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TWO-PARAMETRIC SYSTEMS OF THREE DIMENSIONAL SPACES IN A UNIMODULAR SIX-DIMENSIONAL AFFINE SPACE

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This paper investigates by means of the Cartan method some questions concerning the existence of two-parametric systems Σ of three-dimensional subspaces in a unimodular six-dimensional affine space A_6. There is shown the existence of the above elements both in general and special cases.

The starting point were the well-known results derived in the theory of two-parametric plane systems (briefly called congruences) embedded in the five-dimensional projective space P_5 (see /1/). The specialization of the frame was suitably modified for congruences Σ of three-dimensional spaces in A_6 for every system Σ naturally generating a congruence of planes Σ^∞ in an improper hyperplane N_5^∞ adjoint to A_6. The specialization of the frame is carried out exactly with respect to the congruence Σ^∞.

By the frame in A_6 is meant the point M and the sixtuple of linearly independent vectors e_i, $i = 1,2, \ldots, 6$, satisfying the condition

$$(1) \qquad [e_1\, e_2\, e_3\, e_4\, e_5\, e_6] = 1 \,.$$

The fundamental differentiation formulas are

$$(2) \qquad dM = \omega^j e_j \,, \quad de_i = \omega_i^j e_j \,, \quad i,j = 1,2, \ldots, 6,$$

where the relative components ω^j, ω_i^j are the linear differential forms of the parameters on which the moving frame is depending. These forms satisfy the structure equations of an affine space

$$(3) \qquad d\omega^j = \omega^k \wedge \omega_k^j \,, \quad d\omega_i^j = \omega_i^k \wedge \omega_k^j \,, \quad i,j,k = 1,2,\ldots,6.$$

Every non-zero vector direction belonging to the direction V_6 of the space A_6 determines in the improper hyperplane N_5^{∞} the unique improper point which is in one and only one way determined by an ordered sixtuple of coordinates of the vector given. Thus the hyperplane N_5^{∞} is of a five-dimensional projective space character, wherein every sixtuple of linearly independent points may be regarded as a frame.

For short we denote, hereafter, the points of the hyperplane N_5^{∞} like the vectors in V_6, since the point $A^{\infty} \in N_5^{\infty}$ has the same coordinates as the vector $a \in V_6$ determining the point A^{∞} (for instance, we will write a instead of A^{∞}).

From the foregoing wee see that the ordered sixtuple of linearly independent points $e_i \in N_5^{\infty}$, $(i = 1,2, \ldots, 6)$ for which condition (1) is fulfilled, is a frame in N_5^{∞} . The fundamental differential formula and the structure equations are

$$de_i = \omega_i^j e_j , \quad d\omega_i^j = \omega_i^k \wedge \omega_k^j , \quad i,j,k = 1,2,\dots,6.$$

Performing the differentiation of (1) and substituing the values from (2) for de_i, we get

$$(4) \qquad \sum_j \omega_j^j = 0 , \qquad j = 1,2,\dots,6.$$

Suppose now Σ is a two-parametric system of three-dimensional spaces $S = S(u,v)$ in A_6, where the pair of principal parameters u, v are the analytic functions of variables on which moving frame is depending. The Pfaff forms ω_j^j satisfy equation (4). Every space $S \in \Sigma$ has a plane σ^∞ in the improper hyperplane N_5^∞, i.e. $S \cap N_5^\infty = \sigma^\infty$. The planes σ^∞ generate a two-parametric system Σ^∞ of planes in N_5^∞ which, herafter, will be called the plane congruence in N_5^∞. Likewise the system Σ of three-dimensional spaces in A_6 will be called the congruence of three-dimensional spaces.

If a point M and a triple of linearly independent vectors e_1, e_2, e_3 of the moving frame in A_6 are placed into a three-dimensional subspace $S \subset A_6$, $S \in \Sigma$, then the vertices e_1, e_2, e_3 of the moving frame in N_5^∞ are incident with the current plane σ^∞ of the congruence Σ^∞. Performing the differentiation we obtain

$$(5) \quad d[Me_1e_2e_3] = \sum_{i=1}^{3} \omega_i^i [Me_1e_2e_3] + \sum_{j=1}^{3} \omega^{j+3} [e_{j+3}e_1e_2e_3] +$$

$$+ \sum_k \sum_n \omega_k^n [M\, e_l e_m e_n]$$

where k, l, m are cyclic permutations of the elements 1, 2, 3 and n = 4,5,6.

If the variation is as usually denoted by δ, i.e. differentiating just by the incidental parameters ($\delta u = \delta v = 0$) and the π forms arising from ω are written as $du = dv = 0$ ($\pi_i^j = \omega_i^j(\delta)$). then the variation of (5) yields

$$\delta\left[M\, e_1 e_2 e_3\right] = \sum_{i=1}^{3} \pi_i^i \left[M\, e_1 e_2 e_3\right],$$

so that ω^{j+3}, ω_k^n from relation (5) i.e.

(6) $\omega^4, \omega^5, \omega^6, \omega_1^4, \omega_1^5, \omega_1^6, \omega_2^4, \omega_2^5, \omega_2^6, \omega_3^4, \omega_3^5, \omega_3^6$

are principal forms. There are exactly two independent forms among them. Let

$$\omega_1^4 = \omega_1, \qquad \omega_2^5 = \omega_2$$

be such independent forms. The remaining forms of (6) may be expressed as their combinations with coefficients generally depending both on principal and incidental parameters.

In what follows we will specialize the moving frame with respect to the plane congruence $\sum^{\infty}$ in an improper hyperplane N_5^{∞}. By the focus of the plane $\mathfrak{S}^{\infty} \in \sum^{\infty}$, $\mathfrak{S}^{\infty} = (e_1, e_2, e_3)$ belonging to the focal direction determined by $\omega_1 : \omega_2$ (given f.i. by the relation $q_1 \omega_1 + q_2 \omega_2 = 0$ we mean a point F^{∞} of this plane which in moving in the focal direction satisfies the relation $\left[e_1 e_2 e_3 dF^{\infty}\right] = 0$. As is well-known, there exist three foci in every plane congruence which are supposed not to lie in the same line.

