Setting x in place of t in (15), yields $\tau_1 = \tau_2 = \dfrac{1}{\sqrt{-X'(x)}}$. Naturally, then $\tau_1 \tau_2 = = \dfrac{1}{X'(x)} > 0$, which contradicts (16). Thus sign $X' = 1$.

Let $\tau_1 \neq \tau_2$. Furthermore, let $k_1, k_2 \in C$, $0 \neq k_1 \neq k_2 \neq 0$ and put $y := k_1 u + k_2 v$. Then y is a nontrivial solution of (Q), hence for a $c \in C$, $c \neq 0$:

$$\frac{y[X(t)]}{\sqrt{X'(t)}} = cy(t), \quad t \in R .$$

From the last equality we obtain

$$k_1 \frac{u[X(t)]}{\sqrt{X'(t)}} + k_2 \frac{v[X(t)]}{\sqrt{X'(t)}} = c(k_1 u(t) + k_2 v(t)).$$

Herefrom and from (15) we find

$$k_1(\mathcal{T}_1 - c)u(t) + k_2(\mathcal{T}_2 - c)v(t) = 0.$$

Since u, v are independent solutions of (Q), then $\mathcal{T}_1 = \mathcal{T}_2 = c$, which is a contradiction. This proves that $\mathcal{T}_1 = \mathcal{T}_2 =: \mathcal{T}$ and it follows from (16) that $\mathcal{T}^2 = 1$.

Consequently, we are justified to state the following

Definition 3. Say, a transformator X of (Q), $X'(t) > 0$ for $t \in R$, is a central transformator of (Q) if for every solution y(t) of (Q):

$$\frac{y\,X(t)}{\sqrt{X'(t)}} = \mathcal{T}.y(t), \quad t \in R,$$

where $\mathcal{T}^2 = 1$.

Remark 10. The central transformator of (Q) corresponds in case of the equation (q) to its central dispersion (of the 1st kind) (see [2]).

Remark 11. Let L_Q^c denote the set of central transformators of (Q). It is clear that L_Q^c is a subgroup of the group L_Q^+, $L_Q^c \subset L_Q^+$.

Theorem 5. Let α be a phase of (Q). Then X is a central transformator of (Q) exactly if

$$X \in C^1(R),\ X(R)=R,\ X'(t) \neq 0 \text{ for } t \in R$$

and

$$\alpha[X(t)] = \alpha(t) + k\pi, \quad t \in R,$$

where k is an integer (its value generally depends on the choice of the phase of (Q)).

P r o o f. ($\Longrightarrow$) Let X be a central transformator of (Q). Then (17) holds and

$$\begin{aligned} \frac{\sin\alpha[X(t)]}{\sqrt{\alpha'[X(t)]}\sqrt{X'(t)}} &= \mathcal{T}\,\frac{\sin\alpha(t)}{\sqrt{\alpha'(t)}} \\ \frac{\cos\alpha[X(t)]}{\sqrt{\alpha'[X(t)]}\sqrt{X'(t)}} &= \mathcal{T}\,\frac{\cos\alpha(t)}{\sqrt{\alpha'(t)}}, \quad t \in R, \end{aligned} \qquad (19)$$

where $\tau^2 = 1$. Using (19) we obtain $(\alpha[X(t)])' = \alpha'(t)$, hence $\alpha[X(t)] = \alpha(t) + a$ for $t \in R$, where $a \in C$. Let $\sqrt{(\alpha[X(t)])'} = \tau_1\sqrt{\alpha'(t)}$, where $\tau_1^2 = 1$. Then from (19) we obtain

$$\sin(\alpha(t) + a) = \tau\tau_1 \sin \alpha(t),$$

hence $a = k\pi$, where k is an integer and $\tau\tau_1 = (-1)^k$.

