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ON GEOMETRY OF THE THIRD TANGENT BUNDLE

ALENA VANŽUROVÁ

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Introduction

Starting from certain ideas by I. Kolář, [1], and J. E. White, [3], we study some geometric properties of the third tangent bundle $T_3M = T(T(TM))$ of an arbitrary smooth manifold M. In particular, we use the definition of the bracket $[\xi, \eta]$ of two vector fields ξ, η on M in terms of some geometrical operations on the second tangent bundle of M ([1], [3]), and we deduce the Jacobi identity by means of some geometrical constructions on the third tangent bundle. We make no use of functions in our proof, which might appear to be suitable in more general constructions.

All manifolds and maps are assumed to be smooth, i.e. infinitely differentiable.

1. Preliminaries

1.1. **The vertical functor V.** Consider a fibred manifold $\pi : Y \to X$ and its tangent bundle $p_Y : TY \to Y$. For any $x \in X$, the fibre $\pi^{-1}(x) = Y_x$ is a submanifold of Y. Tangent maps $Th_x : T(Y_x) \to TY$ of canonical inclusions $h_x : Y_x \to Y$ induce an injection $H_Y : \bigcup_{x \in X} T(Y_x) \to TY$. Let $\bigcup_{x \in X} T(Y_x) =: VY$ (the disjoint union). The set VY has a natural structure of a vector bundle over Y with projection $q_Y =: =: p_Y \circ H_Y$. We may regard $q_Y : VY \to Y$ as a subbundle of $p_Y : TY \to Y$. The fibre $V_xY = T(Y_x)$ over $x \in X$ is identified with the set

$$\{A \in TY/(q_Y \circ \pi)(A) = x \quad \text{and} \quad T\pi(A) = 0_{T_xX}\}.$$

Given local coordinates (x^i, y^p) on Y such that (x^i) are some local coordinates on X about x, we have resulting coordinates $(x^i, y^p, X^i = dx^i, Y^p = dy^p)$ on TY

and $(x^i, y^p, Y^p = \mathrm{d}y^p)$ on VY. An element $B = (x^i, y^p, Y^p) \in VY$ is identified with $H_Y(B) = (x^i, y^p, 0, Y^p) \in TY$. Given a morphism $\varphi : Y \to W$ of fibred manifolds $\pi : Y \to X$ and $\varrho : W \to Z$, we define $V\varphi : VY \to VW$ to be the abbreviation (restriction) of $T\varphi$. The coordinate expression of $V\varphi$ is

$$V\varphi: \begin{cases} \varphi: \begin{cases} z^\alpha = f^\alpha(x^i), \\ w^\lambda = \varphi^\lambda(x^i, y^p), \end{cases} \\ W^\lambda = \dfrac{\partial \varphi^\lambda}{\partial y^p} Y^p. \end{cases}$$

1.2. **The tangent bundle of a vector bundle.** Let us consider a vector bundle $\pi : E \to M$. Then $p_E : TE \to E$ is a vector bundle. It can be verified that $T\pi : TE \to TM$ is also a vector bundle, vector operations on fibres being defined as follows. Let $A, B \in TE$ be such that $T\pi(A) = T\pi(B)$. Then A and B may be regarded as tangent vectors, $A = (\partial/\partial t)_0\, \gamma(t)$, $B = (\partial/\partial t)_0\, \delta(t)$, of suitable smooth curves $\gamma(t)$ and $\delta(t) : \mathrm{R} \to E$, chosen so that $\pi(\gamma(t)) = \pi(\delta(t))$ for any t. Let $A \underset{T\pi}{+} B$[1] $=: (\partial/\partial t)_0\, (\gamma(t) + \delta(t))$ and $k \,.\, A =: (\partial/\partial t)_0\, (k \,.\, \gamma(t))$ for $k \in \mathrm{R}$. All axioms of vector space are satisfied. Thus we are given two structures of vector bundle on TE, and the following diagram is commutative:

$$\begin{array}{ccc} TE & \xrightarrow{p_E} & E \\ {\scriptstyle T\pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ TM & \xrightarrow{p_M} & M. \end{array} \tag{1}$$

The coordinate expressions of vector operations with respect to both two structures of vector bundle on TE are following. Given local coordinates (x^i) on M, let (x^i, y^p) be local coordinates on E, (x^i, X^i) on TM, and (x^i, y^p, X^i, Y^p) on TE. Assume two elements A, and B of the same fibre $p_E^{-1}(x^i, y^p)$ over $(x^i, y^p) \in E$, with $A = (x^i, y^p, X^i, Y^p)$ and $B = (x^i, y^p, \bar{X}^i, \bar{Y}^p)$. Then

$$\alpha A \underset{p_E}{+} \beta B = (x^i, y^p, \alpha X^i + \beta \bar{X}^i, \alpha Y^p + \beta \bar{Y}^p).$$

Now let A be as above, and let $C = (x^i, \bar{y}^p, X^i, \bar{Y}^p)$. Then A and C belong to the same fibre $(T\pi)^{-1}(x^i, X^i)$ over $(x^i, X^i) \in TM$, and

$$\alpha A \underset{T\pi}{+} \gamma C = (x^i, \alpha y^p + \gamma \bar{y}^p, X^i, \alpha Y^p + \gamma \bar{Y}^p).$$

The evaluation in local coordinates shows that the following statement holds.

