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METHOD OF SUCCESSIVE APPROXIMATIONS FOR A CERTAIN NON-LINEAR THIRD ORDER BOUNDARY VALUE PROBLEM

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Dedicated to Professor M.Laitoch on occasion of his 65th birthday

Introduction

In this paper we shall investigate a boundary value problem

(1) $$x''' = f(t,x,x'), \quad (t,x,x') \in [a_1,a_3] \times R^2,$$

(2) $$\alpha_2 x'(a_1) - \alpha_3 x''(a_1) = A_1, \quad x(a_2) = A_2, \quad \gamma_2 x'(a_3) + \gamma_3 x''(a_3) = A_3,$$
$$\alpha_2, \alpha_3, \gamma_2, \gamma_3 \geq 0, \quad \alpha_2 + \alpha_3 > 0, \quad \gamma_2 + \gamma_3 > 0, \quad a_1 < a_2 < a_3.$$

Denote $I = [a_1, a_3]$, $I_1 = [a_1, a_2]$, $I_2 = [a_2, a_3]$.

By method of successive approximations we shall prove the existence theorem for (1) and (2). Successive approximations will be formed by means of lower and upper solutions, a monotone operator using Green's functions and their signs. Monotone operator on partially ordered Banach spaces was applied at solving boundary value problems e.g. by K.Schmitt in [1], R. Bellman in [2] and V.Šeda in [3].

Let $G_k(t,s)$, $k = 1,2$ be Green's functions belonging to (1) and (2). G_k are uniquely determined by the following three properties (see [4], [5], [6]).

For arbitrary point $s \in (a_k, a_{k+1})$ there holds:

1. G_k, $\frac{\partial G_k}{\partial t} = G_{kt}$ are continuous in t on I.

2. $\frac{\partial^2 G_k}{\partial t^2} = G_{ktt}$ is continuous in t everywhere on I, except of the point s (a point of incontinuability of the first kind) where $G_{ktt}(s+0,s) - G_{ktt}(s-0,s) = 1$.

3. G_k as a function of t is a solution of $x''' = 0$ on intervals $[a_1,s)$, $(s,a_3]$ and fulfils homogeneous boundary conditions (2) for $A_1 = A_2 = A_3 = 0$.

Further there holds: If $\varphi(t)$ is a solution of $x''' = 0$ and (2), then the solution $x(t)$ of (1) and (2) is a solution of an integro-differential equation

$$(3) \qquad x(t) = \varphi(t) + \sum_{k=1}^{2} \int_{a_k}^{a_{k+1}} G_k(t,s) f(s, x(s), x'(s)) ds$$

and conversely.

Lemma. _For Green's functions G_1 and G_2 there holds:_
if $s \in (a_1,a_2)$, then $G_1 \geqq 0$, $\forall t \in I_1$ and $G_1 \leqq 0$, $\forall t \in I_2$,
if $s \in (a_2,a_3)$, then $G_2 \geqq 0$, $\forall t \in I_1$ and $G_2 \leqq 0$, $\forall t \in I_2$,
if $s \in (a_k,a_{k+1})$, then $G_{kt} \geqq 0$, $\forall t \in I$, $k = 1,2$.

P r o o f. From Green's function properties we get $G_k(a_2,s) = 0$, $s \in (a_k, a_{k+1})$, $k = 1,2$. Now it suffices to prove the third part of Lemma. Its assertion can be obtained from direct calculation of explicit expression of Green's functions. If we denote $\Delta = -2(\alpha_2\gamma_2(a_3-a_1) + \alpha_2\gamma_3 + \alpha_3\gamma_2) < 0$, then we have:

For $s \in (a_1, a_2)$ and $a_1 \leqq t \leqq s$ there is

$$G_{1t}(t,s) = \frac{2}{\Delta}\,(\alpha_2\gamma_2(t-a_1)(a_3-s) + \alpha_2\gamma_3(t-a_1) + \alpha_3\gamma_2(a_3-s) + \alpha_3\gamma_3) \leqq 0.$$

The same result is obtained for $G_{kt}(t,s)$ when $s \in (a_2, a_3)$ and $a_1 \leqq t \leqq s$ because it has formally the same form.

