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ON THE FOCAL POINTS OF SOLUTIONS OF A CERTAIN FOURTH ORDER DIFFERENTIAL EQUATION

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Consider the following differential equation of the fourth order

$$Y^{IV}(t) + 10[q(t)Y'(t)]' + 3[3q^2(t) + q''(t)]Y(t) = 0 \qquad (1)$$

with a function $q(t) \in C^2(-\infty, +\infty)$, $q(t) > 0$ on the interval $I = (-\infty, +\infty)$ obtained by iterating the linear homogeneous differential equation of the second order

$$y''(t) + q(t)y(t) = 0 \qquad (2)$$

[for this reason (1) is also called "iterated"].
It is known if $[u(t), v(t)]$ is a basis of all solutions of (2), then

$$[u^3(t),\ u^2(t)v(t),\ u(t)v^2(t),\ v^3(t)] \qquad (B_4)$$

is a basis of all solutions of (1). As a consequence we see that if the functions u(t), v(t) are oscillatory in the sense

of /2/ at (B_2) /i.e. there lie infinitely many zeros both to the left and to the right from every point $t \in I$/, then all functions generating the basis (B_4) are oscillatory in this sense also. Moreover: if $t_o \in I$ is a simple zero of the function $u(t)$ or $v(t)$, then t_o is an n-fold zero of the function $u^n(t)$ or $v^n(t)$, $n = 1,2,3$.
The differential equation (2) with the oscillatory basis (B_2) will be called oscillatory. We assume hereafter that all solutions of (2) or (1) are nontrivial, only.

Let $t_o \in I$ be an arbitrary firmly chosen point with $u(t_o) = 0$. So it simultaneously holds

$$u'(t_o) \neq 0, \ v(t_o) \neq 0 \qquad \text{(P)}$$

at this point. Then all solutions $Y(t)$ of (1) vanishing at t_o together with the solution $u(t)$ of (2) take the form

$$Y(t,C_1,C_2,C_3) = \sum_{i=1}^{3} C_i u^{4-i}(t) v^{i-1}(t) \ , \qquad \text{(S)}$$

where $C_i \in R$, $i = 1,2,3$, $\sum_{i=1}^{3} C_i^2 > 0$, are arbitrary constants (parameters).
We assume throughout all solutions $y(t) = c_1 u(t) + c_2 v(t)$, $c_j \in R$, $j = 1,2$, $c_1^2 + c_2^2 > 0$, of (2) to be oscillatory. Then every solution $Y(t)$ of (1) from the bundle (S) is also oscillatory, whereby $Y(t_o) = u(t_o)$ since

$$Y(t,C_1,C_2,C_3) = u(t) \sum_{i=1}^{3} C_i u^{3-i}(t) v^{i-1}(t)$$

for every choice $C_i \in R$, $i = 1,2,3$, $\sum_{i=1}^{3} C_i^2 > 0$.
For the zeros of an arbitrary solution $Y(t)$ of (1) from (S) we have

Theorem 1 (see /3/):

1) If $C_3 \neq 0$, then all zeros of the solution $Y(t)$ of (1) from

the bundle (S) coinciding with the zeros of the solution $u(t)$ of (2) are simple

2) If $C_3 = 0$, $C_2 \neq 0$, then all zeros of the solution $Y(t)$ of (1) from the bundle (S) coinciding with the zeros of the solution $u(t)$ of (2) are twofold

3) If $C_3 = C_2 = 0$, $C_1 \neq 0$, then all zeros of the solution $Y(t)$ of (1) from the bundle (S) coinciding with the zeros of the solution $u(t)$ of (2) are threefold.

Remarks.

Since the differential equation (1) is of the fourth order, its solution $Y(t)$ may have threefold zeros at most.