We choose the vectors e_1, e_2, e_3 of the moving frame in A_6 so that the corresponding points $e_i \in N_5^{\infty}$, ($i = 1,2,3$)

coincide with the foci of the plane σ^∞ belonging to the focal directions

(7) $\omega_1 = 0$, $\omega_2 = 0$, $\omega_1 + r\omega_2 \equiv \omega_3 = 0$

under the assumption that $r \neq 0$. Then

(8) $[e_1\ e_2\ e_3\ de_i] = 0$ for $\omega_i = 0$, $i = 1,2,3.$

The condition $[e_1\ e_2\ e_3\ dF^\infty] = 0$ for the focus means with respect to (2) that

$$
(9)\qquad
\begin{aligned}
\alpha\omega_1 + \beta\omega_2^4 + \gamma\omega_3^4 &= 0 \\
\alpha\omega_1^5 + \beta\omega_2 + \gamma\omega_3^5 &= 0 \\
\alpha\omega_1^6 + \beta\omega_2^6 + \gamma\omega_3^6 &= 0 .
\end{aligned}
$$

If the forms ω_2^4, ω_3^4, ω_1^5, ω_3^5, ω_1^6, ω_2^6, ω_3^6 in (9) are expressed as linear combinations of the forms ω_1, ω_2 and on taking account of the conditions $F_i^\infty = e_i$ for the focal directions $\omega_i = 0$, $i = 1,2,3$, we find that equations (8) may be satisfied if

$$\omega_i \wedge \omega_i^{j+3} = 0 , \quad i,j = 1,2,3.$$

By means of the Cartan lemma we get

(10) $\omega_i^{j+3} = a_i^{j+3}\,\omega_i$; $i,j = 1,2,3$; $a_1^4 = a_5^2 = 1$.

In what follows we restrict ourselves to such a case where every of the three foci e_i circumscribes a regular focal surface. The tangent planes of the focal surfaces circumscribed by the foci e_i at the point of tangency e_i are determined by the directions

$$de_1 = \omega_1^1 e_1 + \omega_1^2 e_2 + \omega_1^3 e_3 + \omega_1 (e_4 + a_1^5 e_5 + a_1^6 e_6)$$

$$de_2 = \omega_2^1 e_1 + \omega_2^2 e_2 + \omega_2^3 e_3 + \omega_2 (a_2^4 e_4 + e_5 + a_2^6 e_6)$$

$$de_3 = \omega_3^1 e_1 + \omega_3^2 e_2 + \omega_3^3 e_3 + \omega_3 (a_3^4 e_4 + a_3^5 e_5 + a_3^6 e_6).$$

We choose a moving frame so that the tangent planes lie in the spaces (e_1, e_2, e_3, e_4), (e_1, e_2, e_3, e_5), $(e_1, e_2, e_3, a_3^6 e_6)$; $a_3^6 \neq 0$, which leads to

$$(11) \qquad a_i^{j+3} = 0 \,, \quad i,j = 1,2,3 \;; \quad i \neq j \,.$$

Relations (10) and (11) yield

$$\omega_i^{j+3} = 0 \,, \quad i,j = 1,2,3 \;; \quad i \neq j \,.$$

Exterior differentiation of equation (10) for $i = j = 3$ gives

$$d\omega_3^6 = d(a_3^6 \omega_3) = a_3^6 d\omega_3 + da_3^6 \wedge \omega_3 \,,$$

which results in

$$\omega_1 \wedge \left\{ da_3^6 + a_3^6 (\omega_1^1 + \omega_6^6 - \omega_3^3 - \omega_4^4) \right\} +$$
$$+ \omega_2 \wedge \left\{ d(a_3^6 r) + a_3^6 r (\omega_2^2 + \omega_6^6 - \omega_3^3 - \omega_5^5) \right\} = 0$$

using the Cartan lemma gives

$$(12)\quad \begin{aligned} da_3^6 + a_3^6(\omega_1^1 + \omega_6^6 - \omega_3^3 - \omega_4^4) &= a\omega_1 + b\omega_2 \\ d(a_3^6 r) + a_3^6 r(\omega_2^2 + \omega_6^6 - \omega_3^3 - \omega_4^4) &= c\omega_1 + d\omega_2 \end{aligned}$$

Differentiating by the incidental parameters gives

$$\delta a_3^6 + a_3^6(\pi_1^1 + \pi_6^6 - \pi_3^3 - \pi_4^4) = 0$$

$$\delta(a_3^6 r) + a_3^6 r(\pi_2^2 + \pi_6^6 - \pi_3^3 - \pi_5^5) = 0$$

and with respect to our previons assumption $a_3^6 \neq 0$, we may without any loss of generality choose $a_3^6 = 1$. Exterior differentiating of the remaining equations of (10) on taking account of (11) gives for $i \neq j$

$$(13)\quad \begin{aligned} \omega_1^2 \wedge \omega_2 + \omega_1 \wedge \omega_4^5 &= 0 \\ \omega_1^3 \wedge \omega_3 + \omega_1 \wedge \omega_4^6 &= 0 \\ \omega_1^2 \wedge \omega_1 + \omega_2 \wedge \omega_5^4 &= 0 \\ \omega_2^3 \wedge \omega_3 + \omega_2 \wedge \omega_5^6 &= 0 \\ \omega_3^1 \wedge \omega_1 + \omega_3 \wedge \omega_6^4 &= 0 \\ \omega_3^2 \wedge \omega_2 + \omega_3 \wedge \omega_6^5 &= 0 \end{aligned}$$

Which in applying the Cartan lemma is reduced to

$$
(14)\qquad
\begin{array}{ll}
\omega_1^2 = a_1^2\omega_1 + \alpha_1^2\omega_2 & \omega_2^3 = a_2^3\omega_2 + \alpha_2^3\omega_3 \\
\omega_4^5 = \beta_1^2\omega_1 - a_1^2\omega_2 & \omega_5^6 = \beta_2^3\omega_2 - a_2^3\omega_3 \\
\omega_1^3 = a_1^3\omega_1 + \alpha_1^3\omega_3 & \omega_3^1 = a_3^1\omega_3 + \alpha_3^1\omega_1 \\
\omega_4^6 = \beta_1^3\omega_1 - a_1^3\omega_3 & \omega_6^4 = \beta_3^1\omega_3 - a_3^1\omega_1 \\
\omega_2^1 = a_2^1\omega_2 + \alpha_2^1\omega_1 & \omega_3^2 = a_3^2\omega_3 + \alpha_3^2\omega_2 \\
\omega_5^4 = \beta_2^1\omega_2 - a_2^1\omega_1 & \omega_6^5 = \beta_3^2\omega_3 - a_3^2\omega_2
\end{array}
$$

It can be seen from (12) and (14) that $(\omega_1^1 + \omega_6^6 - \omega_3^3 - \omega_4^4)$, ω_1^2, ω_1^3, ω_2^1, ω_2^3, ω_3^1, ω_3^2, ω_4^5, ω_4^6, ω_5^4, ω_5^6, ω_6^4, ω_6^5, are the principal forms.

