Svatoslav Staněk

On the intersection of groups of increasing dispersions in two second order oscillatory differential equations

Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica, Vol. 24 (1985), No. 1, 61--66

Persistent URL: `http://dml.cz/dmlcz/120165`

ACTA UNIVERSITATIS PALACKIANAE OLOMUCENSIS
FACULTAS RERUM NATURALIUM
1985 MATHEMATICA XXIV Vol. 82

Katedra matematické analýzy a numerické matematiky přírodovědecké fakulty Univerzity Palackého v Olomouci
Vedoucí katedry: Prof. RNDr. Miroslav Laitoch, CSc.

ON THE INTERSECTION OF GROUPS OF INCREASING DISPERSIONS IN TWO SECOND ORDER OSCILLATORY DIFFERENTIAL EQUATIONS

SVATOSLAV STANĚK

(*Received October 12, 1983*)

1. Introduction

O. Borůvka in [4]–[6] and S. Staněk in [9]–[13] investigated under various assumptions laid on the coefficients p, q the intersection of groups in two second order differential equations (p): $y'' = p(t)\,y$ and (q): $y'' = q(t)\,y$, which is equivalent to investigating a set of joint solutions of certain two nonlinear differential equations of the third order of Kummer type. Following the results in [5] we prove below that under the assumption of oscillatory of (p) and (q), $p \neq q$, may be considered as a trivial group or as an infinite cyclic group or as a planar group.

2. Basic concepts and notation

Throughout this discussion the second order linear differential equation of type

$$y'' = q(t)\,y, \qquad q \in C^{\circ}(\mathbb{R}), \tag{q}$$

will be assumed to be oscillatory (on $\mathbf{R}$), i.e. $\pm\infty$ are the cluster points of zeros to every nontrivial solution of (q).

A function $\alpha \in C^{\circ}(\mathbf{R})$ will be called the (first) phase of (q) exactly if there exist independent solutions u, v of (q):

$$\operatorname{tg} \alpha(t) = u(t)/v(t) \qquad \text{for } t \in \mathbf{R} - \{t;\, v(t) = 0\}.$$

Every phase α of (q) has the following three properties

$$\text{(i) } \alpha \in C^{3}(\mathbf{R}); \qquad \text{(ii) } \alpha(\mathbf{R}) = \mathbf{R}; \qquad \text{(iii) } \alpha'(t) \neq 0 \qquad \text{for } t \in \mathbf{R}.$$

The set of phases of the equation $y'' = -y$ constituted a group $\mathfrak{E}$ (the so-called group of fundamental phases) relative to the rule of composition of functions. A function $\varepsilon \in \mathfrak{E}$ exactly if

$$\operatorname{tg} \varepsilon(t) = \frac{a_{11} \cos t + a_{12} \sin t}{a_{21} \cos t + a_{22} \sin t}$$

holds for all t, where both sides of the latter relation are meaningful. Thereby a_{ij} $(i, j = 1, 2)$ are numbers, $\det a_{ij} \neq 0$. For every $\varepsilon \in \mathfrak{E}$ we have $\varepsilon(t + \pi) = \varepsilon(t) + \pi \operatorname{sign} \varepsilon'$ for $t \in \mathbf{R}$.

A function $X \in C^3(\mathbf{R})$, $X'(t) \neq 0$ for $t \in \mathbf{R}$, is called the (first) kind dispersion of (q) if it is a solution of the Kummer differential equation

$$-\{X, t\} + X'^2 q(X) = q(t),$$

where $\{X, t\} := \frac{1}{2} \frac{X'''(t)}{X'(t)} - \frac{3}{4} \left(\frac{X''(t)}{X'(t)} \right)^2$ means the Schwarz derivative of the function X at t.