Specifying the conditions (P) as

$$\begin{aligned} u(0) = 0 \; , \; u'(0) = 1 \\ v(0) = 1 \; , \; v'(0) = 0 \; , \end{aligned} \qquad (P_0)$$

then taking account of (2) we successively obtain for

I. $Y(t) = C_1u^3(t)$:

$$Y'(t) = 3C_1u^2(t)u'(t)$$

$$Y''(t) = 3C_1\left[2u(t)u'^2(t) - q(t)u^3(t)\right]$$

$$Y'''(t) = 3C_1\left[2u'^3(t) - 7q(t)u^2(t)u'(t) - q'(t)u^3(t)\right] ,$$

so that by (P_0) we have

$$Y(0) = Y'(0) = Y''(0) = 0 \; , \; Y'''(0) = 6C_1 \neq 0 \qquad (s_1)$$

II. $Y(t) = C_1u^3(t) + C_2u^2(t)v(t)$:

$$Y'(t) = 3C_1u^2(t)u'(t) + C_2\left[2u(t)u'(t)v(t) + u^2(t)v'(t)\right]$$

$$Y''(t) = 3C_1\left[2u(t)u'^2(t) - q(t)u^3(t)\right] + C_2\left[2u'^2(t)v(t) - \right.$$

$$\left. - 3q(t)u^2(t)v(t) + 4u(t)u'(t)v'(t)\right]$$

$$Y'''(t) = 3C_1[2u'^3(t) - 7q(t)u^2(t)u'(t) - q'(t)u^3(t)] +$$
$$+ C_2[6u'^2(t)v'(t) - 14q(t)u(t)u'(t)v(t) -$$
$$- 7q(t)u^2(t)v'(t) - 3q'(t)u^2(t)v(t)] ,$$

so that by (P_0) we have

$$Y(0) = Y'(0) = 0,\ Y''(0) = 2C_2 \neq 0,\ Y'''(0) = 6C_1 \qquad (s_2)$$

III. $Y(t) = C_1u^3(t) + C_2u^2(t)v(t) + C_3u(t)v^2(t)$:

$$Y'(t) = 3C_1u^2(t)u'(t) + C_2[2u(t)u'(t)v(t) + u^2(t)v'(t)] +$$
$$+ C_3[u'(t)v^2(t) + 2u(t)v(t)v'(t)]$$
$$Y''(t) = 3C_1[2u(t)u'^2(t) - q(t)u^3(t)] + C_2[2u^2(t)v(t) -$$
$$- 3q(t)u^2(t)v(t) + 4u(t)u'(t)v'(t)] +$$
$$+ C_3[2u(t)v'^2(t) - 3q(t)u(t)v^2(t) + 4u'(t)v(t)v'(t)]$$
$$Y'''(t) = 3C_1[2u'^3(t) - 7q(t)u^2(t)u'(t) - q'(t)u^3(t)] +$$
$$+ C_2[6u'^2(t)v'(t) - 14q(t)u(t)u'(t)v(t) -$$
$$- 7q(t)u^2(t)v'(t) - 3q'(t)u^2(t)v(t)] +$$
$$+ C_3[6u'(t)v'^2(t) - 14q(t)u(t)v(t)v'(t) -$$
$$- 7q(t)u'(t)v^2(t) - 3q'(t)u(t)v^2(t)] ,$$

so that by (P_0) we have

$$Y(0) = 0,\ Y'(0) = C_3 \neq 0,\ Y''(0) = 2C_2,\ Y'''(0) = 6C_1 - 7C_3q(0) \qquad (s_3)$$

Conditions (s_1), (s_2) and (s_3) may uniquely characterize the oscillatory bundles I., II. and III. relating to (1).

All zeros of the oscillatory solution $Y(t)$ of (1) from the bundle (S), for which $Y(t_0) = u(t_0)$ in cases 1), 2), 3) from the foregoing Theorem 1 holds, will be called strongly conjugate points of the bundle (S) of solutions with a corresponding multiplicity $\nu = 1$, or $\nu = 2$, or $\nu = 3$.