Taking account of the equalities $d\omega_i = (\omega_i^i - \omega_{i+3}^{i+3})\wedge\omega_i$, $i = 1,2,3$, resulting from the integrability conditions of (3) gives on exterior differention of (14) with some modification:

$$
(15)\qquad
\begin{aligned}
&\omega_1\wedge\left\{da_1^2 + a_1^2(\omega_2^2 - \omega_4^4) + \omega_4^2\right\} + \omega_2\wedge\left\{d\alpha_1^2 + \right.\\
&\quad +\alpha_1^2(2\omega_2^2 - \omega_1^1 - \omega_5^5) - (ra_1^3a_3^2 + a_1^3\alpha_3^2 + \\
&\quad \left. +\alpha_1^3\alpha_3^2)\omega_1\right\} = 0 \\
&\omega_1\wedge\left\{d\beta_1^2 + \beta_1^2(\omega_1^1 + \omega_5^5 - 2\omega_4^4) + (r\beta_1^3\beta_3^2 - a_3^2\beta_1^3 + \right.\\
&\quad \left. + a_1^3a_3^2)\,\omega_2\right\} - \omega_2\wedge\left\{da_1^2 + a_1^2(\omega_2^2 - \omega_4^4) + \right.\\
&\quad \left. + \omega_4^2\right\} = 0
\end{aligned}
$$

$$
\begin{aligned}
&\omega_1 \wedge \{ d a_1^3 + a_1^3(\omega_3^3 - \omega_4^4) + \omega_4^3 + a_1^2 a_2^3 \omega_2 \} + \omega_3 \wedge \{ d\alpha_1^3 + \\
&\qquad + \alpha_1^3(2\omega_3^3 - \omega_1^1 - \omega_6^6) - a_1^2 \alpha_2^3 \omega_1 - \alpha_1^2 \alpha_2^3 \omega_2 \} = 0 \\
&\omega_1 \wedge \{ d\beta_1^3 + \beta_1^3(\omega_1^1 + \omega_6^6 - 2\omega_4^4) + \beta_1^2 \beta_2^3 \omega_2 \} - \omega_3 \wedge \{ d a_1^3 + \\
&\qquad + a_1^3(\omega_3^3 - \omega_4^4) + \omega_4^3 - \beta_1^2 a_2^3 \omega_1 + a_1^2 a_2^3 \omega_2 \} = 0 \\
&\omega_1 \wedge \{ d\alpha_2^1 + \alpha_2^1(2\omega_1^1 - \omega_2^2 - \omega_4^4) - (a_2^3 a_3^1 + a_2^3 \alpha_3^1 + r\alpha_2^3 \alpha_3^1)\omega_2 \} + \\
&\qquad + \omega_2 \wedge \{ d a_2^1 + a_2^1(\omega_1^1 - \omega_5^5) + \omega_5^1 \} = 0 \\
&\omega_1 \wedge \{ d a_2^1 + a_2^1(\omega_1^1 - \omega_5^5) + \omega_5^1 \} - \omega_2 \wedge \{ d\beta_2^1 + \\
&\qquad + \beta_2^1(\omega_2^2 + \omega_4^4 - 2\omega_5^5) + (\beta_2^3 \beta_3^1 - \beta_2^3 a_3^1 + r a_2^3 a_3^1)\omega_1 \} = 0 \\
&\omega_2 \wedge \{ d a_2^3 + a_2^3(\omega_3^3 - \omega_5^5) + \omega_5^3 + a_2^1 a_1^3 \omega_1 \} + \omega_3 \wedge \{ d\alpha_2^3 + \\
&\qquad + \alpha_2^3(2\omega_3^3 - \omega_2^2 - \omega_6^6) - a_2^1 \alpha_1^3 \omega_2 - \alpha_2^1 \alpha_1^3 \omega_1 \} = 0 \\
&\omega_2 \wedge \{ d\beta_2^3 + \beta_2^3(\omega_2^2 + \omega_6^6 - 2\omega_5^5) + \beta_2^1 \beta_1^3 \omega_1 \} - \omega_3 \wedge \{ d a_2^3 + \\
&\qquad + a_2^3(\omega_3^3 - \omega_5^5) + \omega_5^3 - a_1^3 \beta_2^1 \omega_2 + a_2^1 a_1^3 \omega_1 \} = 0 \\
&\omega_1 \wedge \{ d\alpha_3^1 + \alpha_3^1(2\omega_1^1 - \omega_3^3 - \omega_4^4) - \alpha_3^2 \alpha_2^1 \omega_2 \} + \omega_3 \wedge \{ d a_3^1 + \\
&\qquad + a_3^1(\omega_1^1 - \omega_6^6) + \omega_6^1 + a_3^2 a_2^1 \omega_2 + a_3^2 \alpha_2^1 \omega_1 \} = 0 \\
&\omega_1 \wedge \{ d a_3^1 + a_3^1(\omega_1^1 - \omega_6^6) + \omega_6^1 + a_3^2 a_2^1 \omega_2 \} - \omega_3 \wedge \{ d\beta_3^1 + \\
&\qquad + \beta_3^1(\omega_3^3 + \omega_4^4 - 2\omega_6^6) + \beta_3^2 \beta_2^1 \omega_2 - \beta_3^2 a_2^1 \omega_1 \} = 0 \\
&\omega_2 \wedge \{ d\alpha_3^2 + \alpha_3^2(2\omega_2^2 - \omega_3^3 - \omega_5^5) - \alpha_3^1 \alpha_1^2 \omega_1 \} - \omega_3 \wedge \{ d a_3^2 + \\
&\qquad + a_3^2(\omega_2^2 - \omega_6^6) + \omega_6^2 - a_3^1 a_1^2 \omega_1 - a_3^1 \alpha_1^2 \omega_2 \} = 0 \\
&\omega_2 \wedge \{ d a_3^2 + a_3^2(\omega_2^2 - \omega_6^6) + \omega_6^2 + a_3^1 a_1^2 \omega_1 \} - \omega_3 \wedge \{ d\beta_3^2 + \\
&\qquad + \beta_3^2(\omega_3^3 + \omega_5^5 - 2\omega_6^6) + \beta_3^1 \beta_1^2 \omega_1 - a_1^2 \beta_3^1 \omega_2 \} = 0
\end{aligned}
\tag{15}
$$