($\Longleftarrow$) Let (18) hold, where X satisfies assumption (17). Let $X \neq id_R$. If sign $X' = -1$, then $X(x) = x$ for an $x \in R$. Setting $t = x$ in the equality $\alpha'[X(t)].X'(t) = \alpha'(t)$ yields $\alpha'(x)X'(x) = \alpha'(x)$, hence $X'(x) = 1$, which is a contradiction. Consequently there must be sign $X' = 1$. From equality $\alpha'[X(t)].X'(t) = \alpha'(t)$, $t \in R$, we obtain $X \in C^3(R)$. Let $\sqrt{(\alpha[X(t)])'} = \tau_2\sqrt{\alpha'(t)}$ for $t \in R$, where $\tau_2^2 = 1$. Then

$$\frac{\sin \alpha[X(t)]}{\sqrt{(\alpha[X(t)])'}} = \tau_2 \frac{\sin(\alpha(t)+k\pi)}{\sqrt{\alpha'(t)}} = (-1)^k\tau_2 \frac{\sin\alpha(t)}{\sqrt{\alpha'(t)}},$$

$$\frac{\cos \alpha[X(t)]}{\sqrt{(\alpha[X(t)])'}} = \tau_2 \frac{\cos(\alpha(t)+k\pi)}{\sqrt{\alpha'(t)}} = (-1)^k\tau_2 \frac{\cos\alpha(t)}{\sqrt{\alpha'(t)}},$$

which proves the fact that $\frac{y[X(t)]}{\sqrt{X'(t)}} = (-1)^k\tau_2\, y(t)$ for every solution y of (Q), i.e. X is a central transformator of (Q).

<u>Corollary 2</u>. <u>Let j be a positive integer. Then $t + j\pi$ is a central transformator of (Q) exactly if all solutions of (Q) are either $j\pi$-periodic or $j\pi$-halfperiodic.</u>

P r o o f. ($\Longrightarrow$) Let α be a phase of (Q). If $t + j\pi$ is a central transformator of (Q), then (by Theorem 5) there exists an integer k: $\alpha(t+j\pi) = \alpha(t)+k\pi$. Let $\sqrt{\alpha'(t+j\pi)} = \tau\sqrt{\alpha'(t)}$ for $t \in R$, where $\tau^2 = 1$. Setting $u(t) := \frac{\sin\alpha(t)}{\sqrt{\alpha'(t)}}$, $v(t) := \frac{\cos\alpha(t)}{\sqrt{\alpha'(t)}}$, $t \in R$, then u, v are independent solutions of (Q) and because of $u(t+j\pi) = (-1)^k\tau u(t)$, $v(t+j\pi) = (-1)^k\tau v(t)$, all solutions of (Q) are either $j\pi$-periodic or $j\pi$-halfperiodic and this according as the number $(-1)^k\tau$ is equal to 1 or equal to -1.

($\Leftarrow$) Let all solutions of (Q) be either $j\pi$-periodic or $j\pi$-halfperiodic for definiteness let them be $j\pi$-halfperiodic. Let α be a phase of (Q). Then $u(t) := \frac{\sin\alpha(t)}{\sqrt{\alpha'(t)}}$, $v(t) := \frac{\cos\alpha(t)}{\sqrt{\alpha'(t)}}$, $t \in R$, are independent solutions of (Q). By our assumption $u(t+j\pi) = -u(t)$, $v(t+j\pi) = -v(t)$, therefore

$$\frac{\sin\alpha(t+j\pi)}{\sqrt{\alpha'(t+j\pi)}} = -\frac{\sin\alpha(t)}{\sqrt{\alpha'(t)}}, \quad \frac{\cos\alpha(t+j\pi)}{\sqrt{\alpha'(t+j\pi)}} = -\frac{\cos\alpha(t)}{\sqrt{\alpha'(t)}} \text{ for } t \in R .$$

Then $\alpha'(t+j\pi) = \alpha'(t)$, hence $\alpha(t+j\pi) = \alpha(t) + a$, where $a \in C$. Since $\operatorname{tg}\alpha(t+j\pi) = \operatorname{tg}\alpha(t)$, then $a = k\pi$, where k is an integer. Naturally, then $\alpha(t+j\pi) = \alpha(t) + k\pi$ for $t \in R$ and it follows from Theorem 5 that $t+j\pi$ is a central transformator of (Q).

