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GENERALIZATION OF THE SHIRALI—FORD THEOREM IN HERMITIAN LOCALLY MULTIPLICATIVELY CONVEX ALGEBRAŠ

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ABSTRACT

The Shirali Ford theorem, s.f. Shirali S., Ford J. W. M. (1970), which had been an outstanding conjecture for some years, asserts that $\sigma(x^*x) \geqq 0$ for every element x of a Hermitian Banach star algebra A. The present paper will show that this is true also for every complete locally multiplicatively convex Hermitian algebra.

1. NOTATIONS AND PRELIMINARIES

All linear spaces are over the complex field C. The reader is assumed to be familiar with the basic concepts concerning the topological algebras, namely the Banach algebras, Banach star algebras, locally multiplicatively convex (lmc) star algebras, including spectra, Gelfand representation for the commutative case and so on. See Bonsall F., Duncan J. (1973), Michael E. (1952), Najmark A. (1968), Želazko W. (1972). Let us now recall some notations and preliminary facts. Let A be a complete *lmc* algebra. Without any loss of generalitity we can assume the topology of A being given by a family $\{q_\alpha\}_{\alpha\in\Sigma}$ of submultiplicative seminorms on A separating points in A for which Σ is directed in a natural way by the relation $\alpha < \beta$ if and only if q_α is continuous with respect to q_β. It is a well known and widely utilized fact that A is topologicaly isomorphic to the projeptive limit of Banach algebras A_α, $\alpha \in \Sigma$, where A_α denotes the completion of the normed algebra $(A/Ker\, q_\alpha, q_\alpha)$. Speaking more closely $\pi(A) = \lim\limits_{\leftarrow} A_\alpha$, where π denotes the natural isomorphic mapping from A into $\prod\limits_{\alpha\in\Sigma} A_\alpha$, $\pi(x) = (\pi_\alpha(x))_{\alpha\in\Sigma}$ and π_α denotes the natural homomorphism mapping from A onto A_α. Here by the system of Banach algebras $(A_\alpha, \alpha \in \Sigma)$ forms a projective system with respect to the set of continuous homomorphism mappings $\pi_{\alpha\beta} : A_\beta \to A_\alpha$, $\pi_{\alpha\beta}(\pi_\beta(x)) = \pi_\alpha(x)$ for each $x \in A$ whenever $\alpha < \beta$. Throughout the spectrum of an element $x \in A$ will be denoted by $\sigma(x, A)$ pointing out that it is taken with respect to algebra A. The spectral radius will be denoted by $| x |_\sigma^A$. Obviously an element $x \in A$ is regular if and ounly if for each $\alpha \in \Sigma$ $\pi_\alpha(x)$ is regular in the algebra A_α yielding the equality $\sigma(x, A) = \bigcup_{\alpha \in \Sigma} \sigma(\pi_\alpha(x), A_\alpha)$. Then for the spectral radius $| x |_\sigma^A = \sup | \pi_\alpha(x) |_\sigma^A$. Recall now that the spectrum of an element in the Banach algebras is always a compact set of C. Our situation is more general: the spectrum of an element in the *lmc* algebra need not be necessarily bounded.

Definition 1.1. The element x from a star algebra A is said to be Hermitian, if $x^* = x$. The set of all Hermitian elements of A will be denoted by $H(A)$. The element $x \in A$ is said to be normal if $xx^* = x^*x$. The set of all normal elements of A will be denoted by $N(A)$.

Definition 1.2.: The star algebra A is said to be Hermitian if the spectrum $\sigma(x)$ is real for any $x \in H(A)$.

Note 1.3.: Let A be a *lmc* star algebra $\{q_\alpha, \alpha \in \Sigma\}$ the corresponding family of seminorms, x an arbitrary element of A. If no confusion is possible we shall use the following notations: for every $\alpha \in \Sigma$ $\pi_\alpha(x) = x$, $| \pi_\alpha(x) |_\sigma^A = | x |_\sigma^\alpha$, $p(\pi_\alpha(x)) = \sqrt{| (x^*x) |_\sigma^A} = p_\alpha(x)$, $p(x) = \sqrt{| x^*x |_\sigma}$.

In the reminder of this section let us recall two theorems playing a substantial role in our further considerations.

Theorem 1.4.: Let A be a complete *lmc*-algebra with a unit element e. Let $N \subset A$ be the set of pairwise commuting elements. Then N is contained in a maximal closed and commutative subalgebra $B \subset A$. Moreover, for each $x \in B$ we have $\sigma(x, B) = \sigma(x, A)$. Proof: See Štěrbove (1983 AUPO).

Theorem 1.5.: Let A be same as in Theorem 1.4. Denote by $\{q_\alpha, \alpha \in \Sigma\}$ the corresponding directed set of seminorms, defining the topology on A. Then the following conditions are equivalent:

(i) The algebra A is Hermitian.