Lemma 1. Let $\varphi: E \to D$ be a linear morphism, over $f: M \to N$, of a vector bundle $\pi: E \to M$ onto a vector bundle $\varrho: D \to N$, i.e. the diagram

[1]) $\underset{T\pi}{+}$ **denotes addition with respect to the projection** $T\pi$.

$$\begin{array}{ccc} E & \xrightarrow{\varphi} & D \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\varrho} \\ M & \xrightarrow{f} & N \end{array}$$

commutes. Then $T\varphi$ in the following commutative diagram

$$\begin{array}{ccc} TE & \xrightarrow{T\varphi} & TD \\ {\scriptstyle T\pi}\downarrow & & \downarrow{\scriptstyle T\varrho} \\ TM & \xrightarrow{Tf} & TN \end{array}$$

is a linear morphism.

Proof. Let (x^i, y^p, X^i, Y^p) be local coordinates on TE chosen as above. Let us choose coordinates $(z^\alpha, w^\lambda, Z^\alpha, W^\lambda)$ on TD in a similar way. Then coordinate expressions of the maps f, Tf, φ, and $T\varphi$ are

$$T\varphi: \begin{cases} \varphi: \begin{cases} f: \; z^\alpha = f^\alpha(x^i), \\ \quad w^\lambda = f_p^\lambda(x^i, y^p), \end{cases} \\ \quad Z^\alpha = \dfrac{\partial f^\alpha}{\partial x^i} \, . \, X^i, \\ \quad W^\lambda = \dfrac{\partial f_p^\lambda}{\partial x^i} \, . \, X^i \, . \, y^p + f_p^\lambda(x) \, . \, Y^p \end{cases} \qquad Tf: \begin{cases} z^\alpha = f^\alpha(x^i), \\ Z^\alpha = \dfrac{\partial f^\alpha}{\partial x^i} \, . \, X^i \end{cases}$$

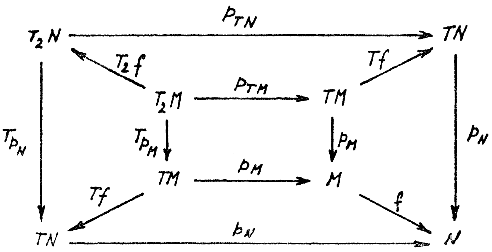
Now it can be easily seen that for any $(x^i, X^i) \in TM$, $T\varphi$ gives a linear map $(y^p, Y^p) \mapsto (w^\lambda, W^\lambda)$ of the fibre $(T\pi)^{-1}\,(x^i, X^i)$ over (x^i, X^i) onto the fibre $(T\varrho)^{-1}\left(f^\alpha(x^i), \dfrac{\partial f^\alpha}{\partial x^i} \, . \, X^i\right)$ over $Tf(x^i, X^i)$. QED.

2. Second tangent bundle

2.1. The second tangent bundle T_2M. Applying previous considerations on the special case p_M: $TM \to M$, we obtain the commutative diagram

$$\begin{array}{ccc} TTM & \xrightarrow{p_{TM}} & TM \\ {\scriptstyle Tp_M}\downarrow & & \downarrow{\scriptstyle p_M} \\ TM & \xrightarrow{p_M} & M \end{array}$$

Let us denote $T(TM)$ by T_2M and $T(Tf)$ by T_2f. According to Lemma 1 and functoriality of T, for any map $f: M \to N$, the diagram



is commutative. If (x^i, u^i, X^i, U^i) denote usual local coordinates on T_2M, the expression of T_2f is

$$T_2f: \begin{cases} Tf: \begin{cases} f : y^k = f^k(x^i), \\ v^k = \dfrac{\partial f^k}{\partial x^i} \,.\, u^i, \end{cases} \\ Y^k = \dfrac{\partial f^k}{\partial x^i} \,.\, X^i, \\ V^k = \dfrac{\partial^2 f^k}{\partial x^i\, \partial x^j} \,.\, u^i \,.\, X^j + \dfrac{\partial f^k}{\partial x^i} \,.\, U^i. \end{cases}$$

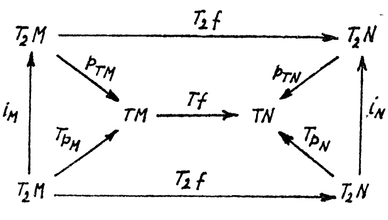
2.2. **The canonical involution as a natural transformation** i: $T_2 \to T_2$. On T_2M, we are given *the canonical involution* i_M: $T_2M \to T_2M$ which may be described as follows. Any $A \in T_2M$ is expressible in the form $A = (\partial/\partial t_2)_0\, ((\partial/\partial t_1)_0\, \delta(t_1, t_2))$ for some smooth local map δ: $\mathrm{R}^2 \to M$. We set $i_M A = (\partial/\partial t_1)_0\, ((\partial/\partial t_2)_0\, \delta(t_1, t_2)$. It can be verified in local coordinates that $i_M A$ depends only on A, and not on the choise of δ. Thus the definition is correct. The map i_M is obviously involutive (that is, $i_M^2 = 1_{TM}$), and in local coordinates,

$$i_M(x^i, u^i, X^i, U^i) = (x^i, X^i, u^i, U^i).$$

The following diagram is commutative:

$$\begin{array}{ccc} T_2M & \overset{i_M}{\longrightarrow} & T_2M. \\ {\scriptstyle T_{p_M}}\downarrow & & \downarrow {\scriptstyle p_{TM}} \\ TM & \overset{1_{TM}}{\longrightarrow} & TM \end{array}$$

Therefore i_M is a linear isomorphism (over identity) of vector bundles T_{p_M} and p_{TM}. Moreover, for any smooth f: $M \to N$, the diagram



is commutative (see [1]). This diagram shows that $i\colon T_2 \to T_2$ is a natural transformation of functors[2)], with some additional property.

