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DIGITAL SIMULATION OF ANALOG COMPUTING DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS AND NONLINEAR DIFFERENTIAL EQUATIONS

KAREL BENEŠ

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This paper deals with the influence of the inaccuracy appearing at the diode multiplier in the setup diagram for computing ordinary differential equations resulting in inaccurate computations. The function of the diode multiplier is simulated by the digital computer. Likewise, the simulation of the whole differential equation is performed by the digital computer in applying the corresponding numerical methods. There are considered differential equations of the first and of the second order, only. The simulation ascertains the theoretical accuracy of the equations considered.

I. Diode multiplier and its function

In analog machines the most used multiplier is the diode multiplier, independent of frequency in the wide band, relatively simple in construction, it brings, however, a certain error in computing. The diode multiplier works on the basis of the relation

$$uv = \left(\frac{u + v}{2} \right)^2 - \left(\frac{u - v}{2} \right)^2 \qquad (1)$$

raising to the second powers is realized by means of functional transformators. In the diode multiplier there are used two squarers, which are the diode functional transformators, where $w = z^2$ is approximated by a certain number of K linear sections generally by five or by ten sections and this is the source of the error of the functional transformators, hence also of the whole multiplier. In general, in polynomial approximating the function f(z) by the polynomial g(z) in the interval of the approximation $\langle a;b \rangle$ the inaccuracy $\mathcal{E}(z) = f(z) - g(z)$ may be written as

$$\mathcal{E}(z) = \frac{f^{(n+1)}(\xi)}{(n+1)\,!} (z-z_0)(z-z_1) \ldots (z-z_n) \qquad (2)$$

where $z_0, z_1, \ldots, z_n$ are the knots of approximation and ξ is a certain point of $\langle a;b \rangle$, n is the degree of the approximating polynomial g(z). The maximal absolute error of the approximation then satisfies the inequality

$$|\mathcal{E}(z)|_{max} \leq \frac{M_{n+1}}{(n+1)\,!} \varphi(z, z_0, z_1, \ldots, z_n) \, , \qquad (3)$$

where

$$M_{n+1} = \max \left| f^{(n+1)}(z) \right| , \quad z \in \langle a;b \rangle \, .$$

$$\varphi(z, z_0, z_1, \ldots, z_n) = \max \left| (z-z_0)(z-z_1)\ldots(z-z_n) \right| ,$$

$$z \in \langle a;b \rangle \, .$$

In approximating the function f(z) by linear sections we get $n = 1$ and the error in the interval $z \in \langle z_{k-1}; z_k \rangle$ is given with respect to (2) by

$$\mathcal{E}(z) = \frac{f''(\xi)}{2!}(z-z_{k-1})(z-z_k) \quad , \tag{4}$$

where ξ is a certain point of the interval $\langle z_{k-1}; z_k \rangle$ and $k = 1, 2, \ldots, K$. Next, we ascertain at which point of the interval $\langle z_{k-1}; z_k \rangle$ the expression $m = (z-z_{k-1})(z-z_k)$ assumes its extreme. From the condition for the extreme value of the function m

$$m' = z - z_k + z - z_{k-1} = 0$$

we find the value z, at which the function m assumes its extreme, i.e.

$$z = \frac{z_k + z_{k-1}}{2} \quad .$$

Substituting this value into the right hand side of (3), we obtain for the upper bound of the inaccuracy in $\langle z_{k-1}; z_k \rangle$

$$\left| \mathcal{E}(z) \right|_{max} \leq \frac{M_2}{2!} \left(\frac{z_k - z_{k-1}}{2} \right)^2 \quad . \tag{5}$$

If an approximation of a quadratic function is concerned, then $M_2 = 2$ and

$$\left| \mathcal{E}(z) \right|_{max} \leq \left(\frac{z_k - z_{k-1}}{2} \right)^2 \tag{6}$$

It becomes apparent from (6) that in this case the error is dependent on the length of the linear section only, and in case of the equidistant decomposition of the breakpoints $z_0, z_1, \dots, z_k$ the maximal error is equal in any section. Hence, this decomposition of breakpoints complies with the requirement of the best uniform approximation. That is why we choose the equidistant distribution of breakpoints in applying the squarer functional transformator. It is an established practice with the diode multipliers to use either five or ten linear sections (i.e. K = 5, or K = 10). In K = 5 the maximal squarer error is $\varepsilon = 0{,}01$, while in K = 10 the maximal error is $\varepsilon = 0{,}0025$. By (1) it is apparent that the above errors may also be eliminated.