Besides the zeros of solution $Y(t)$ of (1) coinciding with the zeros of the function $u(t)$ there may exist further zeros of this solution, namely the weakly conjugate points of the bundle (S) of the solutions $Y(t)$ relative to (1). About their existence and multiplicities decide the properties of coefficients $C_i \in R$, $i = 1,2,3$, in the three-parametric system of functions

$$Y^*(t,C_1,C_2,C_3) = \sum_{i=1}^{3} C_i u^{3-i}(t) v^{i-1}(t)$$

occuring in the bundle (S).

About the occurence and multiplicities of the weakly conjugate points of the bundle (S) of solutions $Y(t)$ relative to (1) we have (see /3/ again) the following

Theorem 2 : If the bundle (S) of oscillatory solutions $Y(t)$ relative to (1) has the form

I. $Y(t,C_1) = C_1 u^3(t)$, $C_1 \neq 0$, then there exist besides the strongly conjugate points having the multiplicity $\nu = 3$, no other zeros of $Y(t)$ relative to (1).

II. $Y(t,C_1,C_2) = u^2(t)[C_1 u(t) + C_2 v(t)]$, $C_2 \neq 0$, then there lies between any two neighbouring strongly conjugate points $T_k, T_{k+1} \in I$, $k = 0, \pm 1, \pm 2, \ldots$, of the bundle (S) of solutions $Y(t)$ relative to (1) with multiplicities $\nu = 2$, exactly one weakly conjugate point of the bundle (S) of solutions $Y(t)$ with multiplicity $\mu = 1$.

III. $Y(t,C_1,C_2,C_3) = u(t)[C_1 u^2(t) + C_2 u(t)v(t) + C_3 v^2(t)]$, $C_3 \neq 0$, and

a) if $C_2^2 - 4C_1C_3 > 0$, then there always lie exactly two distinct weakly conjugate points, both with multiplicity $\mu = 1$, between any two neighbouring strongly conjugate points $T_k, T_{k+1} \in I$, $k = 0, \pm 1, \pm +, \ldots$, of the bundle (S) of solutions $Y(t)$ relative to (1) with multiplicities $\nu = 1$

b) if $C_2^2 - 4C_1C_3 = 0$, then there always lies exactly one weakly conjugate point with multiplicity $\mu = 2$ between any two neighbouring strongly conjugate points T_k, $T_{k+1} \in I$, $k = 0, \pm 1, \pm 2, \ldots$, of the bundle (S) of solutions $Y(t)$ relative to (1) with multiplicities $\nu = 1$

c) if $C_2^2 = 4C_1C_3 < 0$, then the bundle (S) of solutions $Y(t)$ relative to (1) has no weakly conjugate points - all its zeros are strongly conjugate with multiplicity $\nu = 1$.

For short we denote by $y_i(t)$, $i = 1,2$, two arbitrary linearly independent oscillatory solutions of (2), either of which is besides linearly independent of the oscillatory solution $u(t)$ of this equation. With respect to the foregoing Theorem 2, the bundle (S) of all oscillatory solutions $Y(t)$ relative to (1), vanishing together with the function $u(t)$ at an arbitrary firmly chosen point $t_0 \in I$ may then be written as

I. $Y(t) = Cu^3(t)$

II. $Y(t) = Cu^2(t)y_1(t)$

III. a) $Y(t) = Cu(t)y_1(t)y_2(t)$

b) $Y(t) = Cu(t)y_1^2(t)$

c) $Y(t) = Cu(t)[y_1^2(t) + y_2^2(t)]$,

where $C \in R - \{0\}$ is an arbitrary constant.

Let us next denote by $T_0 = t_0$, $T_1 > T_0$ two neighbouring strongly conjugate points of the bundle (S) of solutions $Y(t)$ relative to (1). It then holds for the position of the weakly conjugate points of the bundle (S), i.e. of the zeros of functions $y_i(t)$, $i = 1,2$, on the open interval (T_o, T_1) that

ad I.: $\quad {}^3T_o < {}^3T_1$

ad II.: $\quad {}^2T_o < {}^1t_1 < {}^2T_1$

ad III.: a) $\quad {}^1T_o < {}^1t_1 < {}^1t_2 < {}^1T_1$ or ${}^1T_o < {}^1t_2 < {}^1t_1 < {}^1T_1$

b) $\quad {}^1T_o < {}^2t_1 < {}^1T_1$

c) $\quad {}^1T_o < {}^1T_1$,

where $u(T_o) = u(T_1) = 0$, $y_i(t_i) = 0$, $i = 1,2$, with the superior index $\nu \in \{1,2\}$ on the left at the respective zero of solutions Y(t) of (1) refers to its multiplicity.

