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ON THE INTERSECTION OF GROUPS OF DISPERSIONS OF THE EQUATION $y'' = q(t)y$ AND OF ITS ACCOMPANYING EQUATION

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1. Introduction

O. Borůvka in [3] and the author in [5] investigated a structure of the intersection of the groups of the first kind dispersions of two oscillatory differential equations having the form

$$y'' = q(t)\,y, \qquad q \in C^0(\mathbf{R}). \tag{q}$$

In [1] O. Borůvka introduced the accompanying equation ($\hat{\text{q}}$) to (q). This paper investigates a structure of the intersection of the groups of dispersions relative to (q) and ($\hat{\text{q}}$), i.e. a structure of the intersection for the first and second kind dispersions of (q).

2. Basic concepts and relations

Equation (q) is called oscillatory (on $\mathbf{R}$) if $\pm\infty$ are cluster points of the roots for every solution of (q). We eliminate from consideration the trivial solutions of (q).

Let (q) be an oscillatory equation. A function $X \in C^3(\mathbf{R})$, $X'(t) \neq 0$ for $t \in \mathbf{R}$ is called a dispersion of the first kind of (q) if it is a solution of the nonlinear differential equation

$$-\{X, t\} + X'^2 \,.\, q(X) = q(t),$$

where $\{X, t\} := \frac{1}{2}\frac{X'''(t)}{X'(t)} - \frac{3}{4}\left(\frac{X''(t)}{X'(t)}\right)^2$ is Schwarz's derivative of X at the point t. The set of the first kind dispersions of (q) forms a group with respect to the composition of functions. $\mathscr{L}_q^+$ stands for the group of increasing first kind dispersions of (q). A function $X \in C^3(\mathbf{R})$, $X'(t) \neq 0$ for $t \in \mathbf{R}$, is a first kind dispersion of (q) exactly if to every solution y of (q) there exists only one solution u of this equation, such that $\frac{y[X(t)]}{\sqrt{|X'(t)|}} = u(t)$, $t \in \mathbf{R}$.

Let $\mathscr{S}$ be a subgroup of the group $\mathscr{L}_q^+$. In accordance with [3] we say that $\mathscr{S}$ is a continuous planar group if and only if there exists for each $(t_0, x_0) \in \mathbf{R}\times\mathbf{R}$ only one element $X \in \mathscr{S}$ such that $X(t_0) = x_0$.

Lemma 1. ([3], [5]). *Let* (p), (q) *be oscillatory equations,* $q - p \in C^2(\mathbf{R})$. *Then* $\mathscr{L}_p^+ = \mathscr{L}_q^+$ *exactly if* $p = q$. *If* $p \neq q$, *then* $\mathscr{L}_p^+ \cap \mathscr{L}_q^+$ *is either a continuous planar group or an infinite cyclic group or* $\mathscr{L}_p^+ \cap \mathscr{L}_q^+ = \{\mathrm{id}_\mathbf{R}\}$.

Lemma 2. ([3]). *Let* (p), (q) *be oscillatory equations. Then* $\mathscr{L}_p^+ \cap \mathscr{L}_q^+$ *is a continuous planar group exactly if there exist* $X \in C^3(\mathbf{R})$, $k_1(<0)$, $k_2(<0)$, $X'(t) \neq 0$ *for* $t \in \mathbf{R}$, $X(\mathbf{R}) = \mathbf{R}$, $k_1 \neq k_2$:

$$-\{X, t\} + k_1 \,.\, X'^2(t) = p(t),$$

$$-\{X, t\} + k_2 \,.\, X'^2(t) = q(t), \qquad t \in \mathbf{R}.$$

A function $\alpha \in C^0(\mathbf{R})$ is called a first phase of (q) if there exist independent solutions u, v of (q):

$$\operatorname{tg}\alpha(t) = \frac{u(t)}{v(t)} \qquad \text{for } t \in R - \{t; v(t) = 0\}.$$

A function α is a first phase of (q) exactly if it is a solution of the equation

$$-\{\alpha, t\} - \alpha'^2(t) = q(t).$$

Let $q \in C^2(\mathbf{R})$, $q(t) < 0$ for $t \in \mathbf{R}$. Let us put $\hat{q}(t) := q(t) + \sqrt{-q(t)}\left(\frac{1}{\sqrt{-q(t)}}\right)''$, $t \in \mathbf{R}$. Equation ($\hat{\text{q}}$) is called the accompanying equation to (q). Between the solutions of (q) and those of ($\hat{\text{q}}$) there holds: If u is a solution of (q), then $\frac{u'(t)}{\sqrt{-q(t)}}$ is a solution of ($\hat{\text{q}}$) and vice versa; if z is a solution of (q), then the function $z(t)\sqrt{-q(t)}$ is the derivative of a solution of (q).