Digital simulation of the diode multiplier

The simulation of the multiplier is performed by a program which during the numerical solution of the differential equation, wherein a product of two quantities occurs, will replace the product of these quantities in considering the inaccuracy of the squarers. With respect to (1) it suffices to simulate the squarer only, and in other respects to apply the usual mathematical operations. Thus, for the given value z there should be generated its second power by an approximated broken line in linear sections with the equidistant knots of the approximation. So, let us have a value $z \in \langle 0;1 \rangle$ for which we want to find its approximated value w(z). Suppose, we know in which section it is located, i.e.

$$z_{k-1} \leq z \leq z_k \tag{7}$$

where $k = 1, \dots, K$, $z_0, z_1, \dots, z_k$ are the breakpoints and K is the number of linear sections. In the breakpoints we have $f(z_k) = z_k^2$. Then the values f(z), where (7) holds, lie on the straightline going through the points $\left[z_{k-1}; f(z_{k-1})\right]$, $\left[z_k; f(z_k)\right]$. The equation of the approximating straightline is then

$$f(z) = \frac{f(z_k)-f(z_{k-1})}{z_k - z_{k-1}} (z-z_{k-1}) + f(z_{k-1}) \tag{8}$$

In our case with $z_j = z_j^2$ we obtain

$$f(z) = (z_k + z_{k-1})(z - z_{k-1}) + f(z_{k-1}) =$$

$$= (z_k + z_{k-1})(z - z_{k-1}) + z_{k-1}^2 \quad . \tag{9}$$

For the equidistant distribution of breakpoints we have

$$z_k = kh \quad , \tag{10}$$

where h is the length of the linear section, i.e. $h = \frac{1}{K}$.
If in the computation for the given z the index k is determined, i.e. $z \in \langle z_{k-1}; z_k \rangle$, then, by (9), we determine f(z), i.e.

$$f(z) = \left[kh + (k-1)h\right]\left[z - (k-1)h\right] + \left[(k-1)h\right]^2 =$$

$$= (2kh - h) \left[z - (k-1)h\right] + \left[(k-1)h\right]^2 =$$

$$= \frac{1}{K} (2k-1) \left[z - \frac{1}{K} (k-1)\right] + \left[\frac{1}{K} (k-1)\right]^2 \quad . \tag{10a}$$

To simulate the working of the multiplier we formally modify relation (1) to

$$uv = \left(\frac{|u + v|}{2} \right)^2 - \left(\frac{|u - v|}{2} \right)^2 , \tag{11}$$

so that the fictive squarer is able to generate for non-negative values, only. During this simulation there will enter values u and v into the subprogram for the simulation. First we will calculate the values inside both parentheses of (11), i.e.

$$a_1 = \frac{|u + v|}{2} ,$$

$$a_2 = \frac{|u - v|}{2} . \qquad (12)$$

For the analog computation we assume $|u|, |v| \leq 1$, i.e. $a_{1,2} \in \langle 0;1\rangle$. The index k will be determined on the basis of

$$k = \left[\frac{a_i}{h}\right] + 1 \qquad (13)$$

where the square brackets denote the whole part of the quotient. Returning now to use the notation z for the argument of the approximated quadratic function, we have

$$k = \left[\frac{z}{h}\right] + 1 \quad , \quad \text{i.e.} \quad a_i = z \qquad (14)$$

In closing we insert the values a_i and k into (9) (applying relation (10)). The simulated value of the product may be written as

$$s = f(a_1) - f(a_2) \quad . \qquad (15)$$

II. Simulated solutions of differential equations

In what follows we will show solutions of differential equations with a simulated diode multiplier. We have three types of equations:

A: the solution is monotone convergent

B: the solution is monotone unboundedly increasing

C: the solution is oscillatory

Equation A:

a/ $$y' + 10y^2 = 0 \quad ; \quad y(0) = 0{,}9 \tag{16}$$

The function $y = \frac{1}{10x+C}$, where $C = \frac{10}{9}$, is the solution.