3. Descending map

Let $\pi\colon E \to M$ be a vector bundle. Consider its vertical bundle $q_E\colon VE \to E$. Let (x^i, y^p) be local coordinates on E such that x^i are local coordinate functions on M. Since VE is a subbundle of TE, we have resulting local coordinates $(x^i, y^p, 0, Y^p)$ on VE. Now we shall introduce a map $k_\pi\colon VE \to E$ by the following geometric construction. The vector bundle VE may be regarded as a Whithey sum $E \oplus E$. The fibre $(VE)_x = T(E_x)$ over x of the fibred manifold $VE \to M$ is identified with the direct sum $E_x \times E_x$.
Let $\varkappa\colon VE \to E \oplus E$ denote the corresponding identification. Let p_1, and p_2 be projections of the pullback $E \oplus E$ onto the first or second component, respectively. We have the following commutative diagram:

$$\begin{array}{ccccc} VE & \xrightarrow{\varkappa} & E \oplus E & \xrightarrow{p_2} & E \\ {\scriptstyle q_E}\downarrow & & {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle \pi} \\ E & \xrightarrow{1_E} & E & \xrightarrow{\pi} & M. \end{array}$$

For any $x \in M$ and $a \in E_x$, there exists a canonical isomorphism $\mu_a\colon E_x \to T_a(E_x)$, $\mu_a\colon v \mapsto (\partial/\partial t)_0\,(a + tv)$. Further, each $a \in E_x$ determines a unique translation $\tau_a\colon E_x \to E_x$ of a vector space E_x, $\tau_a\colon v \mapsto v - a$, which sends a to a zero element $0 = 0_{E_x}$ of the fibre. The corresponding tangent map $T\tau_a\colon T(E_x) \to T(E_x)$ maps the tangent space $T_a(E_x)$ onto $T_0(E_x)$.

Now let $C \in VE$ with $(\varkappa \circ p_1)(C) = y$, $(\varkappa \circ p_2)(C) = Y$, and $\pi(y) = \pi(Y) = x$. We set

$$k_\pi(C) =\colon (\mu_0^{-1} \circ T\tau_y \circ \mu_y)(Y).$$

In local coordinates, if $C = (x_0^i, y_0^p, 0, Y_0^p)$, we have $y = (x_0^i, y_0^p)$, $Y = (x_0^i, Y_0^p)$, and

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[2)] Let $\mathscr{K}$ and $\mathscr{L}$ be categories, and let $F, G\colon \mathscr{K} \to \mathscr{L}$ be functors. *A natural transformation* (or *a morphism* of functors) $i\colon F \to G$ is a system of $\mathscr{L}$-morphisms

$$i = \{i_M\colon FM \to GM \text{ in } \mathscr{L}/M \text{ is object of } \mathscr{K}\}$$

such that for any $\mathscr{K}$-morphism $f\colon M \to N$, the following diagram is commutative:

$$\begin{array}{ccc} GM & \xrightarrow{Gf} & GN \\ {\scriptstyle i_M}\downarrow & & \downarrow{\scriptstyle i_N} \\ FM & \xrightarrow{Ff} & FN. \end{array}$$

$$Y \xrightarrow{\mu_y} (x_0^i, y_0^p, Y_0^p) \xrightarrow{T\tau_y} (x_0^i, 0, Y_0^p) \xrightarrow{\mu_0^{-1}} k_\pi(C) = (x_0^i, Y_0^p).$$

The map k_π will be called *the descending map* corresponding to π.

The previous construction yields a natural transformation $k\colon V \to 1$ of functors on the category of vector bundles. In fact, the following assertion holds:

Proposition. Let $\pi\colon E \to M$ and $\varrho\colon D \to N$ be vector bundles. Let $\varphi\colon E \to D$ be a linear morphism over $f\colon M \to N$; that is, the diagram

$$\begin{array}{ccc} E & \xrightarrow{\varphi} & D \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\varrho} \\ M & \xrightarrow{f} & N \end{array}$$

is commutative. Then the following diagram also commutes:

$$\begin{array}{ccc} VE & \xrightarrow{V\varphi} & VD \\ {\scriptstyle k_\pi}\downarrow & & \downarrow{\scriptstyle k_\rho} \\ E & \xrightarrow{\varphi} & D. \end{array}$$

Proof. Let φ be a linear morphism over f. Let $x \in M$. As usual, assume local coordinate system (x^i, y^p) on E such that (x^i) are local coordinates in a neighborhood of $x = (x_0^i)$. Similarly, let (z^α, w^λ) be local coordinates on D chosen so that z^α are coordinate functions about $f(x)$. Since φ maps the fibre E_x linear into $D_{f(x)}$, and coefficients in the corresponding linear combination are constants for x fixed, the coordinate expression of the restriction $V\varphi/E_x = T(\varphi_x)$ is

$$V\varphi/E_x\colon \begin{cases} \varphi_x\colon \begin{cases} f(x)\colon z^\alpha = f^\alpha(x^i), \\ \qquad\quad w^\lambda = a_p^\lambda(x)\, y^p, \end{cases} \\ \qquad\qquad\quad Z^\alpha = 0, \\ \qquad\qquad\quad W^\lambda = a_p^\lambda(x)\, Y^p. \end{cases}$$

