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A NOTE ON HIGHER MONOTONICITY PROPERTIES OF GENERALIZED AIRY FUNCTIONS

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1. Introduction

We consider the differential equation

$$y'' + \left(\beta^2\gamma^2 x^{2\beta-2} + \frac{\beta^2 v^2 - \frac{1}{4}}{x^2}\right) y = 0 \qquad x \in (0, \infty), \tag{1.1}$$

where $\gamma > 0$, and $\beta \in \left(1; \frac{3}{2}\right)$ are parameters.

If we choose in (1.1) $2\beta - 2 = \alpha$ and $\beta^2\gamma^2 = c$, $|v| = \frac{1}{2\beta}$, then we obtain the differential equation

$$y'' + cx^\alpha y = 0 \qquad x \in (0, \infty), \tag{1.2}$$

which was investigated in [5].

When $\beta = \frac{3}{2}$, $\gamma = \frac{2}{3}$, $|v| = \frac{1}{3}$, then (1.1) reduces to the equation

$$y'' + xy = 0 \qquad x \in (0, \infty), \tag{1.3}$$

which is satisfied by the linearly independent Airy functions $\mathrm{Ai}(-x)$, $\mathrm{Bi}(-x)$ of the first kind and of the second kind, respectively.

Remark 1.1. General solution of (1.1) can be written in the form

$$y = \sqrt{x}\, C_v(\gamma x^\beta), \tag{1.4}$$

where $C_v(x)$ denotes any linear combination of the Bessel functions $J_v(x)$ and $Y_v(x)$.

As usual we say that $f(x)$ is completely monotonic on $(0, \infty)$ and write $f(x) \in M_\infty(0, \infty)$ if

$$(-1)^i f^{(i)}(x) \geqq 0 \tag{1.5}$$

for $i = 0, 1, \ldots$ and $x > 0$. If in (1.5) a strict inequality holds for $i = 0, 1, \ldots,$ then we denote it by $f(x) \in M^*_\infty(0, \infty)$.

We say that the sequence $\{x_k\}_{k=0}^\infty$ is completely monotonic if

$$(-1)^i \Delta^i x_k \geqq 0 \qquad i, k = 0, 1, 2, \ldots, \tag{1.6}$$

where the difference operator is defined by $\Delta x_k = x_{k+1} - x_k$ and $\Delta^{n+1} x_k = \Delta(\Delta^n x_k)$; in this case we write $\{x_k\} \in M_\infty$. If the strict inequality holds throughout (1.6) then we write $\{x_k\} \in M^*_\infty$. If in addition $x_k \to 0$ as $k \to \infty$, we write $\{x_k\} \in M^*_{\infty,0}$ and $M_{\infty,0}$ respectively.

From the proof of [2] Theorem 3.1 it follows that

a) *let* $\{x_k\}_{k=0}^\infty$ *denote the sequence of zeros of any nontrivial solution of* (1.1) *and let* $\beta \in \left\langle 1; \frac{3}{2} \right\rangle$ *and* $|v| \geqq \frac{1}{2\beta}$ $\left(\textit{when } \beta = 1 \textit{ then let } |v| > \frac{1}{2}\right)$, *then the sequence* $\{\Delta x_k\} \in M^*_\infty$,

b) *let* $\beta \in \left\langle 1; \frac{3}{2} \right\rangle$ *and* $|v| \geqq \frac{1}{2\beta}$, *then the sequence of areas under each arch between two successive zeros of the graph of any nontrivial solution of* (1.1) *belongs to the class* M^*_∞.

The im of this paper is to enlarge the scope of known higher monotonicity properties of solutions of (1.1).

2. Preliminaries

We recall some results which will be useful in the next section.

Let $y(x)$ be any nontrivial solution of the differential equation

$$y'' + a(x) y' + b(x) y = 0, \qquad x > 0. \tag{2.1}$$

Denote by $\{x_k^{(i)}\}_{k=0}^\infty$ the sequence of consecutive zeros of the i-th derivative $i = 0, 1, 2, \ldots$ of any nontrivial solution $z(x)$ ($z^{(0)}(x) = z(x)$, $x_k^{(0)} = x_k$) which may or may not be linearly independent of $y(x)$.