Example 1. Consider the differential equation

$$y'' = \left(-1 + \frac{\pi^2}{16} e^{4it}\right) y. \qquad (20)$$

The function $\alpha(t) = \frac{\pi}{8} e^{2it}$ is a phase of (Q). From the equality $\alpha(t+\pi) = \alpha(t)$ and from Theorem 5 we find that the function $t+\pi$ is a central transformator of (20). We will show that there exists a phase α_1 of (20) for which $\alpha_1(t+\pi) = \alpha_1(t) + \pi$. Let us put

$$\alpha_1(t) := 4 \int_{\frac{\pi}{8}}^{\frac{\pi}{8}e^{2it}} \frac{dz}{2i\cos^2 z + (2(i-1)\sin z + (1-i)\cos z)^2}, \quad t \in R ,$$

where the integral is written along the curve expressed in a parametric form $z = \frac{\pi}{8} e^{2it}$, $t \in R$. Then α_1 is a phase of (20) (see [13], Theorem 4). Let us set $f(z) :=$

$$= \frac{1}{2i\cos^2 z + (2(i-1)\sin z + (1-i)\cos z)^2}$$ whenever the fraction is meaningful. The singular points of the function f are the roots of the equation

$$2i\cos^2 z + (2(i-1)\sin z + (1-i)\cos z)^2 = 0,$$

which may be written in an equaivalent form

$$(\sin z - \cos z)\sin z = 0.$$

Inside the circle with the center at the origin and the radius $\frac{\pi}{8}$, the function f has the singularity only at the point $z_1=0$, which is the pole of the first order. A direct calculation shows that $\mathrm{Res}\,(f;z_1) = -\frac{i}{8}$ and therefore $\alpha_1(\pi) = \pi$. From the equality

$$\alpha_1'(t+\pi) = \alpha_1'(t) \text{ we obtain } \alpha_1(t+\pi) = \alpha_1(t) + \pi.$$

Let (18) hold for a central transformator X of (Q), where $k \geq 1$. It will become apparent from the following example that, generally, there exists no central transformator ($\neq id_R$) of (Q), to which would correspond a smaller value k (≥ 0) in (18).

Example 2. Let $\alpha(t) := 4t + i\sin 2t$, $Q(t) := -\{\alpha, t\} - \alpha'^2(t)$, $t \in R$. Then α is a phase of (Q) and it follows from $\alpha(t+\pi) = \alpha(t) + 4\pi$ and from Theorem 5 that $t+\pi$ is a central transformator of (Q). If X were such a central transformator of (Q), $X \neq id_R$, that $\alpha[X(t)] = \alpha(t) + k\pi$, where k would be one of the integers 1, 2, 3, then $X(t) = t+\frac{k\pi}{4}$ and it would hold $\sin 2(t+\frac{k\pi}{4}) = \sin 2t$ for $t \in R$, which, however, leads to a contradiction.

Theorem 6. If L_Q^+ is a planar group, then L_Q^c is the trivial group.

P r o o f. Let L_Q^+ be a planar group. Then there exist Y and c satisfying assumptions (12), that α defined by (11) is a phase of (Q). Let $X \in L_Q^c$. According to Theorem 3 there exists an integer k such that (18) holds and we have

$$c.Y[X(t)] = c.Y(t) + k\pi.$$

Then $c_1.Y[X(t)] = c_1.Y(t) + k\pi$, $c_2.Y[X(t)] = c_2.Y(t)$, where $c = c_1+ic_2$. Since $c_1c_2 \neq 0$, then $Y[X(t)] = Y(t)$, hence $X = id_R$ and L_Q^c is the trivial group.

From Theorem 6 immediatelly follows

Corollary 3. *If L_Q^c is not the trivial group, then L_Q^+ is an infinite cyclic group and consequently L_Q^c is also an infinite cyclic group.*

Let L_Q^c be an infinite cyclic group. Then, by Corollary 3, L_Q^+ is also an infinite cyclic group. It becomes apparent from the next example that in such a case L_Q^c may generally be proper subgroup of the group L_Q^+.