(ii) $| \pi_\alpha(x) |_\sigma^A \leqq p(\Pi_\alpha(x)) = p_\alpha(x)$ for each normal element $x \in A$ and for all $\alpha \in \Sigma$. Proof: See Štěrbová (AUPO 1982).

2. THE GENERALIZED SHIRALI–FORD THEOREM

Throughout this section all algebras are supposed to by Hermitian complete *lmc* star algebras with a unit element e.

Lemma 2.1.: For arbitrary $a, bH(A)$ and for arbitrary $\alpha \in \Sigma$ the following inequality holds:

$$|a^2b^2|_\sigma^\alpha \leqq |a^2|_\sigma^\alpha |b^2|_\sigma^\alpha.$$

Proof:

Since $ab(ab)^* \in H(A) \subset N(A)$ Theorem 1.5. yields for every $\alpha \in \Sigma$ $|ab(ab)^*|_\sigma^\alpha \leqq \leqq p_\alpha(ab(ab)^*)$. A simple computation demonstrates further:

$$|ab(ab)^*|_\sigma^\alpha = |abba|_\sigma^\alpha = |a^2b^2|_\sigma^\alpha \leqq p_\alpha(abba) =$$
$$= (|(abba)\,abba)|_\sigma^\alpha)^{1/2} = (|ab^2a^2b^2a|_\sigma^\alpha)^{1/2} = (|(a^2b^2)^2|_\sigma^\alpha)^{1/2}. \qquad (1)$$

Now we use the wellknown relation between the spectral radius and the norm in Banach algebras, see Bonsall F., Duncan J. (1973), to get easily:

$$|a^2b^2|_\sigma^\alpha \leqq (q_\alpha(a^2b^2)^2)^{1/2} \leqq (q_\alpha((a^2)^2))^{1/2} \, . \, (q_\alpha((b^2)^2))^{1/2}. \qquad (2)$$

By the submultiplicativity of the seminorm q_α it follows for every integer $p = 2^m$, $m = 1, 2, \ldots$ by the matematical induction:

$$|a^2b^2|_\sigma^\alpha \leqq (q_\alpha((a^2)^p))^{1/p} \, . \, (q_\alpha((b^2)^p))^{1/p}. \qquad (3)$$

The last inequality (3) together with the same argument as used in (2) implies immediately:

$$|a^2b^2|_\sigma^\alpha \leqq \lim_{m\to\infty} (q_\alpha((a^2)^p))^{1/p} \lim_{m\to\infty} (q_\alpha(b^2)^p))^{1/p}) = |a^2|_\sigma^\alpha |b^2|_\sigma^\alpha \qquad \text{Q. E. D.}$$

Corollary 2.2.: For arbitrary $a, b \in H(A)$ the following inequality holds:

$$|a^2b^2|_\sigma \leqq |a^2|_\sigma |b^2|_\sigma.$$

Proof follows easily if we take suprema with respect to Σ both sides of the inequality in 2.1.

Q. E. D.

Proposition 2.3.: Let $a, b \in H(A)$. If $\sigma(a, A) > 0$, $\sigma(b, A) > 0$, then $\sigma(a + b) \geqq 0$.

Proof:

It is sufficient to show that $-1 \in \sigma(a + b)$ meaning the regularity of the element $(e + a + b)$. The Gelfand representation theory and 1.4. yield the regularity of elements $(e + a)$, $(e + b)$. Further

$$e + a + b = (e + a)(e + b) - ab = (e + a)(e - uv)(e + b), \qquad (1)$$

where

$$u = (e + a)^{-1} \, . \, a, \qquad v = b(e + b)^{-1}.$$

Obviously $u, v \in H(A)$ and their spectra are positive. Further for every $\alpha \in \Sigma$ by 1.4. and the Gelfand representation theory the following inequalities are true

$$|u|_\sigma^\alpha < 1, \qquad |v|_\sigma^\alpha < 1.$$

By Theorem 3.10. Štěrbová (1980) there exist square roots $o_1, o_2 \in A$ of elements u, v so that for every $\alpha \in \Sigma$ $|o_1|_\sigma^\alpha < 1$, $|o_2|_\sigma^\alpha < 1$ and the spectra of o_1, o_2 are positive, too. Now, applying Lemma 2.1. a simple computation gives for every $\alpha \in \Sigma$:

$$| uv |_\sigma^\alpha \leqq | o_1^2 |_\sigma^\alpha \, | o_2^2 |_\sigma^\alpha < 1.$$

The last inequality yields immediately for every $\alpha \in \Sigma$ $1 \notin \sigma_\alpha(uv)$ and so $1 \notin \sigma(uv)$. This means that the element $(e - uv)$ is regular and therefore by (1) the element $(e + a| + b)$ is regular, too. So we got $\sigma(a + b) \geqq 0$.