<u>The accompanying points of solutions Y(t) relative to the differential equation (1)</u>

<u>**D e f i n i t i o n**</u> : By a accompanying point of the solution Y(t) relative to (1) we mean the zero of the derivative $Y'(t)$ of this solution.

From the assumption that every oscillatory solution y(t) relative to (2) is continuously differentiable it follows that every (oscillatory) function of the basis (B_4) relative to (1) is also continuously differentiable and consequently so also is every oscillatory solution Y(t) of this equation from the bundle (S). This evidently implies that the accompanying points of an arbitrary oscillatory solution Y(t) relative to (1) from the bundle (S) having the form I., II. and III. from Theorem 2, exist.

T h e o r e m 3: Let $t_o \in I$ be an arbitrary firmly chosen point, wherein the oscillatory solution $Y(t)$ relative to (1) from the bundle (S) vanishes together with the solution $u(t)$ of the oscillatory differential equation (2).

Let T_1 be the first (neighbouring) strongly conjugate point of the bundle (S) of oscillatory solutions $Y(t)$ relative to (1) lying on the right from the point $t_o = T_o$.

I. Let $Y(t) = Cu^3(t)$, where $C \in R - \{0\}$. Then, there exist exactly three distinct accompanying points on the closed interval $\langle T_o, T_1 \rangle$. Especially there exists on the open interval (T_o, T_1) exactly one simple accompanying point and two additional distinct twofold accompanying points one of which coincides with the point T_o and the other coincides with the point T_1.

II. Let $Y(t) = Cu^2(t)y_1(t)$, where $C \in R - \{0\}$. Then, there exist exactly four distinct accompanying points on the closed interval $\langle T_o, T_1 \rangle$. Especially there exist on the open interval (T_o, T_1) exactly two distinct simple accompanying points and two additional distinct simple accompanying points, one of which coincides with the point T_o and the other coincides with the point T_1.

III. a) Let $Y(t) = Cu(t)y_1(t)y_2(t)$, where $C \in R - \{0\}$. Then, there exist exactly three distinct simple accompanying points on the open interval (T_o, T_1), from which no one coincides with the zeros $t_1, t_2, t_1 \neq t_2$, of the functions $y_i(t)$, $i = 1,2$, lying in the open interval (T_o, T_1).

b) Let $Y(t) = Cu(t)y_1^2(t)$, where $C \in R - \{0\}$. Then, there exist exactly three distinct simple accompanying points on the open interval (T_o, T_1), the middle of which coincides with the twofold zero t_1 of the function $y_1(t)$.

c) Let $Y(t) = Cu(t)[y_1^2(t) + y_2^2(t)]$, where $C \in R - \{0\}$. Then, there exists at least one accompanying point on the open interval (T_o, T_1). If besides $y_1'^2(t) + y_2'^2(t) < q(t)[y_1^2(t) + y_2^2(t)]$, then there exists exactly one simple accompanying point.

P r o o f. In I., II. and III. instead of bundles there will be considered solutions $Y(t)$ relative to (1) only and this without introducing arbitrary multiplicative constant $C \in R\setminus\{0\}$.

Ad I. Since $Y(t) = u^3(t)$, the function $Y'(t) = 3u^2(t)u'(t)$ vanishes on the closed interval $\langle T_0, T_1\rangle$ together with the function $u(t)$ at both boundary points T_0, T_1 of this interval and this with multiplicity $\mu = 2$. In addition to both these zeros there exists exactly one zero of the function $u'(t)$ in the open interval (T_0, T_1) with multiplicity $\mu = 1$. Hence, there lie exactly three distinct zeros of the function $Y'(t)$ on the closed interval $\langle T_0, T_1\rangle$.