The set of dispersion (the set of increasing dispersions) of (q) constitutes a group $\mathscr{L}_q(\mathscr{L}_q^+)$ relative to the rule of composition of functions. If $X \in \mathscr{L}_q$, then $X(\mathbf{R}) = \mathbf{R}$. The dispersions of (q) have the following characteristic property: A function $X \in C^3(\mathbf{R})$, $X'(t) \neq 0$ for $t \in \mathbf{R}$ is a dispersion of (q) exactly if there exists to every solution y of (q) a solution z of this equation such that $\frac{y[X(t)]}{\sqrt{|X'(t)|}} = z(t)$ for $t \in \mathbf{R}$.

Let α be a phase of (q). Then

$$\mathscr{L}_q = \{\alpha^{-1} \varepsilon \alpha;\ \varepsilon \in \mathfrak{E}\},$$
$$\mathscr{L}_q^+ = \{\alpha^{-1} \varepsilon \alpha;\ \varepsilon \in \mathfrak{E}, \operatorname{sign} \varepsilon' = 1\}.$$

All the above concepts and properties may be found in [2] and [3].

Let $p \neq q$ and $X, Y \in \mathscr{L}_p^+ \cap \mathscr{L}_q^+$. It then follows from [6] and [10] that either $X(t) = Y(t)$ for $t \in \mathbf{R}$ or $X(t) \neq Y(t)$ for $t \in \mathbf{R}$. Thus, the group $\mathscr{L}_p^+ \cap \mathscr{L}_q^+$ has the property that the graphs of any two different terms (in so far as such exist) are disjunct.

Let $p \neq q$ and $\mathscr{S} \subset \mathscr{L}_p^+ \cap \mathscr{L}_q^+$ be a subgroup of the group $\mathscr{L}_p^+ \cap \mathscr{L}_q^+$. Let us put $\bigcup \mathscr{S} := \{(t, f(t));\ t \in \mathbf{R}, f \in \mathscr{S}\}$. Say that $\mathscr{S}$ is dense in $\mathbf{R} \times \mathbf{R}$ if $\overline{\bigcup \mathscr{S}} = \mathbf{R} \times \mathbf{R}$ (see [1]). In a special case with $\bigcup \mathscr{S} = \mathbf{R} \times \mathbf{R}$ we will say that the group $\mathscr{S}$ is planar (see [6]).

3. Lemmas

Lemma 1. *Let $p \neq q$ and the group $\mathscr{L}_p^+ \cap \mathscr{L}_q^+$ be dense in* $\mathbf{R} \times \mathbf{R}$. *Then $p(t) \neq q(t)$ for $t \in \mathbf{R}$.*

Proof. Let $p(t_0) = q(t_0)$ for a $t_0 \in \mathbf{R}$. Let $x \in \mathbf{R}$. Then there exists a sequence $\{X_n\}$, $X_n \in \mathscr{L}_p^+ \cap \mathscr{L}_q^+$ such that $\lim_{n \to \infty} X_n(t_0) = x$. It follows from the equalities

$$-\{X_n, t\} + X_n'^2(t)\, p[X_n(t)] = p(t),$$
$$-\{X_n, t\} + X_n'^2(t)\, q[X_n(t)] = q(t),$$

that

$$X_n'^2(t)\,[q[X_n(t)] - p[X_n(t)]] = q(t) - p(t), \qquad t \in \mathbf{R}, \qquad n = 1, 2, 3, \ldots \quad (1)$$

Setting $t = t_0$ in (1) yields

$$q[X_n(t_0)] - p[X_n(t_0)] = 0, \qquad n = 1, 2, 3, \ldots \quad (2)$$

Passing to the limit in (2) ($n \to \infty$) gives $p(x) = q(x)$, whereupon $p = q$, which, however, contradicts with our assumption.

Remark 1. The statement of Lemma 1 was proved in [6] and [12] in assuming the group $\mathscr{L}_p^+ \cap \mathscr{L}_q^+$ to be planar.