Analog program diagram is shown in Figure 1. The run of the

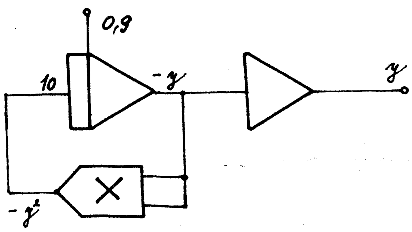

Figure 1.

error $\delta(y) = y - y_{sim}$ resulting from the inaccuracy of the multiplier is given in Figure 1a. (On the understanding that in analog program diagrams the multiplier changes the sign of the product), the number of linear sections of squarers K = 5,10. So far the error $\delta(y)$ for K = 10 is not recorded in the pictures, then $\delta(y) < 0{,}003$.

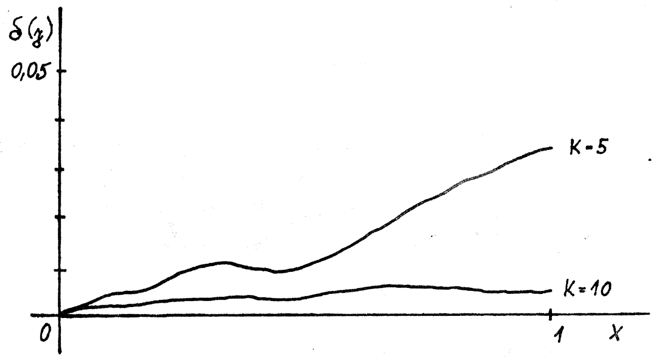

Figure la.

b/ $$y' + 9xy = 0 \quad ; \quad y(0) = 0{,}9 \quad . \tag{17}$$

The function $y = Ce^{-4,5x^2}$, where C = 0,9, is the solution.

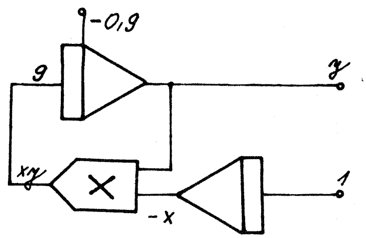

Fig. 2.

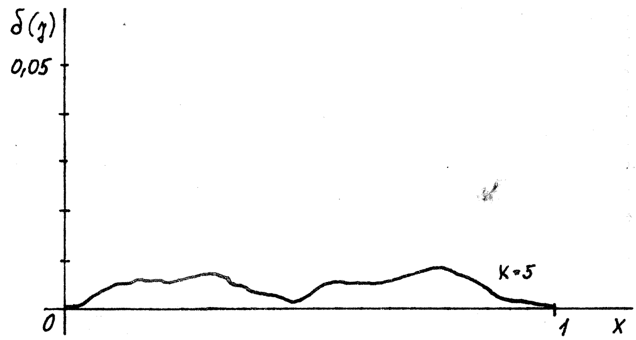

Fig. 2a.

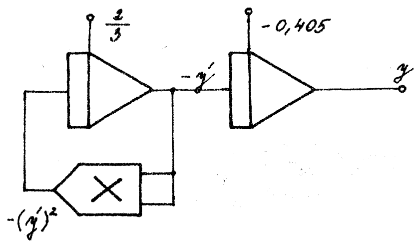

Fig. 3.

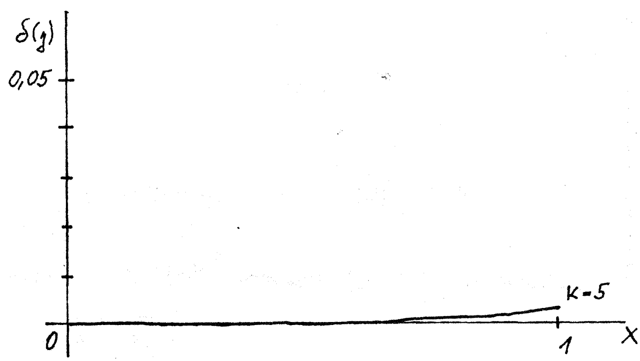

Fig. 3a.

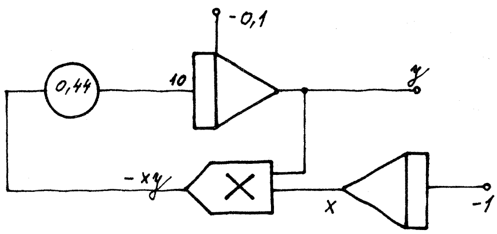

Fig. 4.