Example 3. Let $\alpha(t) := \frac{t}{2} + i\sin 2t$, $Q(t) := -\{\alpha, t\} - \alpha'^2(t)$, $t \in R$. Then α is a phase of (Q) and since $\alpha(t+\pi) = \alpha(t) + \frac{\pi}{2}$, then $t+\pi$ is a transformator of (Q) which is not a central transformator of (Q) as it follows from Theorem 5. From this Theorem and from the equality $\alpha(t+2\pi) = \alpha(t) + \pi$ we find that $t+2\pi$ is a central transformator of (Q). Hence, L_Q^c is a proper subgroup of the group L_Q^+.

Theorem 7. *L_Q^c is an infinite cyclic group exactly if there exist $Y \in C^3(R)$, $Y(R)=R$, $Y'(t) > 0$ for $t \in R$ and s_1, $s_2 \in C^0(R)$, $s_2 \neq 0$, whose the least common period is π such that*

$$\begin{aligned} \operatorname{Re} Q(t) &= -\{Y,t\} + Y'^2(t).s_1[Y(t)], \\ \operatorname{Im} Q(t) &= Y'^2(t).s_2[Y(t)], \quad t \in R, \end{aligned} \tag{21}$$

and all solutions of (S), $S(t) = s_1(t)+is_2(t)$ for $t \in R$, are either $j\pi$-periodic or $j\pi$-halfperiodic, where j is a positive integer.

P r o o f. ($\Longrightarrow$) Let L_Q^c be an infinite cyclic group. Then from the proof ($\Longrightarrow$) to Theorem 4 and Lemma 4 follows the existence of $Y \in C^3(R)$, $Y(R)=R$, $Y'(t) > 0$ for $t \in R$ and h_1, $h_2 \in C^0(R)$, $h_1 \neq h_2$, whose the least common period is equal to π (cf. Remark 2) such that

$$\begin{aligned} \operatorname{Re} Q(t) &= -\{Y,t\} + Y'^2(t).h_1[Y(t)], \\ \operatorname{Re} Q(t) + \operatorname{Im} Q(t) &= -\{Y,t\} + Y'^2(t).h_2[Y(t)], \quad t \in R. \end{aligned} \tag{22}$$

Then, by Remark 5 we have $(L_Q^+ =)\ L^+_{\mathrm{Re}\,Q} \cap L^+_{\mathrm{Re}\,Q + \mathrm{Im}\,Q} = \{ Y^{-1}[Y(t)+k\pi] ;\ k \in Z \}$. Let α be a phase of (Q) and $X \in L_Q^c$, $X(t) > t$ for $t \in R$. It then follows from Theorem 5 and from Corollary 3 that $\alpha[X(t)] = \alpha(t)+k\pi$, $X(t) = Y^{-1}[Y(t)+j\pi]$, where $k \in Z$ and j is a positive integer. Setting $\beta := \alpha \circ Y^{-1}$, then $\beta(t+j\pi) = \beta(t) + k\pi$. Let β be a phase of (S). Then

$$S(t) = -\{\beta, t\} - \beta'^2(t) = -\{\alpha, Y^{-1}(t)\} Y^{-1\,'2}(t) -$$

$$- \{Y^{-1}, t\} - Y^{-1\,'2}(t) . \alpha'^2[Y^{-1}(t)] = \tag{23}$$

$$= Y^{-1\,'2}(t).Q[Y^{-1}(t)] + \{Y, Y^{-1}(t)\} Y^{-1\,'2}(t), \quad t \in R .$$

From (22) we can write

$$Q(t) = -\{Y,t\} + Y'^2(t).h_1[Y(t)] + i(h_2[Y(t)] - h_1[Y(t)])Y'^2(t)$$

and setting $s_1 := h_1$, $s_2 := h_2 - h_1$, yields

$$Q(t) = -\{Y,t\} + (s_1[Y(t)] + is_2[Y(t)])\ Y'^2(t),$$

whence

$$Y^{-1\,'2}(t).Q[Y^{-1}(t)] = -\{Y, Y^{-1}(t)\}\ Y^{-1\,'2}(t) + (s_1(t)+is_2(t)).$$

From (23) we then obtain $S = s_1 + is_2$. The functions s_1, s_2 have the least common period equal to π and from Corollary 3 we know that all solutions of (S) are either $j\pi$-periodic or $j\pi$-halfperiodic. Thus from (22) and from the definition of the functions s_1, s_2 we obtain the validity of (21).