For $s \in (a_1, a_2)$ and $s < t \leqq a_3$ there is

$$G_{1t}(t,s) = \frac{2}{\Delta}\,(\alpha_2(s-a_1) + \alpha_3)(\gamma_2(a_3-t) + \gamma_3) \leqq 0.$$

Similarly, $G_{2t}(t,s)$ has the same form when $s \in (a_2, a_3)$ and $s < t \leqq a_3$ from which we can see that it is non-positive.

A function $\alpha \in C_3(I)$ will be said to be a lower solution of (1) and (2) if

$$\alpha''' \geqq f(t, \alpha, \alpha'),$$

$$\alpha_2\alpha'(a_1) - \alpha_3\alpha''(a_1) \leqq A_1,\ \alpha(a_2) = A_2,\ \gamma_2\alpha'(a_3) + \gamma_3\alpha''(a_2) \leqq A_3.$$

Similarly $\beta \in C_3(I)$ will be an upper solution of (1) and (2) if

$$\beta''' \leqq f(t, \beta, \beta'),$$

$$\alpha_2\beta'(a_1) - \alpha_3\beta''(a_1) \geqq A_1,\ \beta(a_2) = A_2,\ \gamma_2\beta'(a_3) + \gamma_3\beta''(a_3) \geqq A_3.$$

For α and β moreover let

$$\alpha'(t) \leqq \beta'(t), \quad \forall t \in I.$$

Existence theorem

Theorem. Let a function $f(t,x,x')$ have properties:

(i) f is continuous on $I \times \mathbb{R}^2$.

(ii) f is non-decreasing in x on $\mathbb{R}$ for $t \in I_1$ and non-increasing in x on $\mathbb{R}$ for $t \in I_2$.

(iii) f is non-decreasing in x' on $\mathbb{R}$.

(iv) let there exist functions $\alpha, \beta \in C_3(I)$ that are a lower and upper solutions of (1) and (2).

Then there exists at least one solution x of (1) and (2) for which we have

$$\beta(t) \leqq x(t) \leqq \alpha(t), \ \forall t \in I_1, \quad \alpha(t) \leqq x(t) \leqq \beta(t), \ \forall t \in I_2$$
(4)
$$\alpha'(t) \leqq x'(t) \leqq \beta'(t), \quad \forall t \in I$$

and which can be obtained by process of successive approximations. This process will be explained in the following proof.

P r o o f. Define by (3) operator T on $C_1(I)$ as follows:

$$Tx(t) = \varphi(t) + \sum_{k=1}^{2} \int_{a_k}^{a_{k+1}} G_k(t,s) f(s,x(s),x'(s))ds \ . \tag{5}$$

The function Tx fulfils for every $x \in C_1(I)$ boundary conditions (2) and $Tx \in C_3(I)$.

Let α and β satisfy (iv). We shall prove that

$$\alpha'(t) \leqq (T\alpha)'(t), \quad (T\beta)'(t) \leqq \beta'(t), \quad \forall t \in I. \tag{6}$$

First we prove the second inequality. Considering that β is the upper solution of (1) and (2) we obtain for $v = (T\beta)' - \beta'$:

$$- \alpha_2 v(a_1) + \alpha_3 v'(a_1) \geqq 0$$

$$- \gamma_2 v(a_3) - \gamma_3 v'(a_3) \geqq 0,$$

$v'' = (T\beta)'''(t) - \beta'''(t) = f(t,\beta(t),\beta'(t)) - \beta'''(t) \geqq 0$ for every $t \in I$.

From these results we get that $v(t) \leqq 0$ for any $t \in I$; hence the second inequality in (6) holds.

For functions $(T\alpha)'$ and $(T\beta)'$ from the properties of α and β, from Lemma and from (ii) and (iii) there follows (according to (5)) that

(7) $$(T\alpha)'(t) \leqq (T\beta)'(t) \quad \text{for any} \quad t \in I.$$

Further, denote $\alpha_0 = \alpha$ and $\beta_0 = \beta$ and form sequences of functions $\{\alpha_n\}$ and $\{\beta_n\}$ by means of recurrent formulas

$$\alpha_{n+1} = T\alpha_n, \quad \beta_{n+1} = T\beta_n, \qquad n \geqq 0.$$

From inequalities (6) and (7) by induction we obtain

(8) $$\alpha_0'(t) \leqq \alpha_1'(t) \leqq \dots \leqq \alpha_n'(t) \leqq \dots \\ \dots \leqq \beta_n'(t) \leqq \dots \leqq \beta_1'(t) \leqq \beta_0'(t), \\ \forall t \in I.$$

From these inequalities it follows

(9) $$\alpha_0(t) \geqq \alpha_1(t) \geqq \dots \geqq \alpha_n(t) \geqq \dots \\ \dots \geqq \beta_n(t) \geqq \dots \geqq \beta_1(t) \geqq \beta_0(t), \\ \forall t \in I_1$$

and for every $t \in I_2$ converse inequalities are fulfilled.