Moreover we define the following sequences of functions

$$a_0(x) = a(x), \qquad b_0(x) = b(x), \tag{2.2a}$$

$$a_{i+1}(x) = a_i(x) - \frac{b_i'(x)}{b_i(x)}, \qquad i = 0, 1, \ldots \tag{2.2b}$$

$$b_{i+1}(x) = b_i(x) + a_i'(x) - a_i(x)\frac{b_i'(x)}{b_i(x)}, \qquad i = 0, 1, 2, \ldots \tag{2.2c}$$

$$f_i(x) = b_i(x) - \frac{1}{2} a_i'(x) - \frac{1}{4} a_i^2(x), \qquad i = 0, 1, 2, \ldots \tag{2.3}$$

For $\lambda > -1$ and a suitable $W(x)$ we define the quantities

$$R_k^{(i)} = R_k^{(i)}(W, \lambda) = \int_{x_k^{(i)}}^{x_{k+1}^{(i)}} W(x) \exp\left(\frac{\lambda}{2} \int a_i(x)\, dx\right) |y^{(i)}(x)|^\lambda\, dx. \tag{2.4}$$

Lemma 2.1. (see [7] Theorem 5.1). *For $i = 0, 1, \dots$ let $W(x) > 0$ be any completely monotonic function. Let $a_i(x), b_i(x)$ and $f_i(x)$ be defined as in* (2.2) *and* (2.3) *and suppose that the function $f_i(x)$ is such that*

$$f_i'(x) \in M_\infty(a, \infty), f_i(\infty) > 0 \qquad -\infty < a < \infty.$$

Then $\{R_k^{(i)}\} \in M_\infty^*$.

New besides (2.1) consider the equation

$$Y'' + A(x)\, Y' + B(x)\, Y = 0. \tag{2.5}$$

Let A_i, B_i, F_i be determined by formulas analogous to (2.2) and (2.3) and $\{X_k^{(j)}\}_{k=0}^\infty$ denote the sequence of zeros of the j-th derivative of any nontrivial solution $Y(x)$ of (2.5). Then the following result holds.

Lemma 2.2. (see [8] Theorem 1.1). *If $f_i'(x) \in M_\infty(a, \infty)$, $0 < f_i(\infty) = F_j(\infty) < \infty$ and $(f_i - F_j) \in M_\infty(a, \infty)$ for some $i, j \geqq 0$, then the condition*

$$\int^\infty [f_i^{1/2}(x) - F_j^{1/2}(x)]\, \mathrm{d}x < \infty \tag{2.6}$$

is necessary and sufficient to ensure that corresponding to each sequence $\{X_k^{(j)}\}$ there is some solution of (2.1) *whose i-th derivative has zeros $x_k^{(i)}$ $k = 0, 1, \dots$ such that*

$$\{x_k^{(i)} - X_k^{(j)}\} \in M_{\infty,0}. \tag{2.7}$$

Remark 2.1. If in Lemma 2.2 moreover $(f_i - F_j) > 0$ on (a, ∞) holds, then we obtain (2.6) in the form

$$\{x_k^{(i)} - X_k^{(j)}\} \in M_{\infty,0}^*. \tag{2.6'}$$

Finally we establish the following result.

Lemma 2.3. *Let the hypotheses of Lemma* 2.1 *hold and let $y(x) = z(x)$, i.e. $x_k^{(i)}$ are related to $y(x)$. Then $\{P_k^{(i)}\} \in M_\infty^*$ where*

$$P_k^{(i)} = P_k^{(i)}(W) = W(x_k) \exp\left(-\frac{1}{2} \int a_i(x)\, \mathrm{d}x\right)_{x = x_k^{(i)}} | y^{(i+1)}(x_k^{(i)}) |^{-1}. \tag{2.8}$$

For the proof of this lemma see [5] pg. 3 and Remark 3.2.