Lemma 2. *Let* b $(\geqq 0)$ *be a number,* $\varepsilon_n \in \mathfrak{E}$, sign $\varepsilon_n' = 1$, $\varepsilon_n(0) = 0$, $\lim_{n\to\infty} \varepsilon_n'(0) = b$, $\lim_{n\to\infty} |\varepsilon_n''(0)| = \infty$. *Then*

$$\lim_{n\to\infty} \varepsilon_n(t) = 0 \qquad \textit{for } t \in [0, \pi).$$

Proof. Suppose that $\varepsilon_n \in \mathfrak{E}$, sign $\varepsilon_n' = 1$. Then there exists the following sequences of numbers $\{A_n\}$, $\{B_n\}$, $\{C_n\}$ and $\{D_n\}$, $A_nD_n - B_nC_n < 0$ such that

$$\operatorname{tg} \varepsilon_n(t) = \frac{A_n \cos t + B_n \sin t}{C_n \cos t + D_n \sin t}, \quad (3)$$

wherever both sides of (3) are meaningful. Let $\varepsilon_n(0) = 0$. Then $A_n = 0$, $C_n \neq 0$. Furthermore we obtain from (3)

$$\varepsilon_n'(t) = \frac{B_nC_n}{B_n^2 \sin^2 t + (C_n \cos t + D_n \sin t)^2},$$

whence

$$\varepsilon_n''(t) = -2B_nC_n \frac{B_n^2 \sin t \cos t + (C_n \cos t + D_n \sin t)(-C_n \sin t + D_n \cos t)}{[B_n^2 \sin^2 t + (C_n \cos t + D_n \sin t)]^2}.$$

Thus $\varepsilon_n'(0) = \dfrac{B_n}{C_n}$, $\varepsilon_n''(0) = -2\dfrac{B_nD_n}{C_n^2}$. Suppose $\lim_{n\to\infty} \varepsilon_n'(0) = b$, $\lim_{n\to\infty} |\varepsilon_n''(0)| = \infty$. Then $\lim_{n\to\infty} \dfrac{B_n}{C_n} = b$, $\lim_{n\to\infty} \left|\dfrac{B_nD_n}{C_n^2}\right| = \infty$ and consequently $\lim_{n\to\infty} \left|\dfrac{D_n}{C_n}\right| = \infty$, $\lim_{n\to\infty} \left|\dfrac{D_n}{B_n}\right| = \infty$. Since (3) may be written as ($A_n = 0$)

$$\operatorname{tg} \varepsilon_n(t) = \frac{\dfrac{B_n}{C_n} \sin t}{\cos t + \dfrac{D_n}{C_n} \sin t},$$

we obtain $\lim_{n\to\infty} \operatorname{tg} \varepsilon_n(t) = 0$ for $t \in [0, \pi)$. We see that $\lim_{n\to\infty} \varepsilon_n(t) = 0$ for all these t.

Corollary 1. *Let a, b ($\geqq 0$) be numbers, $\varepsilon_n \in \mathfrak{E}$, $\operatorname{sign} \varepsilon'_n = 1$, $\lim_{n\to\infty} \varepsilon_n(0) = a$, $\lim_{n\to\infty} \varepsilon'_n(0) = b$, $\lim_{n\to\infty} |\varepsilon''_n(0)| = \infty$. Then $\lim_{n\to\infty} \varepsilon_n(t) = a$ for $t \in [0, \pi)$.*

Proof. Setting $\tau_n(t) := \varepsilon_n(t) - \varepsilon_n(0)$, $t \in \mathbf{R}$, $n = 1, 2, 3, \ldots$ Then $\tau_n \in \mathfrak{E}$, $\tau_n(0) = 0$, $\lim_{n\to\infty} \tau'_n(0) = \lim_{n\to\infty} \varepsilon'_n(0) = b$ and $\lim_{n\to\infty} |\tau''_n(0)| = \lim_{n\to\infty} |\varepsilon''_n(0)| = \infty$. Following Lemma 2 $\lim_{n\to\infty} \tau_n(t) = 0$ for $t \in [0, \pi)$, whence we get $\lim_{n\to\infty} \varepsilon_n(t) = \lim_{n\to\infty} (\tau_n(t) - \varepsilon_n(0)) = a$ for $t \in [0, \pi)$.