Hence, sequences $\{\alpha_n'\}$ and $\{\beta_n'\}$ are monotone and bounded from above, casually from below. Further, they are uniformly bounded and with regard to expression of their terms by (5) they are even equipollently continous on I, from which it follows that they are uniformly convergent on I. Because $\alpha_n(a_2) = \beta_n(a_2) = A_2$ for every n, there exist functions $x, y \in C_1(I)$ such that

(10) $$\{\alpha_n(t)\} \rightrightarrows x(t), \qquad \{\beta_n(t)\} \rightrightarrows y(t), \\ \{\alpha_n'(t)\} \rightrightarrows x'(t), \qquad \{\beta_n'(t)\} \rightrightarrows y'(t) \qquad \text{on } I.$$

From (8) and (9) we have

$$x(t) \geqq y(t), \ \forall t \in I_1, \quad x(t) \leqq y(t), \ \forall t \in I_2, \\ x'(t) \leqq y'(t), \ \forall t \in I.$$

From uniform convergence given by (10) and on the basis of the properties of G_k, k = 1,2 an f, using (5) we get

$$\{T\alpha_n(t)\} \rightrightarrows Tx(t), \qquad \{T\beta_n(t)\} \rightrightarrows Ty(t) \qquad \text{on } I,$$

from which

$$x = Tx, \qquad y = Ty.$$

Thus, functions x and y are solutions of (1) and (2) whereby for x (4) holds and y has the same property.

Remark 1. Let z(t) be an arbitrary solution of (1) and (2) for which

$$\beta \leqq z \leqq \alpha, \quad \forall t \in I_1, \quad \alpha \leqq z \leqq \beta, \quad \forall t \in I_2,$$
$$\alpha' \leqq z' \leqq \beta', \quad \forall t \in I.$$

Then from equality z = Tz for $z' = (Tz)'$ we obtain

$$\alpha_1' = (T\alpha)' \leqq z' \leqq (T\beta)' = \beta_1', \quad \forall t \in I.$$

By successive repeating this process we get

$$\alpha_n' \leqq z' \leqq \beta_n', \qquad \text{for every } t \in I \text{ and every } n$$

hence according to (10) it holds

$$y \leqq z \leqq x, \quad \forall t \in I_1, \quad x \leqq z \leqq y, \quad \forall t \in I_2,$$
$$x' \leqq z' \leqq y', \quad \forall t \in I.$$

If the case $x(t) = y(t)$ for every $t \in I$ occurs, then there exists the unique solution of (1) and (2) for which (4) holds.

Remark 2. Similar existence results are proved in [7] where the method of modification od differential equation is used.

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SÚHRN

Metóda postupných aproximácií pre istú nelineárnu okrajovú úlohu 3-ho radu

J á n R u s n á k

V práci je dokázána existenčná veta pre okrajovú úlohu typu: $x''' = f(t,x,x')$, $\alpha_2 x'(a_1) - \alpha_3 x''(a_1) = A_1$, $x(a_2) = A_2$, $\gamma_2 x'(a_3) + \gamma_3 x''(a_3) = A_3$. Použitá je metóda postupných aproximácií, ktoré sú utvorené pomocou dolných a horných riešení, monotónneho operátora s použitím Greenových funkcií a ich znamienok.

РЕЗЮМЕ

Метод последовательных приближений для одной нелинейной краевой задачи третьего порядка

Ян Руснак

В этой статье доказана теорема существования для краевой задачи типа: $x''' = f(t,x,x')$, $\alpha_2 x'(a_1) - \alpha_3 x''(a_1) = A_1$, $x(a_2)=A_2$, $\gamma_2 x'(a_3) + \gamma_3 x''(a_3)=A_3$. Использован метод последовательных приближений оформленных при помощи нижних и верхних решений, монотонного оператора с применением функций Грина и их знаков.

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