($\Longleftarrow$) Let Y, s_1 and s_2 satisfy the assumptions of Theorem 7, all solutions of (S) are either $j\pi$-periodic or $j\pi$-halfperiodic where j is a positive integer and (21) holds. Let β be a phase of (S). Following Corollary 3 $\beta(t+j\pi) = \beta(t)+k\pi$, where $k \in Z$. Setting $\alpha := \beta \circ Y$, $X(t) := Y^{-1}[Y(t)+j\pi]$ $(\neq id_R)$ for $t \in R$, then $\alpha[X(t)] = \beta[Y(t)+j\pi] = \beta[Y(t)] + k\pi = \alpha(t) + k\pi$ and

$$-\{\alpha,t\} - \alpha'^2(t) = -\{\beta, Y(t)\} Y'^2(t) - \{Y,t\} -$$

$$- Y'^2(t).\ \beta'^2[Y(t)] = Y'^2(t).S[Y(t)] - \{Y,t\} =$$

$$= -\{Y,t\} + Y'^2(t)(s_1[Y(t)] + is_2[Y(t)]) = Q(t).$$

Consequently α is a phase of (Q) and it follows from Theorem 5 with respect to the equality $\alpha[X(t)] = \alpha(t) + k\pi$, that $X \in L_Q^c$. Hence L_Q^c is a infinite cyclic group.

Theorem 8. *L_Q^c is an infinite cyclic group exactly if the function*

$$\alpha(t) := \int_0^{F(t)} \exp(i\beta(t))ds, \quad t \in R, \tag{24}$$

where $F \in C^3(R)$, $F(R)=R$, $F'(t) > 0$ for $t \in R$, $\beta \in C^2(R)$, $\exp(i\beta(t))$ is an inconstant periodic function (with a period a) and for a non-negative integer k

$$\int_0^a \exp(i\beta(s))ds = k\pi, \tag{25}$$

is a phase of (Q). In this case $F^{-1}[F(t) + a]$ is a central transformator of (Q).

P r o o f. ($\Longleftarrow$) Suppose F, β satisfy the assumptions for $a > 0$ and a non-negative number k given in Theorem 8, and α defined by (24) be a phase of (Q). Then

$$\alpha\left[F^{-1}[F(t)+a]\right] = \int_0^{F(t)+a} \exp(i\beta(s))ds = \int_0^{F(t)} \exp(i\beta(s))ds + $$
$$+ \int_0^a \exp(i\beta(s))ds = \alpha(t) + k\pi,$$

and following Theorem 5 $F^{-1}[F(t)+a]$ ($\neq id_R$) is an element of L_Q^c, hence L_Q^c is an infinite cyclic group.

($\Longrightarrow$) Let $X \in L_Q^c$, $X \neq id_R$. It may be assumed without any loss of generality that $X(t) > t$ for $t \in R$. By Theorem 5 there exist a phase α of (Q) and a non-negative integer k such that

$$\alpha[X(t)] = \alpha(t) + k\pi, \quad t \in R . \tag{26}$$

Since sign $X' = 1$, it follows from (26) ($\alpha = \alpha_1 + i\alpha_2$)

$$X'(t)\sqrt{\alpha_1'^2[X(t)] + \alpha_2'^2[X(t)]} = \sqrt{\alpha_1'^2(t) + \alpha_2'^2(t)} \,,$$

which after integration yields

$$F[X(t)] = F(t) + a, \quad t \in R, \tag{27}$$

where $F(t) := \int_0^t \sqrt{\alpha_1'^2(s) + \alpha_2'^2(s)}\, ds$, $t \in R$, $a = F[X(0)]$. Hereby $a > 0$ follows from $X(0) > 0$. Evidently $F \in C^3(R)$, $F(R) = R$, sign $F' = 1$ and there exists a $\gamma \in C^2(R)$ such that

$$\alpha_1'(t) = F'(t)\cos\gamma(t),$$

$$\alpha_2'(t) = F'(t)\sin\gamma(t), \quad t \in R\,.$$

Here γ is an inconstant function. In the contrary case, equation (Q) possesses a phase $c.F(t)$, where $c \in C$ is an appropriate number and by Corollary 1 L_Q^+ is a planar group. It then follows from Theorem 6 that L_Q^c is the trivial group.

