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A NOTE ON A QUARTIC WITH A TACNODE AND A FLEXNODE

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There is an excellent article by academician Bohumil Bydžovský [1] where the following problem is formulated: Given a plane quartic (**K**) whose inflection-point divisor **I** has the order cancelable by 4. Under what conditions the **I** is determined on (**K**) by a suitable form $\overline{\mathbf{K}}$? This problem is devoted in the same paper [1] for the cases:

(a) (**K**) has two ordinary nodes and no more point-singularities ($\Rightarrow$ ord $\mathbf{I} = 12$),
(b) (**K**) has two flexnodes and no more point-singularities ($\Rightarrow$ ord $\mathbf{I} = 8$),
(c) (**K**) has two cusps and no more point-singularities ($\Rightarrow$ ord $\mathbf{I} = 8$),
(d) (**K**) has two ordinary nodes and one ordinary cusp and no more point-singularities ($\Rightarrow$ ord $\mathbf{I} = 4$)
and obvious case
(e) (**K**) is without point-singularities ($\Rightarrow$ ord $\mathbf{I} = 24$). Additionally, the following cases are studied:
(f) (**K**) has just one ordinary cusp and no more point-singularities ($\Rightarrow$ ord $\mathbf{I} = 16$) [2],
(g) (**K**) has just one flexnode and no more singularities ($\Rightarrow$ ord $\mathbf{I} = 16$) [3],
(h) (**K**) has just one tacnode and no more singularities ($\Rightarrow$ ord $\mathbf{I} = 12$ ("in generally")) [4].

In the article [1] there is described an elementary method for investigating the inflection-point divisor used also in [3], [4] as well as in the present paper devoted to the same problem as [1] – [4] for the plane quartic (**K**) characterized in the title. Moreover we will also evalue the class and the genus of such a curve.

1. Let $\mathbf{S}_2$ be a projective plane (over the field of complex numbers **C**) and (**K**) be quartic considered in $\mathbf{S}_2$ possessing a tacnode as well as a flexnode and no more

point-singularities. Let us choose a projective coordinate frame $(\boldsymbol{A}_0, \boldsymbol{A}_1, \boldsymbol{A}_2, \boldsymbol{E})$ ($\boldsymbol{E}$ is the unity point) as in fig. 1. i.e. let $\boldsymbol{A}_0$ be the tacnode of $(\mathbf{K})$, $\boldsymbol{A}_2$ the flexnode of $(\mathbf{K})$, $\boldsymbol{A}_1$ the intersection-point of the just one tangent $\boldsymbol{a}$ at $\boldsymbol{A}_0$ and of an arbitrary tangent $\boldsymbol{b}$ at $\boldsymbol{A}_2$ and let the unity-point $\boldsymbol{E}$ be situated on the remaining tangent of $\boldsymbol{A}_2$.

With respect to our coordinate frame the quartic $(\mathbf{K})$ is determined by the form $\mathbf{K} = \mathbf{K}(x_0, x_1, x_2)$

$$\mathbf{K}(x_0, x_1, x_2) = ax_0^2x_2^2 - ax_0x_1x_2^2 + ex_1^4, \qquad a \neq 0, e \neq 0. \tag{1}$$

In what follows let for any form $\mathbf{F} = \mathbf{F}(x_0, x_1, x_2)$ ($\deg \mathbf{F} \geqq 1$; over the field $\boldsymbol{C}$) the $(\mathbf{F})$ denote the curve (more exactly: the divisor) in $\boldsymbol{S}_2$ determined by the form $\mathbf{F}$.

The tacnode $\boldsymbol{A}_0$ is the centre of two linear places (branches) $\mathsf{P}_0, \mathsf{P}_0'$ with the common tangent $\boldsymbol{a}$, each is of class 1; the flexnode $\boldsymbol{A}_2$ is the centre of two linear places $\mathsf{P}_2, \mathsf{P}_2'$ with different tangents $\boldsymbol{b}, \boldsymbol{c}$; each of places $\mathsf{P}_2, \mathsf{P}_2'$ of class 2.