Given a vertical tangent vector $C = (x_0^i, y_0^p, 0, Y_0^p)$ at $y = (x_0^i, y_0^i) \in E_x$, we have

$$k_\pi(C) = (x_0^i, Y_0^p),$$

$$V\varphi(C) = (f^\alpha(x), a_p^\lambda(x)\, y_0^p, 0, a_p^\lambda(x)\, Y_0^p),$$

and finally,

$$k_\varrho(V\varphi(C)) = (f^\alpha(x), a_p^\lambda(x)\, Y_0^p) = \varphi(k_\pi(C)).$$

4. Iterated tangent bundles

4.1. The r-th tangent bundle T_rM. Let M be a smooth manifold. We have already investigated tangent bundles TM and T_2M. By iteration, one obtains the r-th tangent bundle defined inductively by

$$T_rM = T(T_{r-1}M) = \underbrace{T(\dots(TM)\dots)}_{r\text{-}time} \qquad \text{for } r \geqq 2.$$

T_rM is a smooth manifold. A smooth map $f: M \to N$ is prolonged to $T_rf: T_rM \to T_rN$ in an obvious way, and T_r is a functor. We also set $T_0M = M$ and $T_0f = f$. According to J. E. White, elements of T_rM will be called tangent r-sectors on M. T_rM admits a structure of vector bundle over $T_{r-1}M$, corresponding projections being

$$\pi_r^s := T_{r-s}p_{T_{s-2}M} : T_rM \to T_{r-1}M \qquad \text{for } s = 2, \dots, r.$$

Any r-sector $A \in T_rM$ is expressible in the form

$$A = \frac{\partial}{\partial t_r}\bigg|_0 \cdots \frac{\partial}{\partial t_1}\bigg|_0 \delta(t_1, \dots, t_r) \tag{2}$$

for a suitable smooth local map $\delta : \mathbb{R}^r \to M$. For any $s \in \{1, \dots, r\}$, we have

$$\pi_r^s A = \frac{\partial}{\partial t_r}\bigg|_0 \cdots \frac{\partial}{\partial t_{s+1}}\bigg|_0 \frac{\partial}{\partial t_{s-1}}\bigg|_0 \cdots \frac{\partial}{\partial t_1}\bigg|_0 \delta(t_1, \dots, t_{s-1}, 0, t_{s+1}, \dots, t_r).$$

4.2. Let S_r denote the symmetric group of r elements. For each $\sigma \in S_r$, let us define a map i_M^σ: $T_rM \to T_rM$ by

$$i_M^\sigma(A) = \frac{\partial}{\partial t_{\sigma(r)}}\bigg|_0 \cdots \frac{\partial}{\partial t_{\sigma(1)}}\bigg|_0 \delta(t_1, \dots, t_r)$$

for all $A \in T_rM$, expressed in the form (2). It can be verified that this definition is correct, and that S_r acts on T_rM on the right, if we set $\sigma(A) =: i_M^\sigma(A)$. For each $\sigma \in S_r$, i^σ: $T_r \to T_r$ is a natural transformation of functors. Further, the group $\{i_M^\sigma\}_{\sigma \in S_r}$ is generated by tangent prolongations of canonical involutions on iterated tangent bundles. In fact, if $i_{T_{s-2}M}$ denotes the canonical involution on T_sM for $s = 2, \dots, r$, and σ_s is the inversion interchanging $r - s + 1$ and $r - s + 2$, that is,

$$\sigma_s = \begin{pmatrix} 1, \dots, r-s, r-s+1, r-s+2, r-s+3, \dots, r \\ 1, \dots, r-s, r-s+2, r-s+1, r-s+3, \dots, r \end{pmatrix},$$

then

$$i_M^{\sigma_2} =: T_{r-2}i_M, \dots, i_M^{\sigma_s} =: T_{r-s}i_{T_{s-2}M}, \dots, i_M^{\sigma_r} =: i_{T_{r-2}M}$$

are required generators.

4.3. It is convenient to introduce local coordinates on T_rM by the following procedure. Let x^i be local coordinates on M and denote by X_0^i the pullbacks of x^i to TM. That is, $X_0^i = x^i \circ p_M$. Besides X_0^i, on TM we have also additional coordinates $X_1^i =: \mathrm{d}x^i$. Let us proceed inductively. Taking pullbacks will be denoted by addition of index 0, addition of 1 denotes differentiation. Hence on T_2M, we have local coordinates X_{00}^i, X_{10}^i (of X_0^i and X_1^i), $X_{01}^i = \mathrm{d}X_0^i$ and $X_{11}^i = \mathrm{d}X_1^i$. For an arbitrary r, we obtain 2^r groups of coordinates on T_rM, each of which consists of m elements $X^i_{j_1,\dots,j_r}$, $i = 1, \dots, m$, where $(j_1, \dots, j_r)$ is a sequence of elements 0 and 1. With respect to these coordinates, the expression of i^σ_M is

$$X^s_{j_1,\dots,j_r}(i_M A) = X^s_{j_{\sigma(1)},\dots,j_{\sigma(r)}}(A). \tag{3}$$

4.4. For each vector bundle π^s_r: $T_r M \to T_{r-1} M$ $(1 \leqq r, 1 \leqq s \leqq r)$, we have the corresponding descending map $k_{\pi^s_r}$: $V(T_r M) \to T_r M$. Moreover, descending maps $k_{\pi^s_{r-l}}$ $(l = 1, \dots, r - 1;\ s = 1, \dots, r - l)$ of lower orders are prolonged to maps $T_l k_{\pi^s_{r-l}}$: $T_l V(T_{r-l} M) \to T_r M$. Together, we obtain $r(r + 1)/2$ maps

$$T_l k_{\pi^s_{r-l}}, \qquad l = 0, \dots, r - 1 \tag{4}$$

with domain $\subset T_{r+1} M$ and image $T_r M$. Each of the maps (4) is "linear" in the following sense.