Ad.II. Since $Y(t) = u^2(t)y_1(t)$, the function $Y'(t) = u(t)[2u'(t)y_1(t) + u(t)y_1'(t)]$. As the function $y_1(t)$ is an arbitrary solution of (2) linearly independent of the solution $u(t)$ of the same equation, then the simple zero t_1 of the function $y_1(t)$ lies on the open interval (T_0, T_1), because - by Sturm's theorem - all zeros of two oscillatory linearly independent solutions of the same 2nd order differential equation mutually separate. Thus the derivative $Y'(t)$ vanishes on the closed interval $\langle T_0, T_1\rangle$ at both neighbouring zeros T_0, T_1 of the function $u(t)$ with multiplicity $\nu = 1$ on the one side, and vanishes both on the open interval (T_0, t_1) and on the open interval (t_1, T_1) at the points with multiplicity $\mu = 1$ on the other side. This fact follows from Rolle's theorem applied to the function $Y(t)$, which is continuous on either of the closed intervals $\langle T_0, t_1\rangle$, $\langle t_1, T_1\rangle$ and differentiable on both open intervals (T_0, t_1), (t_1, T_1), whereby $Y(T_0) = Y(t_1) = Y(T_1) = 0$. Denote by $Y'(t) = u(t)F(t)$, where

$$F(t) = 2u'(t)y_1(t) + u(t)y_1'(t) .$$

To prove the existence of exactly one simple zero of the function $F(t)$ on the open interval (T_0,t_1), let us proceed from the equation

$$2u'(t)y_1(t) + u(t)y_1'(t) = 0 .$$

Since for all $t \in (T_0,t_1)$ it holds $u(t)y_1(t) \neq 0$, we may write

$$u(t)y_1(t)\left[2\frac{u'(t)}{u(t)} + \frac{y_1'(t)}{y_1(t)} \right] = 0 .$$

For the first derivative $f'(t)$ of the function

$$f(t) = 2\frac{u'(t)}{u(t)} + \frac{y_1'(t)}{y_1(t)}$$

we have

$$f'(t) = 2\frac{u''(t)u(t) - u'^2(t)}{u^2(t)} + \frac{y_1''(t)y_1(t) - y_1'^2(t)}{y_1^2(t)} ,$$

which may be written with respect to (2) as

$$f'(t) = -2\frac{q(t)u^2(t) + u'^2(t)}{u^2(t)} - \frac{q(t)y_1^2(t) + y_1'^2(t)}{y_1^2(t)} .$$

Hence, in assuming that $q(t) > 0$ on the interval $I = (-\infty, +\infty)$, we see that $f'(t) < 0$ on the open interval (T_0,t_1). The function $f(t)$ is thus decreasing on the interval (T_0,t_1), whereby

$$\lim_{t\to T_0^+} f(t) = +\infty \quad \text{and} \quad \lim_{t\to t_1^-} f(t) = -\infty$$

(because both $\left|\lim_{t\to T_0^+}\frac{y_1'(t)}{y_1(t)}\right| < \infty$ and $\left|\lim_{t\to t_1^-}\frac{u'(t)}{u(t)}\right| < \infty$).

Thus there exists exactly one zero of the function $f(t)$ with multiplicity $\mathcal{M} = 1$ on the open interval (T_o, t_1). Likewise we could prove the existence of exactly one simple zero of the function $F(t)$ also on the open interval (t_1, T_1). Hence there exist exactly four distinct zeros of the function $Y'(t)$ on the closed interval $\langle T_o, T_1 \rangle$.