Using the Cartan lemma to (15), we find that

$$
\begin{array}{lll}
da_1^2+a_1^2(\omega_2^2-\omega_4^4)+\omega_4^2 & d\alpha_1^2+\alpha_1^2(2\omega_2^2-\omega_1^1-\omega_5^5) & d\beta_1^2+\beta_1^2(\omega_1^1+\omega_5^5-2\omega_4^4) \\
da_1^3+a_1^3(\omega_3^3-\omega_4^4)+\omega_4^3 & d\alpha_1^3+\alpha_1^3(2\omega_3^3-\omega_1^1-\omega_6^6) & d\beta_1^3+\beta_1^3(\omega_1^1+\omega_6^6-2\omega_4^4) \\
da_2^1+a_2^1(\omega_1^1-\omega_5^5)+\omega_5^1 & d\alpha_2^1+\alpha_2^1(2\omega_1^1-\omega_2^2-\omega_4^4) & d\beta_2^1+\beta_2^1(\omega_2^2+\omega_4^4-2\omega_5^5) \\
da_2^3+a_2^3(\omega_3^3-\omega_5^5)+\omega_5^3 & d\alpha_2^3+\alpha_2^3(2\omega_3^3-\omega_2^2-\omega_6^6) & d\beta_2^3+\beta_2^3(\omega_2^2+\omega_6^6-2\omega_5^5) \\
da_3^1+a_3^1(\omega_1^1-\omega_6^6)+\omega_6^1 & d\alpha_3^1+\alpha_3^1(2\omega_1^1-\omega_3^3-\omega_4^4) & d\beta_3^1+\beta_3^1(\omega_3^3+\omega_4^4-2\omega_6^6) \\
da_3^2+a_3^2(\omega_2^2-\omega_6^6)+\omega_6^2 & d\alpha_3^2+\alpha_3^2(2\omega_2^2-\omega_3^3-\omega_5^5) & d\beta_3^2+\beta_3^2(\omega_3^3+\omega_5^5-2\omega_6^6)
\end{array}
$$

are the principal forms. The variations of coefficients of the forms in (14) are

$$
\begin{array}{lll}
\delta a_1^2=a_1^2(\pi_4^4-\pi_2^2)-\pi_4^2 & \delta\alpha_1^2=\alpha_1^2(\pi_1^1+\pi_5^5-2\pi_2^2) & \delta\beta_1^2=\beta_1^2(2\pi_4^4-\pi_1^1-\pi_5^5) \\
\delta a_1^3=a_1^3(\pi_4^4-\pi_3^3)-\pi_4^3 & \delta\alpha_1^3=\alpha_1^3(\pi_1^1+\pi_6^6-2\pi_3^3) & \delta\beta_1^3=\beta_1^3(2\pi_4^4-\pi_1^1-\pi_6^6) \\
\delta a_2^1=a_2^1(\pi_5^5-\pi_1^1)-\pi_5^1 & \delta\alpha_2^1=\alpha_2^1(\pi_2^2+\pi_4^4-2\pi_1^1) & \delta\beta_2^1=\beta_2^1(2\pi_5^5-\pi_2^2-\pi_4^4) \\
\delta a_2^3=a_2^3(\pi_5^5-\pi_3^3)-\pi_5^3 & \delta\alpha_2^3=\alpha_2^3(\pi_2^2+\pi_6^6-2\pi_3^3) & \delta\beta_2^3=\beta_2^3(2\pi_5^5-\pi_2^2-\pi_6^6) \\
\delta a_3^1=a_3^1(\pi_6^6-\pi_1^1)-\pi_6^1 & \delta\alpha_3^1=\alpha_3^1(\pi_3^3+\pi_4^4-2\pi_1^1) & \delta\beta_3^1=\beta_3^1(2\pi_6^6-\pi_3^3-\pi_4^4) \\
\delta a_3^2=a_3^2(\pi_6^6-\pi_2^2)-\pi_6^2 & \delta\alpha_3^2=\alpha_3^2(\pi_3^3+\pi_5^5-2\pi_2^2) & \delta\beta_3^2=\beta_3^2(2\pi_6^6-\pi_3^3-\pi_5^5)
\end{array}
\tag{16}
$$

We can see from the first column of (16) that the moving frame may be further specialized in choosing

$$a_1^2 = a_1^3 = a_2^1 = a_2^3 = a_3^1 = a_3^2 = 0\ , \tag{17}$$

whereby ω_4^2, ω_4^3, ω_5^1, ω_5^3, ω_6^1, ω_6^2 become the principal forms.

As meantioned above, ω^4, ω^5, ω^6 are also principal forms. Hence, they may be written as

$$
\begin{aligned}
&\omega^4 = \alpha^4\omega_1 + \beta^4\omega_2 \\
(18) \qquad &\omega^5 = \alpha^5\omega_1 + \beta^5\omega_2 \\
&\omega^6 = \alpha^6\omega_1 + \beta^6\omega_2
\end{aligned}
$$

Exterior differentiation of (18) yields

$$
\begin{aligned}
&\omega_1\wedge\{d\alpha^4+\alpha^4\omega_1^1-\omega^1\}+\omega_2\wedge\{d\beta^4+\beta^4(\omega_2^2+\omega_4^4-\omega_5^5)+ \\
&\qquad +(\beta^6\beta_3^1-\alpha^5\alpha_2^1-\alpha^6\beta_3^1 r)\omega_1\} = 0 \\
(19) \qquad &\omega_1\wedge\{d\alpha^5+\alpha^5(\omega_1^1+\omega_5^5-\omega_4^4)+(\alpha^6\beta_2^3 r-\beta^4\beta_1^2-\beta^6\beta_3^2)\omega_2\}+ \\
&\qquad +\omega_2\wedge\{d\beta^5+\beta^5\omega_2^2-\omega^2\} = 0 \\
&\omega_1\wedge\{d\alpha^6+\alpha^6(\omega_1^1+\omega_6^6-\omega_4^4)-\omega^3\}+\omega_2\wedge\{d\beta^6+\beta^6(\omega_2^2+\omega_6^6-\omega_5^5)+ \\
&\qquad +(\beta^4\beta_1^3-\alpha^5\beta_2^3)\omega_2-r\omega^3\} = 0
\end{aligned}
$$

Using the Cartan lemma to (19) gives further principal forms

$$
\begin{aligned}
&d\alpha^4+\alpha^4\omega_1^1-\omega^1 && d\beta^4+\beta^4(\omega_2^2+\omega_4^4-\omega_5^5) \\
&d\alpha^5+\alpha^5(\omega_1^1+\omega_5^5-\omega_4^4) && d\beta^5+\beta^5\omega_2^2-\omega^2 \\
&d\alpha^6+\alpha^6(\omega_1^1+\omega_6^6-\omega_4^4)-\omega^3 && d\beta^6+\beta^6(\omega_2^2+\omega_6^6-\omega_5^5)-r\omega^3
\end{aligned}
$$

Differentiating by the incidental parameters gives

$$
\begin{aligned}
&\delta\alpha^4 = -\alpha^4\pi_1^1+\pi^1 && \delta\beta^4 = \beta^4(\pi_5^5-\pi_2^2-\pi_4^4) \\
&\delta\alpha^5 = \alpha^5(\pi_4^4-\pi_1^1-\pi_5^5) && \delta\beta^5 = -\beta^5\pi_2^2+\pi^2 \\
&\delta\alpha^6 = \alpha^6(\pi_4^4-\pi_1^1-\pi_6^6)+\pi^3 && \delta\beta^6 = \beta^6(\pi_5^5-\pi_2^2-\pi_6^6)+r\pi^3
\end{aligned}
$$

whence we see that we may choose $\alpha^4 = \beta^5 = \alpha^6 = 0$, i.e. ω^i, $i = 1,2,3$ are principal forms. Making a formal change in notation and setting $\beta^4 = a^4$, $\alpha^5 = a^5$, $\beta^6 = a^6$ enables us to write with respect to (18) that

$$\omega^4 = a^4\omega_2, \quad \omega^5 = a^5\omega_1, \quad \omega^6 = a^6\omega_2$$