Lemma 2. Let A, and B be $(r + 1)$-sectors of the domain of k, and at the same time, let $\pi^s_{r+1}(A) = \pi^s_{r+1}(B)$ for some $s \in \{1, \dots, r + 1\}$.[3] Then any linear combination $\alpha A \underset{\pi^s_{r+1}}{+} \beta B$ with $\alpha, \beta \in \mathrm{R}$ belongs to the domain of k, and there exists a unique projection π^q_r with the property

$$k(\alpha A \underset{\pi^s_{r+1}}{+} \beta B) = \alpha k(A) \underset{\pi^q_r}{+} \beta k(B).$$

We shall not prove the lemma here.

4.5. In the following text, we shall deal with tangent bundles of orders $r \leqq 3$ only. In the case $r = 2$, coordinates of 2-sectors at a fixed point $x \in M$ will be written into the schema $X^j_{10} - X^j_{11} - X^j_{01}$,[4] which corresponds to the structure

$$TM \overset{p_{TM}}{\longleftarrow} T_2 M \overset{Tp_M}{\longrightarrow} TM$$

of double fibred manifold on $T_2 M$. For p_M: $TM \to M$, we obtain a descending map k_{p_M}: $V(TM) \to TM$ with coordinate expression

$$k_{p_M} : X^j_{10} - X^j_{11} - 0 \to X^j_{11}.$$

It can be easily verified the following:

Lemma 3. Let A, and B be 2-sectors with $T_{p_M}(A) = T_{p_M}(B) = 0$. Then A, B belong to the domain of k_{p_M} and

$$k_{p_M}(\alpha A \underset{Tp_M}{+} \beta B) = \alpha k_{p_M}(A) \underset{p_M}{+} \beta k_{p_M}(B). \tag{5}$$

For $r = 3$, it is convenient to arrange coordinates of a 3-sector A at $x \in M$ into a regular coordinate triangle, see [3],

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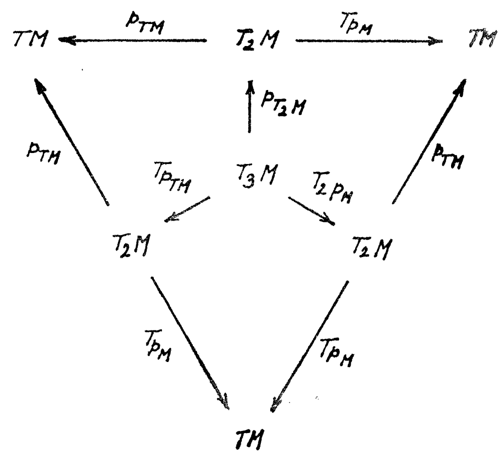
[3]) That is, A, B belong to the same fibre with respect to π^s_{r+1}, and they can be added in this fibre.

[4]) In our considerations, pullbacks $X^j_{0\dots0}$ of coordinates x^j will be omitted.

$$A: \quad \begin{array}{ccc} X^j_{100} & X^j_{110} & X^j_{010} \\ & X^j_{111} & \\ X^j_{101} & & X^j_{011} \\ & X^j_{001} & \end{array} \tag{6}$$

corresponding to the commutative diagram (7):



Commutativity of this diagram follows by functoriality of T and by appliyng of commutative diagram (1) on vector bundles Tp_M: $T_2M \to TM$ and p_{TM}: $T_2M \to TM$, respectively. Coordinate expression of the maps $k_{p_{TM}}$, k_{Tp_M}, and Tk_{p_M} are

$$\begin{array}{ccc} X^j_{100} & X^j_{110} & X^j_{010} \\ & X^j_{111} & \\ 0 & & X^j_{011} \\ & 0 & \end{array} \quad \xmapsto{k_{p_{TM}}} X^j_{100} - X^j_{111} - X^j_{011},$$

$$\begin{array}{ccc} X^j_{100} & X^j_{110} & X^j_{010} \\ & X^j_{111} & \\ X^j_{101} & & 0 \\ & 0 & \end{array} \quad \xmapsto{k_{Tp_M}} X^j_{010} - X^j_{111} - X^j_{101},$$

and

$$\begin{array}{ccccc} X^j_{100} & & X^j_{110} & & 0 \\ & & X^j_{111} & & \\ & X^j_{101} & & 0 & \\ & & & & \\ & & X^j_{001} & & \end{array} \quad \xrightarrow{Tk_{p_M}} X^j_{110} - X^j_{111} - X^j_{001}.$$

Obviously, $k_{T_{p_M}} = k_{p_{TM}} \circ Ti_M$, and

$$Tk_{p_M} = i_M \circ k_{p_{TM}} \circ Ti_M \circ i_{TM}. \tag{8}$$

Hence it suffices to use $k_{p_{TM}}$ only.