Ad III. a) Since $Y(t) = u(t)y_1(t)y_2(t)$, then

$$Y'(t) = u'(t)y_1(t)y_2(t) + u(t)y_1'(t)y_2(t) + u(t)y_1(t)y_2'(t).$$

It holds for the simple zeros $t_1, t_2 \in (T_o, T_1)$, $t_1 \neq t_2$, of the functions $y_1(t)$, $y_2(t)$ - of two linearly independent solutions relative to (2) - that either $t_1 < t_2$ or $t_2 < t_1$. Let us consider only the first case regarding the mutual position of the points t_1, t_2 (for $t_2 < t_1$ we would proceed entirely analogous). These two points separate the closed interval $\langle T_o, T_1 \rangle$ into three closed intervals $\langle T_o, t_1 \rangle$, $\langle t_1, t_2 \rangle$, $\langle t_2, T_1 \rangle$. The function $Y(t)$ is on each of them continuous, it is differentiable on the open intervals (T_o, t_1), (t_1, t_2), (t_2, T_1), whereby

$$Y(T_o) = Y(t_1) = Y(t_2) = Y(T_1) = 0 .$$

So, by Rolle's theorem there exists at least by one zero of the function $Y'(t)$ on every open interval (T_o, t_1), (t_1, t_2), (t_2, T_1).

To prove the existence of exactly one simple zero of the function $Y'(t)$ on the open interval (T_o, t_1), let us proceed from the equation

$$u'(t)y_1(t)y_2(t) + u(t)y_1'(t)y_2(t) + u(t)y_1(t)y_2'(t) = 0.$$

Since for all $t \in (T_o, t_1)$ we have $u(t)y_1(t)y_2(t) \neq 0$, we may write

$$u(t)y_1(t)y_2(t)\left[\frac{u'(t)}{u(t)} + \frac{y_1'(t)}{y_1(t)} + \frac{y_2'(t)}{y_2(t)}\right] = 0 .$$

For the 1st derivative $g'(t)$ of the function

$$g(t) = \frac{u'(t)}{u(t)} + \frac{y_1'(t)}{y_1(t)} + \frac{y_2'(t)}{y_2(t)}$$

it holds

$$g'(t) = \frac{u''(t)u(t) - u'^2(t)}{u^2(t)} + \frac{y_1''(t)y_1(t) - y_1'^2(t)}{y_1^2(t)} + \frac{y_2''(t)y_2(t) - y_2'^2(t)}{y_2^2(t)} ,$$

which with respect to (2) may be written as

$$g'(t) = -\frac{q(t)u^2(t) + u'^2(t)}{u^2(t)} - \frac{q(t)y_1^2(t) + y_1'^2(t)}{y_1^2(t)} - \frac{q(t)y_2^2(t) + y_2'^2(t)}{y_2^2(t)} .$$

Thus, in assuming that $q(t) > 0$ on the interval $I = (-\infty, +\infty)$, it holds $g'(t) < 0$ on the open interval (T_o, t_1). The function $g(t)$ is decreasing on the interval (T_o, t_1), whereby

$$\lim_{t \to T_o^+} g(t) = +\infty \quad \text{and} \quad \lim_{t \to t_1^-} g(t) = -\infty$$

(because $\left|\lim_{t\to T_0^+} \frac{y_1'(t)}{y_1(t)}\right| < \infty$, $\left|\lim_{t\to T_0^+} \frac{y_2'(t)}{y_2(t)}\right| < \infty$,

$\left|\lim_{t\to t_1^-} \frac{u'(t)}{u(t)}\right| < \infty$, $\left|\lim_{t\to t_1^-} \frac{y_2'(t)}{y_2(t)}\right| < \infty$).

Consequently there exists exactly one zero of the function $g(t)$ with multiplicity $\mu = 1$ on the open interval (T_o,t_1). Completely analogous we could prove the existence of exactly one simple zero of the function $g(t)$ both on the open interval (t_1,t_2) and on the open interval (t_2,T_1).