3. Completely monotonic sequences

The main result of this paper is given by the following

Theorem 3.1. *Let $y(x)$ and $z(x)$ be two solutions of* (1.1) *which may or may not be linearly independent and let $\{\bar{x}_k^{(i)}\}_{k=0}^\infty$ and $\{x_k^{(i)}\}_{k=0}^\infty$ for $i = 0, 1, 2, \dots$ be the sequence whose k-th term is a zero of the i-th derivative of $y(x)$ and $z(x)$ respectively. Then each of the sequences whose k-th term is given below is completely monotonic* *on the interval* (p, ∞) *where* $p = \sqrt[2\beta]{\dfrac{\beta^2 v^2 - \frac{1}{4}}{\beta^2 \gamma^2}}$. *Let* $\beta \in \left(1; \frac{3}{2}\right\rangle$ *and* $|v| \geqq \geqq \frac{1}{2\beta}$:

$$y^2(x_k'), \tag{3.1}$$

$$\Delta y'^2(x_k) \equiv \Delta y'^2(x_k''), \tag{3.2}$$

$$\int_{X'_k}^{X'_{k+1}} b_0^{\varkappa} |y'(x)| \, \mathrm{d}x \quad \text{if} \quad \varkappa \leqq -\frac{1}{2}, \tag{3.3}$$

$$(x_k - x_k') \quad \text{provided} \quad x_0 > x_0', \tag{3.4}$$

$$(x_k - \bar{x}_k) \quad \text{provided} \quad x_0 > \bar{x}_0, \tag{3.5}$$

$$(x_k - \bar{x}_k') \quad \text{provided} \quad x_0 > \bar{x}_0', \tag{3.6}$$

$$(x_k' - \bar{x}_k') \quad \text{provided} \quad x_0' > \bar{x}_0'. \tag{3.7}$$

Moreover if $W(x)$ *is any completely monotonic function, then the same is true also for*

$$\frac{W(x_k)}{|y'(x_k)|} \quad \text{for example} \quad |y'(x_k)|^{-1}, \tag{3.8}$$

$$\frac{W(x_k')}{|y''(x_k')|} \quad \text{for example} \quad |y''(x_k')|^{-1}, \tag{3.9}$$

$$\frac{W(x_k'')}{|y'''(x_k'')|} \quad \text{for example} \quad |y'''(x_k)|^{-1}, \tag{3.10}$$

$$W(x_k) \quad \text{for example} \quad x_k^{-\varkappa}, \varkappa > 0, \tag{3.11}$$

$$W(x_k') \quad \text{for example} \quad (x_k')^{-\varkappa}, \varkappa > 0. \tag{3.12}$$

Moreover if $\omega(x)$ *is any function with the completely monotonic first derivative, then the same is true also for*

$$\Delta\omega(x_k) \quad \text{for example} \quad \Delta x_k^{\beta}, \quad 0 < \beta \leqq 1, \tag{3.13}$$

$$x_k' \quad \text{for example} \quad \Delta(x_k')^{\beta}, \quad 0 < \beta \leqq 1, \tag{3.14}$$

$$\left[\frac{\omega(x_{k+1})}{\omega(x_k)}\right]^{\beta} \quad \text{provided} \quad \beta > 0, \omega(x) > 0, \tag{3.15}$$

$$\left[\frac{\omega(x_{k+1}')}{\omega(x_k')}\right]^{\beta} \quad \text{provided} \quad \beta > 0, \omega(x) > 0. \tag{3.16}$$