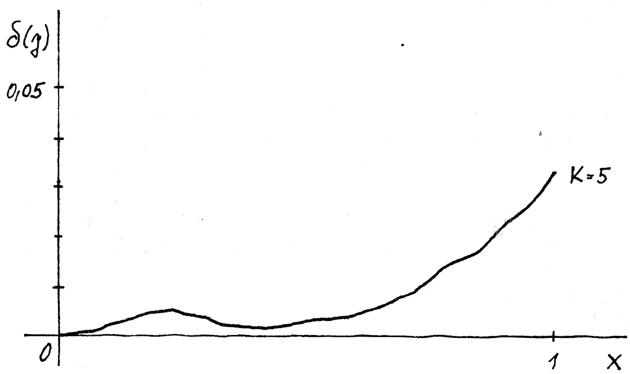

Fig. 4a.

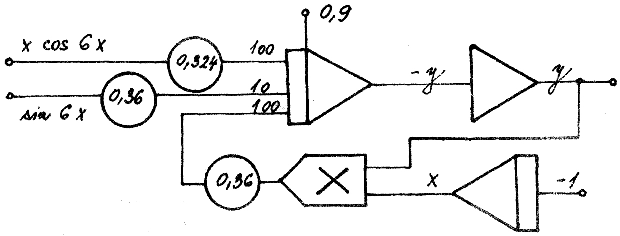

Fig. 5.

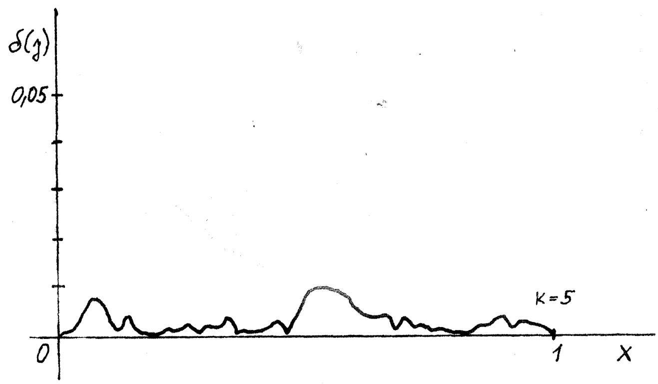

Fig. 5a.

Equation B:

a/ $$y'' + (y')^2 = 0; \quad y(0) = 0{,}405; \quad y'(0) = \frac{2}{3}. \tag{18}$$

The function $y = \ln(x + 1{,}5)$ is the solution.

b/ $$y' - 4{,}4xy = 0 \; ; \quad y(0) = 0{,}1 \tag{19}$$

$$y = Ce^{2{,}2x^2} \; ; \quad C = 0{,}1 \; .$$

Equation C:

a/ $$y' + 36xy = 32{,}4x\cos 6x - 3{,}6 \sin 6x \tag{20}$$

$$y(0) = 0{,}9$$

$$y = 0{,}9 \cos 6x \; .$$

The function $z_1 = x \cos Bx$ is generated by the solution of the differential equation $z_1'' + B^2 z = -2B \sin Bx, z(0) = 0$; $z'(0) = 1$, i.e. in our case by $z_1'' + 36z = -6 \sin 6x$, $z_1(0) = 0$, $z_1'(0) = 1$, where $B = 6$.

The function $z_2 = \sin Bx$ is generated by the solution of the differential equation $z_2'' + B^2 z_2 = 0$, $z_2(0) = 0$, $z_2'(0) = B^2$, i.e. in our case by the solution of the equation $z_2'' + 36\, z_2 = 0$; $z(0) = 0$; $z'(0) = 6$.

The program diagrams for generating the functions z_1 and z_2 are not recorded in Figure 5 for their evidence.

b/ $$y'' + 6y'\cos 6x + 36y \sin 6x = 25{,}2 - 25{,}2 \sin 6x \tag{21}$$

$$y(0) = 0 \; ; \quad y'(0) = 4{,}2$$

The function $y = 0{,}7 \sin 6x$ is the solution.