Hence there lie exactly three distinct zeros of the function $Y'(t)$ with multiplicity $\mu = 1$ on the open interval (T_o,T_1), mutually separating with the zeros t_1,t_2 of the functions $y_1(t)$ and $y_2(t)$.

b) Since $Y(t) = u(t)y_1^2(t)$, then

$$Y'(t) = y_1(t)\left[u'(t)y_1(t) + 2u(t)y_1'(t)\right] .$$

It is readily seen that the function $Y'(t)$ vanishes together with the function $y_1(t)$ at its zero $t_1 \in (T_o,T_1)$ and this with multiplicity $\mu = 1$. The point t_1 divides the closed interval $\langle T_o,T_1\rangle$ into two closed intervals $\langle T_o,t_1\rangle$ and $\langle t_1,T_1\rangle$. The function $Y(t)$ is continuous on either of them and is differentiable on the respective open intervals (T_o,t_1), (t_1,T_1), whereby

$$Y(T_o) = Y(t_1) = Y(T_1) = 0 .$$

By Rolle's theorem there lies at least by one zero of the function $Y'(t)$ on any of the open intervals (T_o,t_1), (t_1,T_1). Denote now $Y'(t) = y_1(t)H(t)$, where

$$H(t) = u'(t)y_1(t) + 2u(t)y_1'(t) .$$

To prove the existence of exactly one simple zero of the function $H(t)$ on the open interval (T_o, t_1), let us proceed from the equation

$$u'(t)y_1(t) + 2u(t)y_1'(t) = 0 .$$

Since it holds for all $t \in (T_o, t_1)$ that $u(t)y_1(t) \neq 0$, we may write

$$u(t)y_1(t)\left[\frac{u'(t)}{u(t)} + 2\,\frac{y_1'(t)}{y_1(t)}\right] = 0 .$$

For the 1st derivative $h'(t)$ of the function

$$h(t) = \frac{u'(t)}{u(t)} + 2\,\frac{y_1'(t)}{y_1(t)}$$

it holds

$$h'(t) = \frac{u''(t)u(t) - u'^2(t)}{u^2(t)} + 2\,\frac{y_1''(t)y_1(t) - y_1'^2(t)}{y_1^2(t)} ,$$

which with respect to (2) may be written as

$$h'(t) = -\,\frac{q(t)u^2(t) + u'^2(t)}{u^2(t)} - 2\,\frac{q(t)y_1^2(t) + y_1'^2(t)}{y_1^2(t)} .$$

Thus, in assuming that $q(t) > 0$ on the interval $I = (-\infty, +\infty)$, it holds $h'(t) < 0$ on the open interval (T_o, t_1). Consequently, the function $h(t)$ is decreasing on the interval (T_o, t_1), whereby

$$\lim_{t \to T_o^+} h(t) = +\infty \qquad \text{and} \qquad \lim_{t \to t_1^-} h(t) = -\infty$$

(because both $\left|\lim_{t\to T_0^+} \frac{y_1'(t)}{y_1(t)}\right| < \infty$ and $\left|\lim_{t\to t_1^-} \frac{u'(t)}{u(t)}\right| < \infty$).

Hence, there exists exactly one zero of the function $H(t)$ with multiplicity $\mu = 1$ on the open interval (T_0, t_1). In analogy to the preceding case could be proved the existence of exactly one simple zero of the function $H(t)$ on the open interval (t_1, T_1), too.

Consequently there exist exactly three distinct zeros of the function $Y'(t)$ with multiplicity $\mu = 1$ on the open interval (T_0, T_1), whereby the middle of them always coincides with the zero t_1 of the function $y_1(t)$.

Let us remark that this case III. b) of the bundle $Y(t)$ - and the zeros of its derivative $Y'(t)$ - is dual to the case II. of the bundle $Y(t)$ of solutions relative to equation (1).

c) Since $Y(t) = u(t)\left[y_1^2(t) + y_2^2(t)\right]$, then

$$Y'(t) = u'(t)\left[y_1^2(t) + y_2^2(t)\right] + 2u(t)\left[y_1(t)y_1'(t) + \right.$$
$$\left. + y_2(t)y_2'(t)\right] .$$

Let us proceed from the equation

$$u'(t)\left[y_1^2(t) + y_2^2(t)\right] + 2u(t)\left[y_1(t)y_1'(t) + y_2(t)y_2'(t)\right] = 0.$$