4. Main result

Theorem 1. *Suppose $p \neq q$. Then the group $\mathscr{L}_p^+ \cap \mathscr{L}_q^+$ is either trivial or infinite cyclic or planar.*

Proof. Supposing $\mathscr{S} := \mathscr{L}_p^+ \cap \mathscr{L}_q^+$ it follows from the main theorem in [5] that the group $\mathscr{S}$ is either trivial or infinite cyclic or it is a dense group in $\mathbf{R}\times\mathbf{R}$. To prove this is suffices to show that $\bigcup \mathscr{S} = \mathbf{R}\times\mathbf{R}$ in assuming $\overline{\bigcup \mathscr{S}} = \mathbf{R}\times\mathbf{R}$. Thus suppose $\overline{\bigcup \mathscr{S}} = \mathbf{R}\times\mathbf{R}$ and $\bigcup \mathscr{S} \neq \mathbf{R}\times\mathbf{R}$. Then there necessarily exists a point $(0, a) \in \mathbf{R}\times\mathbf{R} - \bigcup \mathscr{S}$. By Lemma 8 [1] there exists a planar group $\mathscr{S}_1$ such that $\mathscr{S}$ is its subgroup and the elements $\mathscr{S}_1$ are continuous functions. Let $X(0) = a$ for $X \in \mathscr{S}_1$. From our assumption $\overline{\bigcup \mathscr{S}} = \mathbf{R}\times\mathbf{R}$ we observe that there exists a sequence $\{X_n\}$, $X_n \in \mathscr{S}$ on every compact interval uniformly converging from above to the function X (see Theorem 7.13. [8]):

$$X(t) \leqq \ldots \leqq X_n(t) \leqq \ldots \leqq X_1(t), \qquad t \in \mathbf{R}.$$

Let us prove that $X \in \mathscr{S}$. Since X_n is for every natural number n a joint solution of the Kummer differential equations

$$-\{Z, t\} + Z'^2 p(Z) = p(t),$$
$$-\{Z, t\} + Z'^2 q(Z) = q(t),$$

it suffices with respect to Theorem 2.9. [7] and $\lim_{n\to\infty} X_n(0) = a$ to prove the existence of numbers $b(\geqq 0)$, c, and a selected sequence $\{X_{n_k}\}$ form the sequence $\{X_n\}$ satisfying

$$\lim_{k\to\infty} X'_{n_k}(0) = b, \qquad \lim_{k\to\infty} X''_{n_k}(0) = c. \tag{4}$$

We obtain from Lemma 1 that $p(t) \neq q(t)$ for $t \in \mathbf{R}$ and from (1)

$$X'_n(t) = \sqrt{\frac{q(t) - p(t)}{q[X_n(t)] - p[X_n(t)]}}, \qquad t \in \mathbf{R},\ n = 1, 2, 3, \ldots \tag{5}$$

Thus, $\{X'_n(t)\}$ uniformly converges on every compact interval to a continuous function written as U. It follows from the equality $\lim_{n\to\infty} X'_n(0) = a$ that $U(t) = X'(t)$ for $t \in \mathbf{R}$. Passing to the limit in (5) we get

$$\lim_{n\to\infty} X_n'(0) = \sqrt{\frac{q(0)-p(0)}{q(a)-p(a)}} := b(>0). \tag{6}$$