The places $\mathsf{P}_0, \mathsf{P}_0'$ have the following parametrizations:

$$\begin{array}{ll} \mathsf{P}_0:\ \bar{x}_0 = 1 & \mathsf{P}_0':\ \bar{x}_0' = 1 \\ \quad\ \bar{x}_1 = t & \quad\ \bar{x}_1' = t \\ \quad\ \bar{x}_2 = t^2(m + \ldots) & \quad\ \bar{x}_2' = t^2(m' + \ldots) \\ \quad\ am^2 + e = 0\ (\Rightarrow m \neq 0) & \quad\ am'^2 + e = 0\ (\Rightarrow m' \neq 0) \end{array} \tag{2}$$
$$m \neq m'$$

Similarly, the places $\mathsf{P}_2, \mathsf{P}_2'$ have the parametrizations:

$$\begin{array}{ll} \mathsf{P}_2:\ \bar{x}_0 = t^3(u + \ldots) & \mathsf{P}_2':\ \bar{x}_0' = t + t^3(v + \ldots) \\ \quad\ \bar{x}_1 = t & \quad\ \bar{x}_1' = t \\ \quad\ \bar{x}_2 = 1 & \quad\ \bar{x}_2' = 1 \\ \quad\ au - e = 0\ (\Rightarrow u \neq 0) & \quad\ av + e = 0\ (\Rightarrow v \neq 0) \end{array} \tag{3}$$

2. *The class of* $(\mathbf{K})$. Let us choose a point $\boldsymbol{C} \in \boldsymbol{S}_2$ and let us put for any place P of $(\mathbf{K})$:

$$\varepsilon_{\boldsymbol{C}}(\mathsf{P}) \begin{cases} = 0, \text{ if the tangent of } \mathsf{P} \text{ does not contain } \boldsymbol{C} \\ = \text{the degree (order) of } \mathsf{P}, \text{ if the tangent of } \mathsf{P} \text{ contains } \boldsymbol{C}, \text{ but the centre of } \mathsf{P} \\ \quad \text{is different from } \boldsymbol{C} \\ = \text{the sum of the degree and of the class of } \mathsf{P}, \text{ if } \boldsymbol{C} \text{ is the centre of } \mathsf{P}. \end{cases}$$

Then the order of the divisor $\mathsf{T}_{\boldsymbol{C}} = \Sigma\varepsilon_{\boldsymbol{C}}(\mathsf{P})\,\mathsf{P}$ is well defined and is independent of the point $\boldsymbol{C}$ and is equal to the class τ of $(\mathbf{K})$ (cf. [5] pg. 116–117). On the other hand, if $\boldsymbol{C} = (c_0, c_1, c_2)$ and $\mathbf{F} = \Sigma c_i\mathbf{K}_i$ is the first polar of $\boldsymbol{C}$ with respect to $\mathbf{K}$ ($\mathbf{K}_i$ denotes the partial derivate $\partial\mathbf{K}/\partial x_i$ as usually), then

$$\mathsf{O}_{\mathsf{P}}(\mathbf{F}) = \delta_{\boldsymbol{C}}(\mathsf{P}) + \varepsilon_{\boldsymbol{C}}(\mathsf{P}),$$

where $\delta_{\boldsymbol{C}}(\mathsf{P}) \geqq 0$. It is well known, that for any place P, whose centre is a regular point of $(\mathbf{K})$ it holds $\mathsf{O}_{\mathsf{P}}(\mathbf{F}) = \varepsilon_{\boldsymbol{C}}(\mathsf{P})$ ($\Rightarrow \delta_{\boldsymbol{C}}(\mathsf{P}) = 0$). Now, let us put $\boldsymbol{C} = \boldsymbol{A}_2$, then $\mathbf{F} = \mathbf{K}_2 = 2ax_0x_2(x_0 - x_1)$. Substituting from (2) and (3) into $\mathbf{K}_2$, we establish,

$$O_{P_0}(K_2) = O_{P_0'}(K_2) = 2, \qquad O_{P_2}(K_2) = O_{P_2'}(K_2) = 4.$$

As $\varepsilon_{A_2}(P_0) = \varepsilon_{A_2}(P_0') = 0$, $\varepsilon_{A_2}(P_2) = \varepsilon_{A_2}(P_2') = 3$, we have

$$\delta_{A_2}(P_0) = \delta_{A_2}(P_0') = 2, \qquad \delta_{A_2}(P_2) = \delta_{A_2}(P_2') = 1.$$