Setting $\beta := \gamma \circ F^{-1}$, then $\beta \in C^2(R)$ is an inconstant function and

$$\alpha_1'(t) = F'(t)\cos\beta[F(t)],$$

$$\alpha_2'(t) = F'(t)\sin\beta[F(t)],$$

whence

$$\alpha'(t) = F'(t)\exp(i\beta[F(t)]) \tag{28}$$

and

$$\alpha(t) = \int_0^{F(t)} \exp(i\beta(s))ds + b, \quad t \in R\,,$$

where $b \in C$. Since $\alpha'[X(t)].X'(t) = \alpha'(t)$, it follows from (28) that

$$X'(t).F'[X(t)]\exp(i\beta \circ F \circ X(t)) = F'(t)\exp(i\beta[F(t)])$$

from which and from (27) we obtain

$$\exp(i\beta(t+a)) = \exp(i\beta(t)), \quad t \in R\,.$$

Consequently $\exp(i\beta(t))$ is an inconstant a-periodic function. Besides

$$\alpha[X(t)] = \int_0^{F(t)+a} \exp(i\beta(s))ds + b = \alpha(t) + \int_0^{a} \exp(i\beta(s))ds$$

and with respect to (26) $\int_0^a \exp(i\beta(s))ds = k\pi$. Respecting the fact that $\alpha(t) - b$ is also a phase of (Q), then $\int_0^{F(t)} \exp(i\beta(s))ds$ is a phase of (Q) and it follows from (27) that $X(t) = F^{-1}[F(t) + a]$ for $t \in R$.

<u>Remark 12</u>. Suppose functions F, β satisfy the assumptions given in Theorem 8, and a (> 0) be the least period the (inconstant) function $\exp(i\beta(t))$ (obviously fulfilling (25)). In then follows from the proof of Theorem 8 that $L_Q^c = \{F^{-1}[F(t)+ka] \; ; \; k \in Z\}$.

<u>Corollary 4</u>. *L_Q^c is an infinite cyclic group if and only if*

$$Q(t) = -\{F,t\} - \frac{1}{4}(\beta[F(t)])'^2 - \frac{i}{2}F'(t)\beta''[F(t)] - F'^2(t)\exp(2i\beta[F(t)]), \quad t \in R,$$

where F, β satisfy the assumptions given in Theorem 8.

P r o o f. Following Theorem 8 L_Q^c is an infinite cyclic group if and only if α defined by (24) is a phase of (Q). Now Corollary 4 follows immediately by a modification of the equality

$$Q(t) = -\{\alpha, t\} - \alpha'^2(t).$$

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SOUHRN

Transformace řešení rovnice $y'' = Q(t)y$ s komplexním koeficientem Q reálné proměnné

Svatoslav Staněk

Řekneme, že funkce X je (úplný) transformátor rovnice

$$y'' = Q(t)y, \qquad (Q)$$

kde Q je spojitá (na R) komplexní funkce, jestliže:

(i) $X \in C^3(R)$, $X(R)=R$, $X'(t) \neq 0$ pro $t \in R$,

(ii) pro každé řešení y(t) rovnice (Q) je funkce $\frac{y[X(t)]}{\sqrt{|X'(t)|}}$ opět řešením této rovnice.