Suppose that no subsequence $\{X_{n_k}\}$ may be selected from $\{X_n\}$ that would satisfy (4), with b being degined in (6) and c being a number. Then, necessarily

$$\lim_{n\to\infty} |X_n''(0)| = \infty. \tag{7}$$

Let α be a phase of (q), $\alpha(0) = 0$, $\alpha'(0) = 1$, $\alpha''(0) = 0$. There exists a sequence $\{\varepsilon_n\}$, $\varepsilon_n \in \mathfrak{E} : X_n = \alpha^{-1}\varepsilon_n\alpha$. Since $a = \lim_{n\to\infty} X_n(0) = \lim_{n\to\infty} \alpha^{-1}\varepsilon_n\alpha(0) = \lim_{n\to\infty} \alpha^{-1}\varepsilon_n(0)$, we get

$$\lim_{n\to\infty} \varepsilon_n(0) = \alpha(a). \tag{8}$$

From the equalities

$$X_n' = \frac{\varepsilon_n'\alpha\,\alpha'}{\alpha'\alpha^{-1}\varepsilon_n\alpha},$$

$$X_n'' = -\frac{\alpha''\alpha^{-1}\,\varepsilon_n\alpha\,(\varepsilon_n'\alpha\,.\,\alpha')^2}{(\alpha'\alpha^{-1}\varepsilon_n\alpha)^3} + \frac{\varepsilon_n''\alpha\,\alpha'^2 + \varepsilon_n'\alpha\,\alpha''}{\alpha'\alpha^{-1}\varepsilon_n\alpha},$$

and from (6), (7) we obtain

$$\lim_{n\to\infty} \varepsilon_n'(0) = b\,\alpha'(a), \qquad \lim_{n\to\infty} |\varepsilon_n''(0)| = \infty. \tag{9}$$

Since (8) and (9) hold for $\{\varepsilon_n\}$, it follows from Corollary 1 that $\lim_{n\to\infty} \varepsilon_n(t) = \alpha(a)$ for $t \in [0, \pi)$, hence $X(t) = \lim_{n\to\infty} X_n(t) = \lim_{n\to\infty} \alpha^{-1}\varepsilon_n\alpha(t) = a$ for $t \in [0, \alpha^{-1}(\pi))$. Then, naturally, $X'(0) = 0$ contrary to $X'(0) = \lim_{n\to\infty} X_n'(0) = b > 0$. Thus $X \in \mathscr{S}$ and the point $(0, a) \in \bigcup\mathscr{S}$. So we proved that $\bigcup\mathscr{S} = \mathbf{R}\times\mathbf{R}$.

REFERENCES

[1] Blanton, G. and Baker, J. A.: *Iteration group generated by C^n functions.* Arch. Math. (Brno), 3, XVIII, 1982, 121–128.

[2] Borůvka, O.: *Linear Differential Transformations of the Second Order.* The English Universities Press, London, 1971.

[3] Борувка, О.: *Теория глобальных свойств обыкновенных линейных дифференциальных уравнений второго порядка.* Дифференциальные уравнения, N° 8, т. XII, 1976, 1347—1383.

[4] Borůvka, O.: *Sur une classe de groupes continus á un paramètre formes des fonctions réelles d'une variable.* Ann. Polon. Math., XLII (1982), 27–37.

[5] Borůvka, O.: *Sur les Transformations Simultanées de Deux Équations Différentielles Linéarie du Deuxième Ordre dans Elles-mêmes.* Applicable Analysis, 1983, No 1—4, 187–200.

[6] Borůvka, O.: *Lectures at the seminar of the Institute of Mathematics of the Czechoslovak Academy of Science in Brno.*

[7] Hartman, P.: *Ordinary Differential Equations* (in Russian). Moscow, 1970.

[8] Rudin, W.: *Principle of Mathematical Analysis* (in Russian), Moscow, 1966.

[9] Staněk, S.: *On some properties of soluecontions of the sd order linear differential equation with periodic coefficients having a common oneparametric continuous group of dispersion.* Acta Univ. Palackianae Olomucensis F. R. N., 69, 1981, 93–99.

[10] Staněk, S.: *On a structure of the intersection of the set of dispersions of two second-order linear differential equations.* Acta Univ. Palackianae Olomucensis F. R. N., 73, 1982, 79–85.