Since ω^i, $i = 1,2,3$ are principal forms, they may be expressed as combinations of ω_1, ω_2, i.e.

$$\begin{aligned} \omega^1 &= \alpha^1\omega_1 + \beta^1\omega_2 \\ \omega^2 &= \alpha^2\omega_1 + \beta^2\omega_2 \\ \omega^3 &= \alpha^3\omega_1 + \beta^3\omega_2 \end{aligned} \tag{20}$$

further extension of the above relations gives

$$\omega_1 \wedge \{d\alpha^1 + \alpha^1(2\omega_1^1 - \omega_4^4) - (\alpha_2^1\beta^2 + \alpha_3^1\beta^3)\omega_2 + a^5\omega_5^1\} +$$

$$+\omega_2 \wedge \{d\beta^1 + \beta^1(\omega_1^1 + \omega_2^2 - \omega_5^5) + a^4\omega_4^1 + a^6\omega_6^1\} = 0$$

$$\omega_1 \wedge \{d\alpha^2 + \alpha^2(\omega_1^1 + \omega_2^2 - \omega_4^4) + a^5\omega_5^2 + (\alpha^1\alpha_1^2 + \alpha^3\alpha_3^2)\omega_2\} +$$

$$+\omega_2 \wedge \{d\beta^2 + \beta^2(2\omega_2^2 - \omega_5^5) + a^4\omega_4^2 + a^6\omega_6^2\} = 0$$

$$\omega_1 \wedge \{d\alpha^3 + \alpha^3(\omega_1^1 + \omega_3^3 - \omega_4^4) + a^5\omega_5^3 + (\alpha^1\alpha_1^3 r + \alpha^2\alpha_2^3 r - \beta^1\alpha_1^3 - \beta^2\alpha_2^3)\omega_2\} +$$

$$+\omega_2 \wedge \{d\beta^3 + \beta^3(\omega_2^2 + \omega_3^3 - \omega_5^5) + a^6\omega_6^3 + a^4\omega_4^3\} = 0$$

next principal forms are

$$d\alpha^1 + \alpha^1(2\omega_1^1 - \omega_4^4) \qquad d\beta^1 + \beta^1(\omega_1^1 + \omega_2^2 - \omega_5^5) + a^4\omega_4^1$$

$$d\alpha^2 + \alpha^2(\omega_1^1 + \omega_2^2 - \omega_4^4) + a^5\omega_5^2 \qquad d\beta^2 + \beta^2(2\omega_2^2 - \omega_5^5)$$

$$d\alpha^3 + \alpha^3(\omega_1^1 + \omega_3^3 - \omega_4^4) \qquad d\beta^3 + \beta^3(\omega_2^2 + \omega_3^3 - \omega_5^5) + a^6\omega_6^3$$

and the variation of coefficients

$$\delta\alpha^1 = \alpha^1(\pi_4^4 - 2\pi_1^1) \qquad \delta\beta^1 = \beta^1(\pi_5^5 - \pi_1^1 - \pi_2^2) - a^4\pi_4^1$$

$$\delta\alpha^2 = \alpha^2(\pi_4^4 - \pi_1^1 - \pi_2^2) - a^5\pi_5^2 \qquad \delta\beta^2 = \beta^2(\pi_5^5 - 2\pi_2^2)$$

$$\delta\alpha^3 = \alpha^3(\pi_4^4 - \pi_1^1 - \pi_3^3) \qquad \delta\beta^3 = \beta^3(\pi_5^5 - \pi_2^2 - \pi_3^3) - a^6\pi_6^3$$

where we see that we may choose $\beta^1 = \alpha^2 = \beta^3 = 0$, where upon ω_1^4, ω_5^2, ω_6^3 become the principal forms. To simplify our notation we set analogous to the preceding case $\beta^1 = a^1$, $\alpha^2 = a^2$, $\beta^3 = a^3$ and obtain with respect to (20) that

$$\omega^1 = a^1\omega_1, \quad \omega^2 = a^2\omega_2, \quad \omega^3 = a^3\omega_1 .$$

It is a necessary condition for the existence of the two-parametric congruences Σ in the three-dimensional subspaces of the space A_6 that the system of the Pfaff equations

$$
(21)\qquad
\begin{array}{lll}
\omega_1^5 = 0 & \omega_1^2 = \alpha_1^2\omega_2 & \omega_4^5 = \beta_1^2\omega_1 \\
\omega_1^6 = 0 & \omega_1^3 = \alpha_1^3\omega_3 & \omega_4^6 = \beta_1^3\omega_1 \\
\omega_2^4 = 0 & \omega_2^1 = \alpha_2^1\omega_1 & \omega_5^4 = \beta_2^1\omega_2 \\
\omega_2^6 = 0 & \omega_2^3 = \alpha_2^3\omega_3 & \omega_5^6 = \beta_2^3\omega_2 \\
\omega_3^4 = 0 & \omega_3^1 = \alpha_3^1\omega_1 & \omega_6^4 = \beta_3^1\omega_3 \\
\omega_3^5 = 0 & \omega_3^2 = \alpha_3^2\omega_2 & \omega_6^5 = \beta_3^2\omega_3 \\
\omega_3^6 = \omega_1 + r\omega_2 & & \\
 & \omega^4 = a^4\omega_2 & \\
 & \omega^5 = a^5\omega_1 & \\
 & \omega^6 = a^6\omega_2 &
\end{array}
$$

is in involution. The integrability conditions with respect to (13), (15), (19) on taking account of (7) and (17) are

$$\omega_1 \wedge \omega_4^5 - \omega_2 \wedge \omega_1^2 = 0$$

$$\omega_1 \wedge \left(\omega_4^6 - \omega_1^3\right) - \omega_2 \wedge r\omega_1^3 = 0$$

$$\omega_1 \wedge \omega_2^1 - \omega_2 \wedge \omega_5^4 = 0$$

$$\omega_1 \wedge \omega_2^3 + \omega_2 \wedge \left(r\omega_2^3 - \omega_5^6\right) = 0$$

$$\omega_1 \wedge \left(\omega_6^4 - \omega_3^1\right) + \omega_2 \wedge r\omega_6^4 = 0$$

$$\omega_1 \wedge \omega_6^5 + \omega_2 \wedge \left(r\omega_6^5 - \omega_3^2\right) = 0 \tag{22}$$