Proof: Formulas (2.2) and (2.3) in case of equation (1.1) give

$$a_0 = a = 0; \qquad b_0 = b; \qquad f_0(x) = b_0 = \beta^2 \gamma^2 x^{2\beta - 2} - \frac{\beta^2 v^2 - \frac{1}{4}}{x^2},$$

$$a_1 = -\frac{b_0'}{b_0}; \qquad b_1 = b_0; \qquad f_1(x) = b_0 + \frac{1}{2}\frac{b_0''}{b_0} - \frac{3}{4}\left(\frac{b_0'}{b_0}\right)^2,$$

$$a_2 = -2\frac{b_0'}{b_0}; \qquad b_2 = b_0 - \frac{b_0''}{b_0} + 2\left(\frac{b_0'}{b_0}\right)^2; \qquad f_2(x) = b_0.$$

Let us choose in 2.4 $i = 1, \lambda = 2$ and

$$W_1(x) = \left(\frac{b'}{b} + 2a\right)\exp\left[-\int a\,dx\right] = \frac{b_0'}{b_0}.$$

It is clear, for the general rules for calculation with higher monotonic functions (see e.g. [9]) that under the hypotheses of Theorem 3.1 $W_1(x) \in M_\infty$ and $f_1'(x) \in \in M_\infty(p, \infty)$. In our case we have ([7] Theorem 7.1)

$$R_k'(W_1, 2) = -\Delta y^2(x_k')$$

and since $y^2(x_k') > 0$ for $k = 0, 1, \ldots$, lemma 2.1 implies that $\{y^2(x_k')\}\ M_\infty^*$ when $x_0' > p$.

To prove (3.2) we choose in (2.4) $i = 2$, $\lambda = 2$ and

$$W(x) = W_2(x) = \left(\frac{b_1'}{b_1} + 2a_1\right)\exp\left[-\int a_1\,dx\right] =$$

$$= \left(\frac{b_0'}{b_0} - 2\frac{b_0'}{b_0}\right)\exp\int\frac{b_0'}{b_0}\,dx = -b_0(x).$$

Thus we have $(-W_2(x)) \in M_\infty(p, \infty)$ and, in the same way as above

$$R_k''(-W_2, 2) = -R_k''(W_2, 2) = \Delta y'^2(x_k'') = \Delta y'^2(x_k).$$

Moreover since $f_2(x) \in M_\infty(p, \infty)$, an application of lemma 2.1 gives (3.2).

Property (3.3) is a consequence of lemma 2.1 with $i = \lambda = 1$ and with $W(x) = 1$ when $\varkappa = -\frac{1}{2}$.

To prove (3.4) we set $i = 0$, $j = 1$ in lemma 2.2, in which case (2.5) and (2.1) become identical. It is simple to see that

$$f_0 - f_1 = -\frac{1}{2}\frac{b_0''}{b_0} + \frac{3}{4}\left(\frac{b_0'}{b_0}\right)^2 \in M_\infty(p, \infty).$$

Now let us denote

$$\mathscr{F}(x) = f_0^{1/2} - f_1^{1/2}.$$

We can write $\mathscr{F}(x)$ in the form

$$\mathscr{F}(x) = \frac{f_0 - f_1}{f_0^{1/2} + f_1^{1/2}}.$$

We can find easy that

$$\lim_{x\to\infty} \mathscr{F}(x)\,x^\beta$$

exists and is finite when $\beta > 1$ and with respect to [3] pg. 549 we can see that (2.6) is valid, i.e. $\int^{\infty} (f_0 - f_1)\, dx < \infty$.

The proof of (3.5), (3.6) and (3.7) is analogous.

To prove (3.8) we use lemma 2.3 with $i = 0$. Since $a_0 = 0$, we have

$$P_k^{(0)}(W) = W(x_k) \mid y'(x_k) \mid^{-1}$$

and

$$P_k^{(0)}(W) = \mid y'(x_k) \mid^{-1}.$$

To prove (3.9) we set $\bar{W}(x) = W(x) \dfrac{1}{\sqrt{b_0(x)}}$. Since $W \in M_\infty$, $\dfrac{1}{\sqrt{b_0}} \in M_\infty(p, \infty)$, we have $\bar{W}(x) \in M_\infty(p, \infty)$ and