Q. E. D.

Theorem 2.4.: Let $x \in A$ be given with $p(x) < \infty$. Then $\sigma(x^*x) \geqq 0$.
Proof:
The above proposition with the fact that $p(x) < \infty$ enables us to prove this version of Shirali – Ford theorem analogous to Shirali S., Ford J. W. m. (1970):

Let $\delta = \sup \{-\lambda : \lambda \in \sigma(a^*a),\ a \in A,\ p(a) \leqq 1.$ Assume $\delta > 0$. Then there exists $a \in A$, $\zeta \in \sigma(a^*a)$ such that $p(a) < 1$, $-\zeta > \frac{1}{4}\delta$. Let $b = (2a)(e + aa)^{-1}$, which exists because of $| a^*a |_\sigma < 1$. Then

$$e - b^*b = (e - a^*a)^2 (e + a^*a)^{-2}.$$

Hence by the Gelfand representation theory of commutative algebras

$$\sigma(b^*b) = \{1 - (\Phi(\lambda))^2;\ \lambda \in \sigma(a^*a)\},$$

where

$$\Phi(\lambda) = \frac{1 - \lambda}{1 + \lambda}. \tag{2}$$

Thus $\sigma(b^*b) \subset (-\infty, 1\rangle$.
Let $b = h + ik$, where $h, k \in H(A)$. Then

$$bb^* = 2h^2 + 2k^2 - b^*b.$$

By Proposition 2.3. we have $\sigma(2h^2 + 2k^2 - b^*b + e) \geqq 0$ and therefore $\sigma(bb^*) \subset$ $\subset \langle -1, \infty)$. By proposition 3.5. cited in Bonsall F., Duncan J. (1973), we no have

$$\sigma(b^*b) \subset \langle -1, 1\rangle, \qquad p(b) \leqq 1.$$

By definition of δ

$$-\{1 - (\Phi(\zeta))^2\} \leqq \delta, \qquad \Phi(\zeta) = (1 + \delta)^{1/2}.$$

Since $\Phi(\Phi(\zeta)) = \zeta$, and Φ is decreasing, it follows that $\zeta \geqq \Phi\{(1 + \delta)^{1/2}\}$, and

$$-\xi \leqq \frac{(1 + \delta)^{1/2} - 1}{(1 + \delta)^{1/2} + 1} \leqq \frac{(1/2)\delta}{2} = \frac{\delta}{4}.$$

This contradiction shows that $\delta \leqq 0$, as required.

Q. E. D.

Now we are able to state the main result.

Theorem 2.5.: Let arbitrary $x \in A$ be given. Then $\sigma(x^*x) \geqq 0$.
Proof:
Let us suppose that the required inequality does not hold for some $x \in A$. By our assumption made at the beginning of this section, the algebra A is Hermitian wherefor there exists $y = x(e + (x^*x)^2)^{-1}$: it next holds

$$yy^* = x(e + (x^*x)^2)^{-2} . (e + (x^*x)^2)^{-1}x^* = x(e + (x^*x)^2)^{-2}x^*$$

and

$$\sigma(yy^*)/0 = \sigma(x(e + (x^*x)^2)^{-2}x^*)/0 = \sigma(x^*x(e + (x^*x)^2)^{-2})/0.$$

Using the Gelfand representation theory of commutative algebras and Theorem 1.4. we easily get $\sigma(yy^*) < 0$ and

$$| yy^* |_\sigma \leqq \sup_{t \in (-\infty, \infty)} \frac{t}{(1 + t^2)^2} = \frac{9}{16\sqrt{3}} < 1.$$

(We used the fact that the function $\frac{t}{(1 + t^2)^2}$ assumes its maximum at the point $1/\sqrt{3}$ as shown in Sa-Do-Šin (1959).) Hence

$$\sup \{ -\lambda : \lambda \in \sigma(a^*a), a \in A, p(a) < 1 \} = \delta > 0,$$

in contrary to Theorem 2.4.

Q. E. D.

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ZOBECNĚNÁ SHIRALI—FORDOVA VĚTA V lmc-ALGEBŘE

Souhrn

Shirali—Fordova věta tvrdí, že pro každý prvek x Hermiteovské Banachovy algebry s involucí A je spektrum prvku x^*x nezáporné. V předložené práci je tento výsledek dokázaný pro každou úplnou Hermiteovskou lokálně m-konvexní algebru s involucí.

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ОБ ТЕОРЕМЕ ШИРАЛИ—ФОРДА В ПОЛУНОРМИРОВАННЫХ СИММЕТРИЧЕСКИХ КОЛЬЦАХ С ИНВОЛЮЦИЕЙ

Резюме

Классическая теорема Ширали—Форда утверждает, что для каждого элемента x симметриеческого польного нормированного кольца с инволюцией спектр элемента выполняется неравенство $\sigma(x^*x) \geqq 0$. В настоящей статье показывается, что это утверждение правдиво и для полных симметрических полунормированных колец с инволюцией.