Lemma 4. Let A, and B be 2-sectors on M, and let both A and B belong to the domain of $k_{p_{TM}}$. Let $\alpha, \beta \in \mathrm{R}$. If A, B belong to the same fibre with respect to the projection $T_2 p_M$, then

$$k_{p_{TM}}(\alpha A \underset{T_2 p_M}{+} \beta B) = \alpha k_{p_{TM}}(A) \underset{T p_M}{+} \beta k_{p_{TM}}(B). \tag{9}$$

If A, B are elements of the same fibre with respect to Tp_{TM}, then

$$k_{p_{TM}}(\alpha A \underset{Tp_{TM}}{+} \beta B) = \alpha k_{p_{TM}}(A) \underset{p_{TM}}{+} \beta k_{p_{TM}}(B). \tag{10}$$

If A, B belong to the same fibre with respect to p_{T_2M}, then

$$k_{p_{TM}}(\alpha A \underset{p_{T_2M}}{+} \beta B) = \alpha k_{p_{TM}}(A) \underset{p_{TM}}{+} \beta k_{p_{TM}}(B).$$

4.6. The action of the symmetric group S_3 on T_3M may be described in terms of canonical involutions as follows. If $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$, then $i^\sigma_M = i_{TM} \circ Ti_M$ corresponds to the permutation $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$, $Ti_M \circ i_{TM}$ corresponds to $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$, $i_{TM} \circ Ti_M$ corresponds to $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$. For $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$, $i^\sigma_M = Ti_M \circ i_{TM} \circ Ti_M = i_{TM} \circ Ti_M \circ \circ i_{TM}$. Finally, the identical permutation induces the identity 1_{T_3M}. In local coordinates, i_{TM}, Ti_M, and $Ti_M \circ i_{TM} \circ Ti_M$ are "axial symmetries" of the regular coordinate triangle (6), while $i_{TM} \circ Ti_M$ and $Ti_M \circ i_{TM}$ are "rotations" with angles $\pi/3$ and $-\pi/3$, respectively:

$$i_{TM}(A)\colon \begin{array}{ccccc} X^j_{100} & & X^j_{101} & & X^j_{001} \\ & & X^j_{111} & & \\ & X^j_{110} & & X^j_{011} & \\ & & & & \\ & & X^j_{010} & & \end{array} \qquad Ti_M(A)\colon \begin{array}{ccccc} X^j_{010} & & X^j_{110} & & X^j_{100} \\ & & X^j_{111} & & \\ & X^j_{011} & & X^j_{101} & \\ & & & & \\ & & X^j_{001} & & \end{array}$$

$Ti_M \circ i_{TM}(A)$:

X^j_{001} X^j_{101} X^j_{100}
X^j_{111}
X^j_{011} X^j_{110}

X^j_{010}

$i_{TM} \circ Ti_M(A)$:

X^j_{010} X^j_{011} X^j_{001}
X^j_{111}
X^j_{110} X^j_{101}

X^j_{100}

$Ti_M \circ i_{TM} \circ Ti_M(A)$:

X^j_{001} X^j_{011} X^j_{010}
X^j_{111}
X^j_{101} X^j_{110}

X^j_{100}

Consequently, by (3) the coordinates X^j_{111} are not changed by any i^σ_M.

5. Vector fields and Jacobi identity

5.1. A smooth map ξ: $M \to TM$ is called *a vector field* on a manifold M, if $p_M \circ \xi = 1_M$; that is, ξ is a smooth section of the projection p_M: $TM \to M$. For a vector field ξ: $M \to TM$, prolongations $T\xi$: $TM \to T_2M$ and $T_2\xi$: $T_2M \to T_3M$ satisfy $Tp_M \circ T\xi = 1_{TM}$ and $T_2p_M \circ T_2\xi = 1_{T_2M}$, respectively.

Remark. $T\xi$, or $T_2\xi$ is not a vector field on TM, or T_2M, respectively. But let us observe that $i_M \circ T\xi$ is a vector field on TM, and $Ti_M \circ i_{TM} \circ T_2\xi$ is a vector field on T_2M. In general,

$$T_{r-1}i_M \circ \dots \circ T_{r-s}i_{T_{s-1}M} \circ \dots \circ i_{T_{r-1}M} \circ T_r\xi := \mathscr{T}_r\xi \qquad \text{for } 1 \leqq s \leqq r$$

is a vector field on T_rM with the flow $\mathbf{exp}\, t(\mathscr{T}_r\xi) = T_r(\mathbf{exp}\, t\xi)$ (see also [1]).

Let ξ, and η be two vector fields on M with coordinate expressions $\xi = \xi^j(\partial/\partial x^j)$ and $\eta = \eta^j(\partial/\partial x^j)$. Let us consider compositions

$$T\eta \circ \xi : \eta^i - \frac{\partial \eta^i}{\partial x^j}\xi^j - \xi^i,$$

and

$$i_M \circ T\xi \circ \eta : \eta^i - \frac{\partial \xi^i}{\partial x^j}\eta^j - \xi^i.$$

Given a point $x \in M$, $(T\eta \circ \xi)(x)$ and $(i_M \circ T\xi \circ \eta)(x)$ are *2*-sectors of the same fibre with respect to p_{TM}. Hence the difference

$$T\eta \circ \xi \underset{p_{TM}}{-} i_M \circ T\xi \circ \eta : \eta^i - \left(\frac{\partial \eta^i}{\partial x^j}\xi^j - \frac{\partial \xi^i}{\partial xj}\eta^j\right) - 0$$

belongs to the domain of k_{p_M}. It was shown in [1], that

$$k_{p_M}(T\eta \circ \xi \underset{p_{TM}}{-} i_M \circ T\xi \circ \eta) := T\eta \circ \xi \dot{-} T\xi \circ \eta$$

coincides with the classical bracket $[\xi, \eta]$ of vector fields. To simplify the notation, we set

$$\overline{[\xi, \eta]} =: T\eta \circ \xi \underset{p_{TM}}{-} i_M \circ T\xi \circ \eta.$$

Hence

$$[\xi, \eta] = k_{p_M} \circ \overline{[\xi, \eta]}. \tag{11}$$

5.2. ***Jacobi identity***. Three vector fields ξ, η, and ζ on M satisfy the classical well-known formula

$$[[\xi, \eta], \zeta] + [[\eta, \zeta], \xi] + [[\zeta, \xi], \eta] = 0, \tag{12}$$

so called Jacobi identity, where 0 denotes the zero vector field on M. The previous considerations enable us to prove it in a new way, without refering to functions.