$$\omega_1 \wedge \{\omega_1^1 + \omega_6^6 - \omega_3^3 - \omega_4^4\} + \omega_2 \wedge \{dr + r(\omega_2^2 + \omega_6^6 - \omega_3^3 - \omega_5^5)\} = 0$$

$$\omega_1 \wedge \omega_4^2 + \omega_2 \wedge \{d\alpha_1^2 + \alpha_1^2(2\omega_2^2 - \omega_1^1 - \omega_5^5) - \alpha_1^3\alpha_3^2\omega_1\} = 0$$

$$\omega_1 \wedge \{\omega_4^3 + d\alpha_1^3 + \alpha_1^3(2\omega_3^3 - \omega_1^1 - \omega_6^6) - \alpha_1^2\alpha_2^3\omega_2\} +$$
$$+ \omega_2 \wedge r\{d\alpha_1^3 + \alpha_1^3(2\omega_3^3 - \omega_1^1 - \omega_6^6)\} = 0$$

$$\omega_1 \wedge \{d\alpha_2^1 + \alpha_2^1(2\omega_1^1 - \omega_2^2 - \omega_4^4) - r\alpha_2^3\alpha_3^1\omega_2\} + \omega_2 \wedge \omega_5^1 = 0$$

$$\omega_1 \wedge \{d\alpha_2^3 + \alpha_2^3(2\omega_3^3 - \omega_2^2 - \omega_6^6)\} + \omega_2 \wedge \{\omega_5^3 + r[d\alpha_2^3 +$$
$$+ \alpha_2^3(2\omega_3^3 - \omega_2^2 - \omega_6^6) - \alpha_2^1\alpha_1^3\omega_1]\} = 0$$

$$\omega_1 \wedge \{\omega_6^1 + d\alpha_3^1 + \alpha_3^1(2\omega_1^1 - \omega_3^3 - \omega_4^4) - \alpha_3^2\alpha_2^1\omega_2\} + \omega_2 \wedge r\omega_6^1 = 0$$

$$\omega_1 \wedge \omega_6^2 + \omega_2 \wedge \{ r\omega_6^2 + d\alpha_3^2 + \alpha_3^2(2\omega_2^2 - \omega_3^3 - \omega_5^5) - \alpha_3^1\alpha_1^2\omega_1 \} = 0$$

$$\omega_1 \wedge \{ d\beta_1^2 + \beta_1^2(\omega_1^1 + \omega_5^5 - 2\omega_4^4) + \beta_1^3\beta_3^2 r\omega_2 \} - \omega_2 \wedge \omega_4^2 = 0$$

$$\omega_1 \wedge \{ -\omega_4^3 + d\beta_1^3 + \beta_1^3(\omega_1^1 + \omega_6^6 - 2\omega_4^4) + \beta_1^2\beta_2^3\omega_2 \} - \omega_2 \wedge r\omega_4^3 = 0$$

$$\omega_1 \wedge \omega_5^1 - \omega_2 \wedge \{ d\beta_2^1 + \beta_2^1(\omega_2^2 + \omega_4^4 - 2\omega_5^5) + \beta_2^3\beta_3^1\omega_1 \} = 0$$

$$\omega_1 \wedge \omega_5^3 - \omega_2 \wedge \{ -r\omega_5^3 + d\beta_2^3 + \beta_2^3(\omega_2^2 + \omega_6^6 - 2\omega_5^5) + \beta_2^1\beta_1^3\omega_1 \} = 0$$

$$\omega_1 \wedge \{ \omega_6^1 - d\beta_3^1 - \beta_3^1(\omega_3^3 + \omega_4^4 - 2\omega_6^6) - \beta_3^2\beta_2^1\omega_2 \} -$$

(22)

$$- \omega_2 \wedge r \{ d\beta_3^1 + \beta_3^1(\omega_3^3 + \omega_4^4 - 2\omega_6^6) \} = 0$$

$$\omega_1 \wedge \{ d\beta_3^2 + \beta_3^2(\omega_3^3 + \omega_5^5 - 2\omega_6^6) \} - \omega_2 \wedge \{ \omega_6^2 - r [d\beta_3^2 +$$

$$+ \beta_3^2(\omega_3^3 + \omega_5^5 - 2\omega_6^6) - \beta_3^1\beta_1^2\omega_1 \} = 0$$

$$\omega_1 \wedge \omega^1 - \omega_2 \wedge \{ da^4 + a^4(\omega_2^2 + \omega_4^4 - \omega_5^5) + (a^6\beta_3^1 - a^5\beta_2^1)\omega_1 \} = 0$$

$$\omega_1 \wedge \{ da^5 + a^5(\omega_1^1 + \omega_5^5 - \omega_4^4) - (a^4\beta_1^2 + a^6\beta_3^2)\omega_2 \} - \omega_2 \wedge \omega^2 = 0$$

$$\omega_1 \wedge \omega^3 - \omega_2 \wedge \{ da^6 + a^6(\omega_2^2 + \omega_6^6 - \omega_5^5) - r\omega^3 - (a^5\beta_2^3 - a^4\beta_1^3)\omega_1 \} = 0$$

The above system (22) contains 16 independent equations (the first six equations are dependent on equations 8 through 19 in the second and third column of (21)). The forms in the first six equations of (22) are principal and thus dependent on ω_1, ω_2. From the remaining 16 independent equations of (22) it is necessary to exclude the forms

$$\begin{aligned}
&\omega_4^2 \\
&\omega_4^3 + d\alpha_1^3 + \alpha_1^3(2\omega_3^3 - \omega_1^1 - \omega_6^6) - \alpha_1^2\alpha_2^2\omega_2 \\
&\omega_5^1 \\
&\omega_5^3 + r[d\alpha_2^3(2\omega_3^3 - \omega_2^2 - \omega_6^6) - \alpha_2^1\alpha_1^3\omega_1] \\
&\omega_6^1 - d\beta_3^1 - \beta_3^1(\omega_3^3 + \omega_4^4 - 2\omega_6^6) - \beta_3^2\beta_2^1\omega_2 \\
&\omega_6^2 - r[d\beta_3^2 + \beta_3^2(\omega_3^3 + \omega_5^5 - 2\omega_6^6) + \beta_3^1\beta_1^2\omega_1]
\end{aligned}$$

which are dependent on the other forms of (22). System (22) contains 44 forms in all from which we exclude 18, i.e. there remain 26 linearly independent forms. We will express these independent forms by means of the Cartan lemma to the relations of (22):