Since it holds for all $t \in (T_0, T_1)$ that $u(t)\left[y_1^2(t) + y_2^2(t)\right] \neq 0$, we may write

$$u(t)\left[y_1^2(t) + y_2^2(t)\right]\left[\frac{u'(t)}{u(t)} + 2\,\frac{y_1(t)y_1'(t) + y_2(t)y_2'(t)}{y_1^2(t) + y_2^2(t)}\right] = 0 .$$

For the 1st derivative $p'(t)$ of the function

$$p(t) = \frac{u'(t)}{u(t)} + 2\,\frac{y_1(t)y_1'(t) + y_2(t)y_2'(t)}{y_1^2(t) + y_2^2(t)}$$

it holds

$$p'(t) = \frac{u''(t)u(t) - u'^2(t)}{u^2(t)} +$$

$$+ 2\frac{y_1'^2(t) + y_2'^2(t) + y_1(t)y_1''(t) + y_2(t)y_2''(t)}{y_1^2(t) + y_2^2(t)} -$$

$$- 2\left[\frac{y_1(t)y_1'(t) + y_2(t)y_2'(t)}{y_1^2(t) + y_2^2(t)}\right]^2 \quad .$$

which with respect to (2) may be written as

$$p'(t) = - \frac{q(t)u^2(t) + u'^2(t)}{u^2(t)}$$

$$- 2 \left\{ \frac{q(t)[y_1^2(t) + y_2^2(t)] - [y_1'^2(t) + y_2'^2(t)]}{y_1^2(t) + y_2^2(t)} + \right.$$

$$\left. + 2\left[\frac{y_1(t)y_1'(t) + y_2(t)y_2'(t)}{y_1^2(t) + y_2^2(t)}\right]^2 \right\} \quad .$$

If $y_1'^2(t) + y_2'^2(t) < q(t)\,[y_1^2(t) + y_2^2(t)]$, where $q(t) > 0$, then $p'(t) < 0$ on the open interval (T_o, T_1). This implies that the function $p(t)$ is decreasing on the interval (T_o, T_1), whereby

$$\lim_{t \to T_o^+} p(t) = +\infty \quad , \quad \lim_{t \to T_1^-} p(t) = -\infty$$

(because both $\left| \lim_{t \to T_o^+} \frac{[y_1^2(t) + y_2^2(t)]^{\cdot}}{y_1^2(t) + y_2^2(t)} \right| < \infty$ and

$$\left| \lim_{t \to T_1^-} \frac{[y_1^2(t) + y_2^2(t)]^{\cdot}}{y_1^2(t) + y_2^2(t)} \right| < \infty \;).$$

Consequently, there exists exactly one zero of the function $p(t)$ [and so also the function $Y^{\circ}(t)$] on the open interval (T_o, T_1) with multiplicity $\mu = 1$.

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SOUHRN

PRŮVODNÍ BODY ŘEŠENÍ JISTÉ DIFERENCIÁLNÍ ROVNICE 4.ŘÁDU

VLADIMÍR VLČEK

Práce navazuje na předchozí autorovy výsledky dosažené při vyšetřování nulových bodů oscilatorických řešení iterované diferenciální rovnice 4.řádu.

V práci jsou studovány nulové body derivací řešení se zaměřením na rozložení a násobnosti průvodních bodů v souvislosti s polohou nulových bodů řešení uvažované diferenciální rovnice.

РЕЗЮМЕ

СОПРОВОДИТЕЛЬНЫЕ ТОЧКИ РЕШЕНИЙ ОДНОГО ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ 4-ГО ПОРЯДКА

ВЛАДИМИР ВЛЧЕК

Работа исходит из результатов, достигнутых автором в предыдущем исследованию нулевых точек колеблющихся решений итерированного дифференциального уравнения 4-го порядка.

В работе изучаются нулевые точки производных этих решений с учетом разложения и кратности сопроводительных точек в связи с положением нулевых точек решений рассматриваемого дифференциального уравнения.

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