For any place P different from P_0, P_0', P_2, P_2' it holds $\delta_{A_2}(P) = 0$. Thus

$$\sum_P O_P(K_2) = \sum_P [\delta_{A_2}(P) + \varepsilon_{A_2}(P)] = \sum_P \delta_{A_2}(P) + \sum_P \varepsilon_{A_2}(P) =$$
$$= [\delta_{A_2}P_0) + \delta_{A_2}(P_0') + \delta_{A_2}(P_2) + \delta_{A_2}(P_2')] + \tau = 6 + \tau.$$

Since $\sum_P O_P(K_2) = 4 \,.\, 3 = 12$, then $\tau = 6$.

3. The inflection-point divisor of **(K)**. For any place P of the curve (K) let us denote by τ_P its class. Then the divisor $I = \sum (\tau_P - 1) P$, where the summation through all places P of (K) except the place P_0, P_0', P_2, P_2' is the s.c. inflection-point

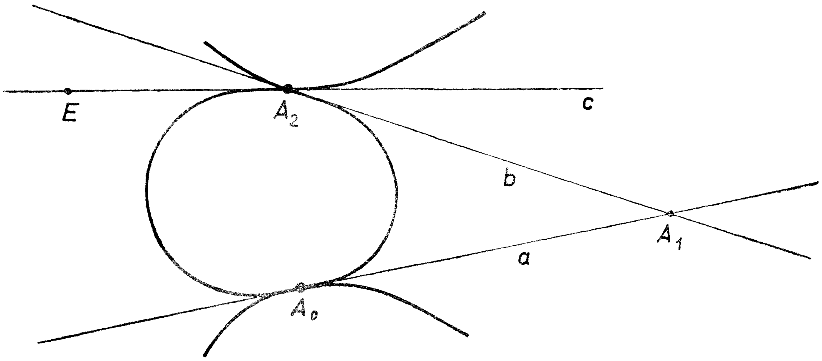

Fig. 1

divisor. Let H be the Hessian of the form K, then for any place P of (K) whose centre is a regular point of (K) (i.e. $P \neq P_0, P_0', P_2, P_2'$) the relation

$$O_P(H) = \tau_P - 1$$

is true. Therefore, if i means the order of I, then

$$\sum_P O_P(H) = i + \sum_{P'} O_{P'}(H),$$

where the summation on the left side runs through all places of (K), the summation on the right side runs trough P_0, P_0', P_2, P_2' only.

By a simple calculating we get

$$H - 6a^2x_2^2K = 18a^2ex_1^2x_2^2 \,.\, Q(x_0, x_1, x_2), \tag{4}$$

where

$$Q(x_0, x_1, x_2) = -8x_0^2 + 8x_0x_1 - 3x_1^2. \tag{5}$$

Let $\mathbf{M}$ denote the set of all places P of $(\mathbf{K})$ different from $\mathsf{P}_0, \mathsf{P}_0', \mathsf{P}_2, \mathsf{P}_2'$ and let $\mathbf{M}'$ denote the set $\{\mathsf{P}_0, \mathsf{P}_0', \mathsf{P}_2, \mathsf{P}_2'\}$. For any place $\mathsf{P} \notin \mathbf{M}$ we have