$$\begin{aligned}
\omega_1^1 + \omega_6^6 - \omega_3^3 - \omega_4^4 &= A_1\omega_1 + A_2\omega_2 \\
dr + r(\omega_2^2 + \omega_6^6 - \omega_3^3 - \omega_5^5) &= A_3\omega_1 + A_4\omega_2 \\
\omega_4^2 &= A_5\omega_1 + A_6\omega_2 \\
d\alpha_1^2 + \alpha_1^2(2\omega_2^2 - \omega_1^1 - \omega_5^5) - \alpha_1^3\alpha_3^2\omega_1 &= A_7\omega_1 + A_8\omega_2 \\
r\{d\alpha_1^3 + \alpha_1^3(2\omega_3^3 - \omega_1^1 - \omega_6^6)\} &= A_9\omega_1 + A_{10}\omega_2 \\
d\alpha_2^1 + \alpha_2^1(2\omega_1^1 - \omega_2^2 - \omega_4^4) - r\alpha_2^3\alpha_3^1\omega_2 &= A_{11}\omega_1 + A_{12}\omega_2 \\
\omega_5^1 &= A_{13}\omega_1 + A_{14}\omega_2
\end{aligned} \tag{23}$$

$$
\begin{aligned}
d\alpha_2^3+\alpha_2^3(2\omega_3^3-\omega_2^2-\omega_6^6) &= A_{15}\omega_1+A_{16}\omega_2\\
\omega_6^1+d\alpha_3^1+\alpha_3^1(2\omega_1^1-\omega_3^3-\omega_4^4)-\alpha_3^2\alpha_2^1\omega_2 &= A_{17}\omega_1+A_{18}\omega_2\\
r\omega_6^1 &= A_{19}\omega_1+A_{20}\omega_2\\
\omega_6^2 &= A_{21}\omega_1+A_{22}\omega_2\\
r\omega_6^2+d\alpha_3^2+\alpha_3^2(2\omega_2^2-\omega_3^3-\omega_5^5)-\alpha_3^1\alpha_1^2\omega_1 &= A_{23}\omega_1+A_{24}\omega_2\\
d\beta_1^2+\beta_1^2(\omega_1^1+\omega_5^5-2\omega_4^4)+\beta_1^3\beta_3^2 r\omega_2 &= A_{25}\omega_1+A_{26}\omega_2\\
-\omega_4^3+d\beta_1^3+\beta_1^3(\omega_1^1+\omega_6^6-2\omega_4^4)+\beta_1^2\beta_2^3\omega_2 &= A_{27}\omega_1+A_{28}\omega_2\\
r\omega_4^3 &= A_{29}\omega_1+A_{30}\omega_2\\
d\beta_2^1+\beta_2^1(\omega_2^2+\omega_4^4-2\omega_5^5)+\beta_2^3\beta_3^1\omega_1 &= A_{31}\omega_1+A_{32}\omega_2\\
\omega_5^3 &= A_{33}\omega_1+A_{34}\omega_2\\
-r\omega_5^3+d\beta_2^3+\beta_2^3(\omega_2^2+\omega_6^6-2\omega_5^5)+\beta_2^1\beta_1^3\omega_1 &= A_{35}\omega_1+A_{36}\omega_2\\
r\{d\beta_3^1+\beta_3^1(\omega_3^3+\omega_4^4-2\omega_6^6)\} &= A_{37}\omega_1+A_{38}\omega_2\\
d\beta_3^2+\beta_3^2(\omega_3^3+\omega_5^5-2\omega_6^6) &= A_{39}\omega_1+A_{40}\omega_2\\
\omega^1 &= A_{41}\omega_1+A_{42}\omega_2\\
da^4+a^4(\omega_2^2+\omega_4^4-\omega_5^5)+(a^6\beta_3^1-a^5\beta_2^1)\,\omega_1 &= A_{43}\omega_1+A_{44}\omega_2\\
da^5+a^5(\omega_1^1+\omega_5^5-\omega_4^4)-(a^4\beta_1^2+a^6\beta_3^2)\,\omega_2 &= A_{45}\omega_1+A_{46}\omega_2\\
\omega^2 &= A_{47}\omega_1+A_{48}\omega_2\\
\omega^3 &= A_{49}\omega_1+A_{50}\omega_2\\
da^6+a^6(\omega_2^2+\omega_6^6-\omega_5^5)-r\omega^3-(a^5\beta_2^3-a^4\beta_1^3)\,\omega_1 &= A_{51}\omega_1+A_{52}\omega_2
\end{aligned}
\tag{23}
$$

From the Cartan lemma then follow for the coefficients the relations:

$A_2 = A_3$, $A_6 = A_7$, $A_{12} = A_{13}$, $A_{18} = A_{19}$, $A_{22} = A_{23}$, $A_{26} = -A_5$,

$A_{28} = -A_{29}$, $A_{31} = -A_{14}$, $A_{34} = -A_{35}$, $A_{42} = -A_{43}$, $A_{46} = -A_{47}$,

$A_{50} = -A_{51}$, $A_9 = \frac{1}{r}(A_{10} + A_{30}) - \alpha^2_1 \alpha^3_2$, $A_{16} = rA_{15} + A_{33} - \alpha^1_2 \alpha^3_1$;

$A_{37} = \frac{1}{r}(A_{38} - A_{20}) + \beta^2_3 \beta^1_2$, $A_{40} = rA_{39} - A_{21} - \beta^1_3 \beta^2_1$.

There are expressed linearly independent forms of (23) by means of 52 coefficients from which 16 may be expressed by the remaining 36 coefficients. If we let q denote the number of linearly independent forms, s_1 the number of linearly independent equations, N the number of independent coefficients and if $s_2 = q - s_1$ and $Q = s_1 + 2s_2$ is the Cartan number, then the condition for the existence is $Q = N$. In our case $q = 26$, $s_1 = 16$, $s_2 = 10$, $Q = 36$, $N = 36$, i.e. $Q = N$ and therefore the objects under consideration exist. The general solution of the system depends on 10 functions of 2 arguments.

The condition for the existence was proved in a general case in assuming $a^4 \neq 0$, $a^5 \neq 0$, $a^6 \neq 0$. Let us however observe the question of existence in special cases

$$
\begin{array}{ll}
 & \text{a)} \quad a^4 = 0,\ a^5 \neq 0,\ a^6 \neq 0 \\
 & \text{b)} \quad a^4 \neq 0,\ a^5 = 0,\ a^6 \neq 0 \\
 & \text{c)} \quad a^4 \neq 0,\ a^5 \neq 0,\ a^6 = 0 \\
(24) & \text{d)} \quad a^4 = 0,\ a^5 = 0,\ a^6 \neq 0 \\
 & \text{e)} \quad a^4 = 0,\ a^5 \neq 0,\ a^6 = 0 \\
 & \text{f)} \quad a^4 \neq 0,\ a^5 = 0,\ a^6 = 0 \\
 & \text{g)} \quad a^4 = 0,\ a^5 = 0,\ a^6 = 0
\end{array}
$$

Because of the symmetry there is on need to investigate the existence in all cases a) - g). It suffices to focus our attention to one case a) - c), to one case d) - f) and to one case g).