$$P_k'(\bar{W}) = W(x_k') \frac{1}{\sqrt{b_0(x_k')}} \left(\exp \frac{1}{2} \int \frac{b_0'}{b_0} \, dx \right)_{x = x_x'} \mid y''(x_k') \mid^{-1} = W(x_k') \mid y''(x_k') \mid^{-1},$$

$$P_k'(1) = \mid y''(x_k') \mid^{-1}.$$

Similarly to prove (3.10) we set $\bar{W}(x) = W(x) \dfrac{1}{b_0(x)}$ and

$$P_k''(\bar{W}) = \bar{W}(x_k'') \mid y'''(x_k'') \mid^{-1}.$$

Since $x_k'' = x_k$, it follows $P_k''(1) = \mid y'''(x_k) \mid^{-1}$.

Properties (3.11), (3.13) and (3.15) follow from ([6] Corollaries 3.1, 3.2 and 3.3).

Finally, properties (3.12), (3.14) and (3.16) can be proved in a similar way by using lemma 2.1.

Remark 3.1. Property (3.2) implies that the local extrema of the derivative of $\mid y(x) \mid$ increase, which contrasts with (3.1) (see e.g. [1] pg. 446). So (3.1) and (3.2) describe more precisely this fact which is known for the differential equation (1.2) (see [5]).

Remark 3.2. It is known that if $g(x) \in M_\infty(p, \infty)$ and $g'(x) < 0$ on (p, ∞), then $(-1)^j g^{(j)}(x) > 0$ for $j = 0, 1, \ldots$ This situation occurs in our case for the completely monotonic sequences.

Remark 3.3. It is known that the diameter $r(x)$ of the osculating circle to the curve $y = y(x)$ is given by the formula

$$r(x) = \frac{(1 + \mid y'(x) \mid^2)^{3/2}}{y''(x)}. \tag{3.17}$$

If $r_k = r_k(x_k')$ then (3.9) and (3.17) imply

$$\{r_k\} \in M_\infty,$$

which means that the diameters of the circles which osculate at the local extrema form a completely monotonic sequence on (p, ∞).

Similarly if $\bar{r}_k = r_k(x_k'')$, we have

$$\{\bar{r}_k\} \in M_\infty,$$

i.e. the same is true with respect to the locus of the derivative of $y(x)$.

Remark 3.4. It is known (see [4] pg 52) that the general solution of

$$y'' + \frac{2\alpha - 2\beta\nu + 1}{x} y' + \left[\beta^2\gamma^2 x^{2\beta-2} + \frac{\alpha(\alpha - 2\beta\nu)}{x^2}\right] y = 0 \qquad x \in (0, \infty), \tag{3.18}$$

can be expressed in the form

$$y(x) = x^{\beta\nu-\alpha} C_\nu(\gamma x^\beta),$$

where $C_\nu(x)$ is the Bessel function mentioned in (1.4).

By easy calculation we find that $f_0(x)$ defined by (2.3) has the form

$$f_0(x) = \beta^2\gamma^2 x^{2\beta-2} - \frac{\beta^2\nu^2 - \frac{1}{4}}{x^2}$$

which equals the quantity b_0 from the proof of theorem 3.1. Therefore, there holds for $\lambda > -1$

$$R_k^{(0)} = R_k(W, \lambda) = \int_{x_k}^{x_{k+1}} W(x)\left[\exp\frac{\lambda}{2}\int\frac{2\alpha - 2\beta\nu + 1}{x}\,\mathrm{d}x\right] |y(x)|^\lambda\,\mathrm{d}x =$$

$$= \int_{x_k}^{x_{k+1}} W(x)\, x^{\lambda(\alpha - \beta\nu + \frac{1}{2})} |y(x)|^\lambda\,\mathrm{d}x. \tag{3.19}$$

If $W(x) \in M_\infty(0, \infty)$, then by lemma 2.1 we have $\{R_k(W, \lambda)\} \in \mathrm{M}_\infty^*$. If $\lambda\left(\alpha - \beta\nu + \frac{1}{2}\right) \geqq 0$, then in (3.19) we can choose $W(x) = \frac{1}{x^{\lambda(\alpha - \beta\nu + \frac{1}{2})}}$ which belong to $M_\infty(0, \infty)$ and it holds

$$\left\{\int_{x_k}^{x_{k+1}} |y(x)|^\lambda\,\mathrm{d}x\right\} \in M_\infty^*$$

on $(0, \infty)$.