$$\mathsf{O}_\mathsf{P}(\mathbf{H}) = \mathsf{O}_\mathsf{P}(\mathbf{Q}).$$

As $\boldsymbol{i} = \sum_{\mathsf{P}\in\mathbf{M}} \mathsf{O}_\mathsf{P}(\mathbf{H})$, then $\boldsymbol{i} = \sum_{\mathsf{P}\in\mathbf{M}} \mathsf{O}_\mathsf{P}(\mathbf{Q})$, consequently $\sum_{\mathsf{P}} \mathsf{O}_\mathsf{P}(\mathbf{Q}) = \sum_{\mathsf{P}\in\mathbf{M}} \mathsf{O}_\mathsf{P}(\mathbf{Q}) + \sum_{\mathsf{P}\in\mathbf{M}'} \mathsf{O}_\mathsf{P}(\mathbf{Q}) = \boldsymbol{i} + \mathsf{O}_{\mathsf{P}_0}(\mathbf{Q}) + \mathsf{O}_{\mathsf{P}_0'}(\mathbf{Q}) + \mathsf{O}_{\mathsf{P}_2}(\mathbf{Q}) + \mathsf{O}_{\mathsf{P}_2'}(\mathbf{Q})$. It follows from (2), (3) and (5) that $\mathsf{O}_{\mathsf{P}_0}(\mathbf{Q}) = \mathsf{O}_{\mathsf{P}_0'}(\mathbf{Q}) = 0$, $\mathsf{O}_{\mathsf{P}_2}(\mathbf{Q}) = \mathsf{O}_{\mathsf{P}_2'}(\mathbf{Q}) = 2$, since $\sum_{\mathsf{P}} \mathsf{O}_\mathsf{P}(\mathbf{Q}) = 8$, $\boldsymbol{i} + 4 = 8$, hence $\boldsymbol{i} = 4$.

Obviously the curve $(\mathbf{Q})$ consists of two different lines $(\boldsymbol{l}_2)$, $(\boldsymbol{l}_2')$ through the point $\boldsymbol{A}_2$, namely:

$$(\boldsymbol{l}_2) : 2x_0 - (1 + \mathrm{i}\sqrt{1/2})\, x_1 = 0;$$
$$(\boldsymbol{l}_2') : 2x_0 - (1 - \mathrm{i}\sqrt{1/2})\, x_1 = 0, \tag{6}$$

each of them is different from tangents $\boldsymbol{b}$, $\boldsymbol{c}$ at $\boldsymbol{A}_2$. $(\boldsymbol{l}_2)$ resp. $(\boldsymbol{l}_2')$ intersect $(\mathbf{K})$ into divisor $\mathsf{P}_2 + \mathsf{P}_2' + \mathsf{D}_2$, resp. $\mathsf{P}_2 + \mathsf{P}_2' + \mathsf{D}_2'$, where D_2 and D_2' are the divisors of order 2 and obviously

$$\mathsf{I} = \mathsf{D}_2 + \mathsf{D}_2'. \tag{7}$$

Thus we have:

Proposition 1. *The inflection-point divisor* I *on the quartic* $(\mathbf{K})$ *is of order 4 and of form (7), where* D_2, D_2' *are two divisors of order 2 such that the divisors* $\mathsf{P}_2 + \mathsf{P}_2' + \mathsf{D}_2$, $\mathsf{P}_2 + \mathsf{P}_2' + \mathsf{D}_2'$ *are on the* $(\mathbf{K})$ *determined by two different lines* $(\boldsymbol{l}_2)$, $(\boldsymbol{l}_2')$ *through the point* $\boldsymbol{A}_2$. *Consequently*: I *is not determined by any linear form.*

But, if we substitute (6) into (1), we find the homogeneous coordinate of inflection points:

$$\boldsymbol{J}_1 = (-1/4\sqrt{3a}\,(2 + \mathrm{i}\sqrt{2}),\ -\sqrt{3a},\ 2\sqrt{2e}),$$
$$\boldsymbol{J}_2 = (1/4\sqrt{3a}\,(2 + \mathrm{i}\sqrt{2}),\ \sqrt{3a},\ 2\sqrt{2e})$$

on the line $(\boldsymbol{l}_2)$,

$$\boldsymbol{J}_1' = (-1/4\sqrt{3a}\,(2 - \mathrm{i}\sqrt{2}),\ -\sqrt{3a},\ 2\sqrt{2e}),$$
$$\boldsymbol{J}_2' = (1/4\sqrt{3a}\,(2 - \mathrm{i}\sqrt{2}),\ \sqrt{3a},\ 2\sqrt{2e})$$

on the line $(\boldsymbol{l}_2')$.