Proof of Jacobi identity. By (11), $[[\xi, \eta], \zeta] = k_{p_M} \circ \overline{[[\xi, \eta] \circ \zeta]}$. Further,

$$\overline{[[\xi, \eta], \zeta]} = T\zeta \circ k_{p_M} \circ \overline{[\xi, \eta]} \underset{p_{TM}}{-} i_M \circ T(k_{p_M} \circ \overline{[\xi, \eta]}) \circ \zeta.$$

Now $T\zeta \circ k_{p_M} = k_{p_{TM}} \circ VT\zeta$, since k is a natural transformation, and thus the diagram

$$\begin{array}{ccc}
VTM & \xrightarrow{VT\zeta} & VT_2M \\
{\scriptstyle k_{p_M}}\downarrow & & \downarrow{\scriptstyle k_{p_{TM}}} \\
TM & \xrightarrow{T\zeta} & T_2M \\
{\scriptstyle p_M}\downarrow & & \downarrow{\scriptstyle p_{TM}} \\
M & \xrightarrow{\zeta} & TM
\end{array}$$

is commutative. But $\overline{[\xi, \eta]} \in VTM$, hence

$$T\zeta \circ k_{p_M} \circ \overline{[\xi, \eta]} = k_{p_{TM}} \circ T_2\zeta \circ [\xi, \eta].$$

By (8), and functoriality of T,

$$i_M \circ T(k_{p_M} \circ \overline{[\xi, \eta]}) = i_M \circ Tk_{p_M} \circ T\overline{[\xi, \eta]} = k_{p_{TM}} \circ Ti_M \circ i_{TM} \circ T\overline{[\xi, \eta]}.$$

Hence we conclude by (10) that

$$\overline{[[\xi, \eta], \zeta]} = k_{p_{TM}} \circ (T_2\zeta \circ \overline{[\xi, \eta]}) \underset{p_{TM}}{-} k_{p_{TM}} \circ (Ti_M \circ i_{TM} \circ T\overline{[\xi, \eta]} \circ \zeta),$$

$$[[\xi, \eta], \zeta] = k_{p_{TM}} \circ (T_2\xi \circ \overline{[\xi, \eta]} \underset{Tp_{TM}}{-} Ti_M \circ i_{TM} \circ T\overline{[\xi, \eta]} \circ \zeta).$$

It remains to show that the assumptions of (10) are satisfied. But that follows immediately, if we express projections of our 3-sectors by diagrams of the type (7):

$$\begin{array}{ccc} \zeta & T\zeta \circ \eta & \eta \\ & T_2\zeta \circ \overline{[\xi, \eta]} & \\ 0 & & \overline{[\xi, \eta]} \\ & 0 & \end{array} \qquad \begin{array}{ccc} \zeta & i_M \circ T\eta \circ \zeta & \eta \\ & Ti_M \circ i_{TM} \circ T\overline{[\xi, \eta]} \circ \zeta & \\ 0 & & \overline{[\xi, \eta]} \\ & 0 & \end{array}$$

Further, we have

$$T_2\zeta \circ \overline{[\xi, \eta]} = T_2\zeta \circ T\eta \circ \xi \underset{p_{T_2M}}{-} i_{TM} \circ T_2\zeta \circ T\xi \circ \eta,$$

since

$$\begin{array}{ccc} \zeta & T\zeta \circ \eta & \eta \\ & T\zeta \circ \overline{[\xi \circ \eta]} & \\ 0 & & \overline{[\xi, \eta]} \\ & 0 & \end{array}$$

and

$$\begin{array}{ccc} \zeta & T\zeta \circ \eta & \eta \\ & T_2\zeta \circ T\eta \circ \xi & \\ T\zeta \circ \xi & & T\eta \circ \xi \\ & \xi & \end{array} \qquad \begin{array}{ccc} \zeta & T\zeta \circ \eta & \eta \\ & i_{TM} \circ T_2\zeta \circ T\xi \circ \eta & \\ T\zeta \circ \xi & & i_M \circ T\xi \circ \eta \\ & \xi & \end{array}$$

Similarly,

$$T\overline{[\xi, \eta]} \circ \zeta = T_2\eta \circ T\xi \circ \zeta \underset{Tp_{TM}}{-} Ti_M \circ T_2\xi \circ T\eta \circ \zeta,$$

because

$$
\begin{array}{ccc}
\eta & \overbrace{[\xi,\eta]} & 0 \\
 & T\overbrace{[\xi,\eta]}\circ\zeta & \\
 & T\eta\circ\zeta \quad\quad 0 & \\
 & \zeta &
\end{array}
$$

and

$$
\begin{array}{ccc}
\eta & T\eta\circ\xi & \xi \\
 & T_2\eta\circ T\xi\circ\zeta & \\
T\eta\circ\zeta & & T\xi\circ\zeta \\
 & \zeta &
\end{array}
\qquad
\begin{array}{ccc}
\eta & i_M\circ T\xi\circ\eta & \xi \\
 & Ti_M\circ T_2\xi\; T\eta\circ\zeta & \\
T\eta\circ\zeta & & T\xi\circ\zeta \\
 & \zeta &
\end{array}
$$