REFERENCES

[1] Abramowitz, M., Stegun, I. A.: *Handbook of Mathematical Functions*. Nat. Bureau of Standards 1964.

[1] Blaško, R., Háčik, M., Vereš, O.: *Poznámka o vlastnostiach vyššej monotónnosti istých Sturm-Liouvilleových funkcii, zvlášť Besselových funkcii*. Práce a štúdie VŠDS v Žiline (to appear)

[3] Kluvánek, I., Mišík, L., Švec, M.: *Matematika II*. SVTL Bratislava 1961.

[4] Korenev, B. G.: *Úvod do teorie Besselových funkcí*. SNTL Praha 1977.

[5] Laforgia, A., Vosmanský, J.: *Higher monotonicity properties of generalized Airy functions*. Consiglio Nazionale delle Ricerche (Italy) 1982 pp. 1–11.

[6] Lorch, L., Muldoon, M. E., Szego, P.: *Higher monotonicity properties of certain Sturm-Liouville functions III.* Canad. J. of Math. 22, 1970, pp. 1238–1265.
[7] Vosmanský, J.: *Certain higher monotonicity properties of i-th derivatives of solutions of* $y'' + a(t)y' + b(t)y = 0$. Archivum Mathematicum (Brno) 10, 1974, pp. 87–102.
[8] Vosmanský, J.: *Distribution of zeros of solutions of linear second order differential equations* Transl. from Differ. uravnenija 15, 1979, pp. 2115–2124.
[9] Vosmanský, J.: *Monotonic properties of zeros and extremants of the diff. equation* $y'' + q(t)y = 0$. Archivum Mathematicum (Brno) 6, 1971, pp. 37–74.

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Súhrn

POZNÁMKA O VLASTNOSTIACH VYŠŠEJ MONOTÓNNOSTI ZOVŠEOBECNENÝCH AIRYHO FUNKCIÍ

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V tomto článku sa skúma diferenciálna rovnica

$$y'' + \left[\beta^2\gamma^2 x^{2\beta-2} - \frac{\beta^2\nu^2 - \frac{1}{4}}{x^2}\right] y = 0 \qquad x \in (0, \infty), \tag{1.1}$$

kde $\gamma > 0$, $\beta \in \left(1, \frac{3}{2}\right\rangle$, ktorú rovnicu môžeme považovať za difer. rovnicu pre zovšeobecnené Airyho funkcie.

Pomocou známych postačujúcich podmienok sú odvodené vlastnosti kompletnej monotónnosti postupností (3.1)–(3.16) (veta 3.1); tieto výsledky rozširujú časť výsledkov práce [5]. Sú uvedené dva aplikačné príklady.

Резюме

ЗАМЕТКА О СВОЙСТВАХ МОНОТОННОСТИ ВЫСШЕГО ПОРЯДКА ОБОБЩЕННЫХ ФУНКЦИЙ ЭЙРИ

МИЛОШ ГАЧИК И ПЕТЕР ИВАН

В этой статье изучается дифференциальное уравнение

$$y'' + \left[\beta^2\gamma^2x^{2\beta-2} - \frac{\beta^2\nu^2 - \frac{1}{4}}{x^2}\right]y = 0 \qquad x \in (0, \infty) \tag{1.1}$$

которое становится (в определенном смысле) уравнением для обобщенных функций Эйри.

При помощи знакомых достаточных условий здесь дедуцированы свойства комплетной монотонности последовательностей (3.1)–(3.16) что является расширением части результатов работы [5]. Тоже здесь введены примеры для апликаций.