Clearly $\boldsymbol{J}_1, \boldsymbol{J}_2, \boldsymbol{J}_1', \boldsymbol{J}_2'$ are pairwise different points. Let us again consider the form

$$\mathbf{K}(x_0, x_1, x_2) = ax_0^2x_2^2 - ax_0x_1x_2^2 + ex_1^4$$

and the form

$$\mathbf{Q}(x_0, x_1, x_2) = -8x_0^2 + 8x_0x_2 - 3x_1^2.$$

We may easily find that the form $\mathbf{G} = 3\mathbf{K} + ex_1^2\mathbf{Q}$ has the decomposition

$$\mathbf{G}(x_0, x_1, x_2) = x_0(x_0 - x_1)\,\mathbf{Q}'(x_0, x_1, x_2)$$

where

$$\mathbf{Q}'(x_0, x_1, x_2) = -8ex_1^2 + 3ax_2^2 = (\sqrt{3a}\, x_2 + 2\sqrt{2e}\, x_1) \cdot (\sqrt{3a}\, x_2 - 2\sqrt{2e}\, x_1),$$

which means, that the points of inflection of $(\mathbf{K})$ lie pairwise on the lines

$$(\mathbf{l}_1): \quad 2\sqrt{2e}x_1 + \sqrt{3a}x_2 = 0, \qquad \text{namely } \mathbf{J}_1 \text{ and } \mathbf{J}_1',$$
$$(\mathbf{l}_1'): \quad -2\sqrt{2e}x_1 + \sqrt{3a}x_2 = 0, \qquad \text{namely } \mathbf{J}_2 \text{ and } \mathbf{J}_2'.$$

The intersection-point of $(\mathbf{l}_1)$ and $(\mathbf{l}_1')$ is the tacnode $\mathbf{A}_0$.

Now, the quadratic forms $\mathbf{Q}$ and $\mathbf{Q}'$ determine a pencil $\mathbf{P}$ of quadratic forms and therefore a pencil $(\mathbf{P})$ of conics. We can easily establish the remaining singular conic $(\mathbf{Q}'')$ of $(\mathbf{P})$; namely

$$\mathbf{Q}''(x_0, x_1, x_2) = 64ex_0^2 - 64ex_0x_1 + 16ex_1^2 + 3ax_2^2,$$

with the singular point $\mathbf{C} = (1, 2, 0)$ lying on the tacnodal tangent. Let $\mathbf{E}_2$ be the intersection-point of the flexnodal tangent $\mathbf{A}_2\mathbf{E}$ with the tacnodal tangent $\mathbf{A}_0\mathbf{A}_1$. Then $\mathbf{E}_2 = (1, 1, 0)$ and the cross-ratio $(\mathbf{A}_0\mathbf{C}\mathbf{A}_1\mathbf{E}_2) = -1$. We have thus proved:

Theorem. *Let the quartic* $(\mathbf{K})$ *have just two point-singularities, namely a tacnode* $\mathbf{A}$ *with the tacnodal tangent* $\mathbf{a}$ *and a flexnode* $\mathbf{B}$ *with the flexnode tangents* $\mathbf{b}$, $\mathbf{c}$. *Let* $\mathbf{B}'$, $\mathbf{E}'$ *be the intersection-points of* $\mathbf{b}$, $\mathbf{c}$ *with the line* $\mathbf{a}$. *The* $(\mathbf{K})$ *has just four distinct inflection points being the vertices of a complet quadrangle. The diagonal points of this quadrangle are just the tacnode* $\mathbf{A}$, *the flexnode* $\mathbf{B}$ *and the point* $\mathbf{C}$ *on* $\mathbf{a}$ *such that the quadruple* $(\mathbf{ACB'E'})$ *is harmonic.*

4. *The genus of* $(\mathbf{K})$. If we transform the curve $(\mathbf{K})$ by a quadratic transformation given by

$$x_0 = y_1y_2, \qquad x_1 = y_0y_2, \qquad x_2 = y_0y_1,$$

we get as a geometrical image (in sence of [5]) the curve

$$ay_1y_2^2 - ay_0y_2^2 + ey_0^2y_1 = 0,$$

which is a cubic curve with one node namely $\mathbf{A}_1 \Rightarrow$ the genus p of $(\mathbf{K})$ equals 0.