Množina všech rostoucích transformátorů rovnice (Q) tvoří vzhledem k operaci skládání funkcí grupu, kterou označíme L_Q^+. Užitím výsledků o algebraické struktuře průniku grup rostoucích dispersí dvou různých diferenciálních rovnic typu $y'' = q(t)y$, kde q je spojitá (na R) reálná funkce, je dokázáno, že L_Q^+ je buď planární grupa (tj. ke každému bodu $(t_0, x_0) \in R \times R$ existuje jediná funkce $X \in L_Q^+$ taková, že $X(t_0) = x_0$), nebo nekonečná cyklická grupa a nebo triviální grupa (věta 1). Ve větě 3 resp. ve větě 4 jsou uvedeny nutné a postačující podmínky kladené na koeficient Q rovnice (Q), aby L_Q^+ byla planární grupa resp. nekonečná cyklická grupa.

Řekneme, že transformátor X rovnice (Q), sign $X' = 1$, je centrální transformátor této rovnice, jestliže pro každé řešení y(t) rovnice (Q) je

$$\frac{y[X(t)]}{\sqrt{X'(t)}} = \Upsilon . Q(t), \qquad t \in R,$$

kde $\Upsilon^2 = 1$. Množina centrálních transformátorů rovnice (Q) tvoří podgrupu L_Q^c grupy L_Q^+, $L_Q^c \subset L_Q^+$. Jestliže L_Q^+ je planární grupa, pak L_Q^c je triviální grupa (věta 6), tedy L_Q^c je buď nekonečná cyklická grupa a nebo triviální grupa (důsledek 3). Ve větě 7 (větě 8) jsou uvedeny nutné a postačující podmínky

kladené na koeficient Q (na fázi rovnice (Q)), aby L_Q^c byla nekonečná cyklická grupa.

Věta 2 resp. věta 5 dává do souvislosti fáze rovnice (Q) a transformátor resp. centrální transformátor rovnice (Q).

РЕЗЮМЕ

Преобразования решений уравнения $y'' = Q(t)y$ с комплексным коэффициентом Q вещественной переменной

Сватослав Станек

Функция X называется полным трансформатором уравнения

$$y'' = Q(t)y, \qquad (Q)$$

где Q непрерывная (на R)комплексная функция, если:

(i) $X \in C^3(R)$, $X'(t) \neq 0$ для $t \in R$, $X(R) = R$;

(ii) для каждого решения $y(t)$ уравнения (Q) функция $\frac{y[X(t)]}{\sqrt{|X'(t)|}}$ снова решением этого уравнения.

Множество возрастающих трансформаторов уравнения (Q) является относительно операции сложения функций группой, которую обозначаем L_Q^+.

Применением результатов алгебраической структуры пересечения групп возрастающих дисперсий двух различных дифференциальных уравнений типа

$$y'' = q(t)y,$$

где q непрерывная (на R) вещественная функция, показано, что L_Q^+ или планарная группа (т.е. для любой точки $(t_0, x_0) \in R \times R$ найдется только одна функция $X \in L_Q^+$, что $X(t_0) = x_0$) или бесконечная циклическая группа или тривиальная группа (теорема 1). В теореме 3 соответственно в теореме 4 приведены необходимые и достаточные условия которым должен удовлетворять коэффициент Q уравнения (Q), чтобы L_Q^+ являлась пла-

нарной группой соответственно бесконечной циклической группой.

Трансформатор X уравнения (Q), sign $X' = 1$, называется центральным трансформатором этого уравнения, если для каждого решения y(t) уравнения (Q) имеет место $\frac{y[X(t)]}{\sqrt{X'(t)}} = \gamma . y(t), t \in R$, где $\gamma^2 = 1$.

Множество центральных трансформаторов уравнения (Q) является подгруппой L_Q^c группы L_Q^+. Если L_Q^+ планарная группа, то L_Q^c тривиальная группа (теорема 6), следовательно L_Q^c или бесконечная циклическая группа или тривиальная группа (следствие 3). В теореме 7 (в теореме 8) приведены необходимые и достаточные условия которым должен удовлетворять коэффициент Q уравнения (Q) (фаза уравнения (Q)), чтобы L_Q^c была бесконечная циклическая группа.

В теореме 2 соответственно в теореме 5 показана связь между фазой уравнения (Q) и трансформатором соответственно центральным трансформатором уравнения (Q).