For example, if we consider case (24) a), then the equation $\omega^4 = a^4\omega_2$ in system (21) changes to the form $\omega^4 = 0$, whereby the remaining equations of (21) are unaltered. Consequently the last three equations of (22) turn to the form

$$
\begin{aligned}
&\omega_1 \wedge \omega^1 - \omega_2 \wedge (a^6\beta_3^1 - a^5\beta_2^1)\omega_1 = 0 \\
(25) \qquad &\omega_1 \{da^5 + a^5(\omega_1^1 + \omega_5^5 - \omega_4^4) - a^6\beta_3^2\omega_2\} - \omega_2 \wedge \omega^2 = 0 \\
&\omega_1 \wedge \omega^3 - \omega_2 \wedge \{da^6 + a^6(\omega_2^2 + \omega_6^6 - \omega_5^5) - r\omega^3 - a^5\beta_2^3\omega_1\} = 0
\end{aligned}
$$

Using the Cartan lemma we get from (25)

$$
\begin{aligned}
\omega^1 &= A_{41}\omega_1 + A_{42}\omega_2 \\
da^5 + a^5(\omega_1^1 + \omega_5^5 - \omega_4^4) - a^6\beta_3^2\omega_2 &= A_{43}\omega_1 + A_{44}\omega_2 \\
(26) \qquad \omega^2 &= A_{45}\omega_1 + A_{46}\omega_2 \\
\omega^3 &= A_{47}\omega_1 + A_{48}\omega_2 \\
da^6 + a^6(\omega_2^2 + \omega_6^6 - \omega_5^5) - r\omega^3 - a^5\beta_2^3\omega_1 &= A_{49}\omega_1 + A_{50}\omega_2
\end{aligned}
$$

by which we replace the last six relations in (23). Then $A_{42} = a^5\beta^1_2 - a^6\beta^1_3$, $A_{44} = -A_{45}$, $A_{48} = -A_{49}$ hold for the coefficients on the right-hand sides of (26). Hence we have $q = 25$, $s_1 = 16$, $s_2 = 9$, $Q = 34$, $N = 34$, i.e. $Q = N$. The same values will be obtained in (24) b) and (24) c) on carrying out analogous changes in systems (21), (22) and in relations (23). Thus we proved the existence in cases (24) a), (24) b) and (24) c), whereby the solution of the system depends on 9 functions of 2 arguments.

In relations (24) d) through (24) f) we will notice only the case d). In system (21) two equations $\omega^4 = a^4\omega_2$, $\omega^5 = a^5\omega_1$ will change to $\omega^4 = 0$, $\omega^5 = 0$, while the others will remain unchanged. The last three equations of (22) will change to

$$\omega_1 \wedge \omega^1 - \omega_2 \wedge a^6\beta_3^1\omega_1 = 0$$

$$\omega_1 \wedge a^6\beta_3^2\omega_2 + \omega_2 \wedge \omega^2 = 0$$

$$\omega_1 \wedge \omega^3 - \omega_2 \wedge \{da^6 + a^6(\omega_2^2 + \omega_6^6 - \omega_5^5) - r\omega^3\} = 0$$

other equations of system (22) remain unchanged. Applying the Cartan lemma we obtain

$$\omega^1 = A_{41}\omega_1 + A_{42}\omega_2$$

$$\omega^2 = A_{43}\omega_1 + A_{44}\omega_2$$

$$\omega^3 = A_{45}\omega_1 + A_{46}\omega_2$$

$$da^6 + a^6(\omega_2^2 + \omega_6^6 - \omega_5^5) - r\omega^3 = A_{47}\omega_1 + A_{48}\omega_2$$

where for the coefficients the relations $A_{42} = -a^6\beta_3^1$, $A_{43} = a^6\beta_3^2$, $A_{46} = -A_{47}$ hold. Cosequently $q = 24$, $s_1 = 16$, $s_2 = 8$, $Q = 32$, $N = 32$, i.e. $Q = N$. The same result will be obtained in case (24) e) and (24) f). We see that system (21) is in involution also in cases (24) d), (24) e), (24) f) and the solution depends on 8 functions of 2 variables.

Finally, if in case (24) g) $a^4 = a^5 = a^6 = 0$, then the last three equations of system (21) reduce to $\omega^j = 0$, $j = 4,5,6$ and the last three equations of system (22) are

$$\omega_1 \wedge \omega^1 = 0$$

$$\omega_2 \wedge \omega^2 = 0$$

$$\omega_1 \wedge \omega^3 + \omega_2 \wedge r\omega^3 = 0$$

Applying the Cartan lemma, we obtain

$$\omega^1 = A_{41}\omega_1$$

$$\omega^2 = A_{42}\omega_2$$

$$\omega^3 = A_{43}\omega_1 + A_{44}\omega_2$$

whereby $A_{44} = rA_{43}$. Consequently $q = 23$, $s_1 = 16$, $s_2 = 7$, $Q = 30$, $N = 30$, hence $Q = N$. In this case the solution depends on 7 functions of 2 arguments.

This proves the existence both in a general and in all special cases.

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SOUHRN

DVOJPARAMETRICKÉ SOUSTAVY TROJROZMĚRNÝCH PROSTORŮ V UNIMODULÁRNÍM ŠESTIROZMĚRNÉM AFINNÍM PROSTORU

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Práce se zabývá studiem dvojparametrické soustavy Σ trojrozměrných podprostorů šestirozměrného unimodulárního afinního prostoru A_6. Současně se soustavou Σ se uvažuje dvojparametrická soustava nevlastních rovin tvořících prostorů soustavy Σ vnořená do nevlastní nadroviny prostoru A_6. Užitím Cartanovy metody je ke každému tvořícímu prostoru soustavy Σ přiřazen vhodně zvolený systém pohyblivých reperů, odvozena soustava diferenciálních rovnic uvažované soustavy Σ a nalezeny její podmínky integrability. Hlavní výsledky práce spočívají v odvození vztahů jednajících o existenci a obecnosti uvažované soustavy Σ, a to jak v obecném případě, tak i případech speciálních.

РЕЗЮМЕ

ДВУХПАРАМЕТРИЧЕСКИЕ СИСТЕМЫ ТРЕХМЕРНЫХ ПРОСТРАНСТВ В УНИМОДУЛЯРНОМ ШЕСТИМЕРНОМ АФФИННОМ ПРОСТРАНСТВЕ

ИОСЕФ СРОВНАЛ

В работе рассмотрен вопрос существования систем Σ трехмерных пространств (так называемых конгруэнций)в унимодулярном аффинном пространстве A_6. С помощью метода Картана приводится система дифференциальных уравнений конгруэнции Σ и показывается, что она в инволюции. Доказано существование исследуемых объектов в частных и в общем случаях.

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