Let us set

$$
\lambda(\xi,\eta,\zeta) =: (T_2\zeta\circ T\eta\circ\xi \underset{p_{T_2M}}{-} i_{TM}\circ T_2\zeta\circ T\xi\circ\eta)
$$

$$
\underset{Tp_M}{-} Ti_M\circ i_{TM}\circ(T_2\eta\circ T\xi\circ\zeta \underset{Tp_{TM}}{-} Ti_M\circ T_2\xi\circ T\eta\circ\zeta).
$$

The corresponding diagram of projections is

$$
\begin{array}{ccc}
\zeta & \overbrace{[\eta,\zeta]} & 0 \\
 & \lambda(\xi,\eta,\zeta) & \\
 & 0 \quad\quad 0 & \\
 & 0. &
\end{array}
\tag{13}
$$

Now $\overbrace{[[\xi,\eta],\xi]} = k_{p_{TM}}\circ\lambda(\xi,\eta,\zeta)$ with

$$
\zeta \xleftarrow{p_{TM}} \overbrace{[[\xi,\eta],\zeta]} \xrightarrow{Tp_M} 0. \tag{14}
$$

By (13) and (14), the assumptions of (5) and (9) are satisfied. Therefore

$$
\mathscr{C}\sum_{p_M}{}^{5)}[[\xi,\eta],\zeta] = \mathscr{C}\sum_{p_M} k_{p_M}\circ\overbrace{[[\xi,\eta],\zeta]} =
$$

5) $\mathscr{C}\sum$ will denote a cyclic summation. Thus $\mathscr{C}\sum[[\xi,\eta],\ \zeta] = [[\xi,\eta],\zeta] + [[\eta,\xi],\zeta] + [[\zeta,\xi],\eta]$.

$$= \mathscr{C} \sum_{p_M} k_{p_M} \circ k_{p_{TM}} \circ \lambda(\xi, \eta, \zeta) =$$

$$= k_{p_M} \circ (\mathscr{C} \sum_{T p_M} k_{p_{TM}} \circ \lambda(\xi, \eta, \zeta)) \qquad \text{(by (5))}$$

$$= k_{p_M} \circ k_{p_{TM}} \circ \mathscr{C} \sum_{T_2 p_M} \lambda(\xi, \eta, \zeta) = \qquad \text{(by (9))}.$$

Here

$$\mathscr{C} \sum_{p_M} \zeta \qquad \mathscr{C} \sum_{T p_M} \overbrace{[\eta, \zeta]} \qquad 0$$

$$\mathscr{C} \sum_{T_2 p_M} \lambda(\xi, \eta, \zeta)$$

$$0 \qquad 0$$

$$0.$$

By the definition of $\lambda(\xi, \eta, \zeta)$, it is clear that

$$X^j_{111} \circ \mathscr{C} \sum_{T_2 p_M} \lambda(\xi, \eta, \zeta) = 0,$$

since the maps i^σ_M, $\sigma \in S_3$ do not change coordinates X^j_{111}, and each term of the form $i^\sigma_M \circ T_2\zeta \circ T\eta \circ \xi$ occures in the sum exactly two times, always with opposite signs. Hence

$$\mathscr{C} \sum_{p_M} \zeta \rightleftarrows k_{p_{TM}} \circ \mathscr{C} \sum_{T_2 p_M} \lambda(\xi, \eta, \zeta) = 0 \rightleftarrows 0.$$

Consequently, $\mathscr{C} \sum_{p_M} [[\xi, \eta, \zeta]$ is a zero vector field on TM. QED.

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O GEOMETRII TŘETÍHO TEČNÉHO BANDLU

Shrnutí

Článek navazuje na práce I. Koláře a knihu J. E. Whitea, [3]. Je příspěvkem ke studiu geometrických vlastností třetího tečného bandlu $T_3M = T(T(TM))$ hladké variety M. Původních výsledků je dosaženo při popisu a interpretaci operace symetrické grupy na třetím tečném bandlu a při studiu operace sestupu na libovolném r-krát iterovaném tečném bandlu. V závěru je pomocí těchto nových metod podán nový důkaz Jacobiho identity pro vektorová pole. Metodický význam tohoto nového postupu spočívá v tom, že se neužívá funkcí, což může mít význam pro některá další zobecnění této problematiky.

О ГЕОМЕТРИИ ТРЕТЬЕГО КАСАТЕЛЬНОГО РАССЛОЕНИЯ

Резюме

Статья исходит из идей И. Коларжа и Й. Э. Вайта. Рассматриваются здесь некоторые геометрические свойства третьего касательного расслоения $T_3M = T(T(TM))$ произвольного гладкого многообразия M, а именно его группа симметрий. Также определяется новая операция „сбросивания" на касательном пространстве r-го порядка T_rM, и устанавливаются ее свойства. Достигнутые результаты применяются к доказательству тождества Якоби. Для скобки $[\xi, \eta]$ двух векторных полей ξ и η на M употребляется определение основанное на геометрических операциях на втором касательном расслоении. Это новое доказательство не пользуется функциями, что может оказаться полезным для некоторых более общих рассуждений.