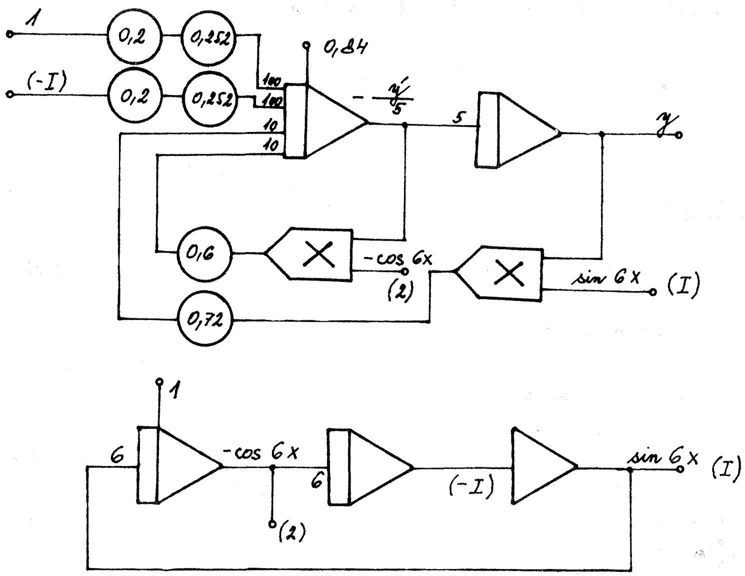
1
(-I)
0,2
0,2
0,252
0,252
0,84
100
100
10
10
5
y
0,6
0,72
-cos 6x
(2)
sin 6x
(I)
1
6
-cos 6x
6
(-I)
sin 6x
(I)
(2)

Fig. 6.

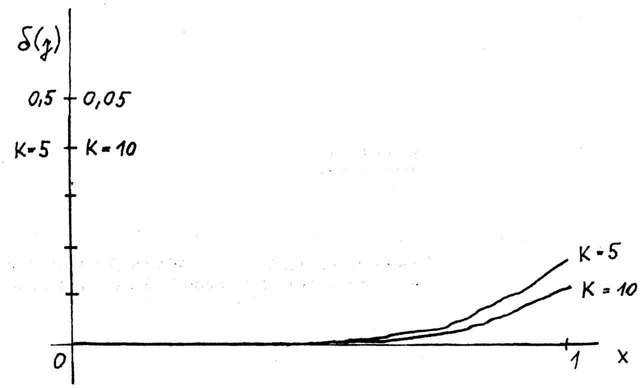

Fig. 6a.

III. Evaluation of results

a/ The diode multiplier with K = 5.
In equations A is the error lesser than 0,01. Only if the resulting function rapidly increases or decreases, the error is greater (equation (16)). Similarly, in equations B is the error small. Only in equation (19) is the error greater due to the rapid increase of the value of the solution. In equation C greater inaccuracy is stated, if more multipliers are used.

b/ The diode multiplier with K = 10.

Applying the multiplier with ten linear sections (K = 10), we see that this multiplier negligibly affects the accuracy of the solution, since in (21) - which is the worst case, the error is about 0,01, in other cases it is even lesser than 0,005.

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SOUHRN

ČÍSLICOVÁ SIMULACE ANALOGOVÉHO ŘEŠENÍ DIFERENCIÁLNÍCH ROVNIC S PROMĚNNÝMI KOEFICIENTY A NELINEÁRNÍCH DIFERENCIÁLNÍCH ROVNIC

KAREL BENEŠ

Práce se zabývá experimentálním zjištěním vlivu nepřesnosti diodové násobičky na přesnost analogového řešení uvedených diferenciálních rovnic. Jsou uvedeny grafy průběhů chyby při řešení některých diferenciálních rovnic.

РЕЗЮМЕ

НУМЕРИЧЕСКАЯ СИМУЛЯЦИЯ АНАЛОГОВО РЕШЕНИЯ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ С ПЕРЕМЕННЫМИ КОЭФФИЦИЕНТАМИ И НЕЛИНЕЙНЫХ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ

КАРЕК БЕНЕШ

Работа занимается экспериментальным установитием влияния ошибки диодного умножительного блока на точность решения этих дифференциальных уравнений. В работе даны ходы ошибок при решению некоторых дифференциальных уравнений.

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