5. We will finish our contribution with the following remark: In the considered plane $\mathbf{S}_2$ let two different point $\mathbf{A}, \mathbf{B}$ and three different lines $\mathbf{a}, \mathbf{b}, \mathbf{c}$, such that $\mathbf{A} \in \mathbf{a}$, $\mathbf{B} \in \mathbf{b}$, $\mathbf{c}$, $\mathbf{A} \notin \mathbf{b}, \mathbf{c}$, $\mathbf{B} \notin \mathbf{a}$ be given. Let us denote by (Σ) the system of all quartics having $\mathbf{A}$ as the tacnode, $\mathbf{B}$ as the flexnode, the line $\mathbf{a}$ as the tangent at $\mathbf{A}$ and the lines $\mathbf{b}$, $\mathbf{c}$ as tangents at $\mathbf{B}$. If we choose the coordinate frame as shown in part 1, then the quartic $(\mathbf{K}) \in (\Sigma)$ will be expressed by equation (1). Conversely, any quartic expressed by (1) fulfilling the conditions $a \neq 0$, $e \neq 0$ belongs to the system (Σ). This means that (Σ) together with the curves (more exactly: the divisors)

$$x_0x_2^2(x_0 - x_1) = 0 \quad (e = 0), \qquad x_1^4 = 0 \quad (a = 0),$$

forms a pencil of curves of degree 4. All complete quadrangles whose vertices are just the inflection points of the quartics of (Σ) have fixed diagonal points.

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POZNÁMKA O KVARTICE S TAKTNODÁLNÍM A FLEKTNODÁLNÍM BODEM

Souhrn

V článku se vyšetřuje třída, inflexní body a rod irreducibilní kvartiky, která má jeden taktnodální bod (dvojnásobný bod, který je středem dvou různých lineárních větví o společné tečně) a jeden flektnodální bod (dvojnásobný bod, který je středem dvou různých lineárních větví druhé třídy o různých tečnách). Hlavní výsledek: Vyšetřovaná kvartika má právě čtyři navzájem různé inflexní body tvořící úplný čtyřroh, jehož diagonálními body jsou taktnodální bod, flektnodální bod a další bod na tečně v taktnodálním bodě takový, že spolu s taktnodálním bodem harmonicky oddělují průsečíky taktnodální tečny s tečnami v bodě flektnodálním.

ЗАМЕЧАНИЕ ОБ АЛГЕБРАИЧЕСКОЙ КРИВОЙ 4-ОГО ПОРЯДКА ОБЛАДАЮЩЕЙ ОДНОЙ ТОЧКОЙ САМОКАСАНИЯ И ОДНОЙ ДВОЙНОЙ ТОЧКОЙ ЯВЛЯЮЩЕЙСЯ ЦЕНТРОМ ДВУХ РАЗЛИЧНЫХ ВЕТВЕЙ ПЕРВОГО ПОРЯДКА И ВТОРОГО КЛАССА С РАЗЛИЧНЫМИ КАСАТЕЛЬНЫМИ

Резюме

В статье рассматриваются класс, точки перегиба и род неразложимой плоской кривой четвертого порядка с особенными точками введенными в заголовке. Основной результат: Рассматриваемая кривая обладает четырми различными точками перегиба. Эти точки образуют полный четырехвершиник, диалоналными точками которого являются особые точки кривой и точка касательной в точке самокасания такая, что в месте с точкой самокасания гармонически сопряжена с точкой пересечения этой касательной с касательными в остальной особой точке.