[11] Staněk, S.: *On the intersection of groups of dispersions of the eqιation $y'' = q(t)\,y$ and its accompanying equation.* Acta Univ. Palackianae Olomucensis F. R. N., 79, 1984, 33—37.

[12] Staněk, S.: *On the intersection of the set of solutions of two Kummer's differential equations.* Acta Univ. Palackianae Olomucensis F. R. N., 79, 1984, 25–32.

[13] Staněk, S.: *Common increasing dispersions of certain linear second order differential equations.* Acta Univ. Palackianae Olomucensis F. R. N., 82, 1985, 51–60.

PRŮNIK GRUP ROSTOUCÍCH DISPERSÍ DVOU OSCILATORICKÝCH LINEÁRNÍCH DIFERENCIÁLNÍCH ROVNIC 2. ŘÁDU

Souhrn

Řekneme, že funkce $X \in C^3(\mathbf{R})$, $X'(t) \neq 0$ pro $t \in \mathbf{R}$, je disperse (1. druhu) oscilatorické rovnice (q) : $y'' = q(t)\,y$, $q \in C^0(\mathbf{R})$, jestliže X je řešením rovnice

$$-\{X, t\} + X'^2\, q(X) = q(t),$$

kde $\{X, t\} = \frac{1}{2}\frac{X'''(t)}{X'(t)} - \frac{3}{4}\left(\frac{X''(t)}{X'(t)}\right)^2$. Množina rostoucích dispersí oscilatorické rovnice (q) tvoří vzhledem k operaci skládání funkce grupu $\mathscr{L}_q^+$. Nechť $\mathscr{S} \subset \mathscr{L}_q^+$ je podgrupa grupy $\mathscr{L}_q^+$. Řekneme, že $\mathscr{S}$ je planární grupa, jestliže ke každému bodu $(x_0, y_0) \in \mathbf{R}\times\mathbf{R}$ existuje jediná funkce $X \in \mathscr{S}$ taková, že $X(x_0) = y_0$. Nechť (p) : $y'' = p(t)\,y$, $p \in C^0(\mathbf{R})$ je oscilatorická rovnice. Je dokázáno, že $\mathscr{L}_p^+ \cap \mathscr{L}_q^+$ je buď triviální grupa, nebo nekonečná cyklická grupa anebo planární grupa.

О ПЕРЕСЕЧЕНИИ ГРУПП ВОЗРАСТАЮЩИХ ДИСПЕРСИЙ ДВУХ КОЛЕБЛЮЩИХСЯ ЛИНЕЙНЫХ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ 2-ОГО ПОРЯДКА

Резюме

Функция $X \in C^3(\mathbf{R})$, $X'(t) \neq 0$ для $t \in \mathbf{R}$, называется дисперсией (1-ого рода) колеблющегося уравнения (q) : $y'' = q(t)y$, $q \in C^\circ(\mathbf{R})$, если X решением уравнения

$$-\{X, t\} + X'^2\, q(X) = q(t),$$

где $\{,X\,t\} = \frac{1}{2}\frac{X'''(t)}{X'(t)} - \frac{3}{4}\left(\frac{X''(t)}{X'(t)}\right)^2$. Множество возрастающих дисперсий колеблющегося уравнения (q) является относительно операции сложения функций группой $\mathscr{L}_q^+$. Пусть $\mathscr{S} \subset \mathscr{L}_q^+$ подгруппа группы $\mathscr{L}_q^+$. $\mathscr{S}$ называется планарной группой если для каждой точки $(x_0, y_0) \in \mathbf{R} \times \mathbf{R}$ существует только одна функция $X \in \mathscr{S}$, что $X(x_0) = y_0$. Пусть (p) : $y'' = p(t)y$ колеблющиеся уравнение. Показывается, что $\mathscr{L}_p^+ \cap \mathscr{L}_q^+$ или тривиальная группа, или бесконечная группа, или планарная группа.