The first phase of ($\hat{\text{q}}$) and the first kind dispersion of ($\hat{\text{q}}$) are called the second phase of (q) and the second kind dispersion of (q), respectively.

Let α and β be increasing (decreasing) first and second phases of (q), respectively. Then the function $\alpha(t) - \beta(t)$ is bounded on $\mathbf{R}$.

All the above definitions and properties are presented in [1] and [2].

3. Main results

Theorem 1. *Let* (q) *be an oscillatory equation and* ($\hat{\text{q}}$) *be its accompanying equation,* $\hat{q} - q \in C^2(\mathbf{R})$. *Then* $\mathscr{L}_q^+ = \mathscr{L}_{\hat{q}}^+$ *exactly if* $q(t)$ *is a constant* (<0) *If* $q(t)$ *is not a constant, then* $\mathscr{L}_q^+ \cap \mathscr{L}_{\hat{q}}^+$ *is either an infinite cyclic group or* $\mathscr{L}_q^+ \cap \mathscr{L}_{\hat{q}}^+ = \{\mathrm{id}_\mathbf{R}\}$.

Proof. From Lemma 1 it follows that $\mathscr{L}_q^+ = \mathscr{L}_{\hat{q}}^+$ exactly if $q = \hat{q}$ which is true only if $q(t) = a$ constant (<0) (cf. [4]).

Let $q \neq \hat{q}$. With reference to Lemma 1 it suffices to show that $\mathscr{L}_q^+ \cap \mathscr{L}_{\hat{q}}^+$ is not a continuous planar group. In the contrary case there exist function $X \in C^3(\mathbf{R})$, $X'(t) \neq 0$ for $t \in \mathbf{R}$, $X(\mathbf{R}) = \mathbf{R}$, and the numbers k_1 (< 0), k_2 (< 0), $k_1 \neq k_2$:

$$-\{X, t\} + k_1 \,.\, X'^2(t) = q(t),$$
$$-\{X, t\} + k_2 \,.\, X'^2(t) = \hat{q}(t), \qquad t \in \mathbf{R}.$$

Let us put $\alpha(t) := \sqrt{-k_1} \,.\, X(t)$, $\beta(t) := \sqrt{-k_2} \,.\, X(t)$, $t \in \mathbf{R}$. Then α is a first phase of (q) and β is its a second phase. In consequence of

$$\lim_{|t| \to \infty} |\alpha(t) - \beta(t)| = \lim_{|t| \to \infty} |(\sqrt{-k_1} - \sqrt{-k_2})\, X(t)| = \infty,$$

we are led to a contradiction in that the function $\alpha(t) - \beta(t)$ must be bounded on $\mathbf{R}$.

Theorem 2. *Let* (q) *be an oscillatory equation and* ($\hat{\text{q}}$) *be its accompanying equation. Let* $X \in \mathscr{L}_q^+ \cap \mathscr{L}_{\hat{q}}^+$. *Then* $X(t) = t + a$ *and* a *is a constant exactly if* $X''(t_1) = X'''(t_1) = 0$, $X''(t_2) = X'''(t_2) = 0$ *where* t_1, t_2 *are not conjugate points of* ($\hat{\text{q}}$).

Proof. Let $X \in \mathscr{L}_q^+ \cap \mathscr{L}_{\hat{q}}^+$ and $X(t) = t + a$, $a \in \mathbf{R}$. Then $X''(t) = X'''(t) = 0$ for $t \in \mathbf{R}$.

Let $X \in \mathscr{L}_q^+ \cap \mathscr{L}_{\hat{q}}^+$ and $X''(t_1) = X'''(t_1) = 0$, $X''(t_2) = X'''(t_2) = 0$, where t_1, t_2 are not conjugate points of ($\hat{\text{q}}$). Then there exists to every solution y of (q) only one solution u of (q) and only one solution z of (q) such that

$$\frac{y[X(t)]}{\sqrt{X'(t)}} = u(t), \qquad t \in \mathbf{R}, \tag{1}$$

$$\frac{y'[X(t)]}{\sqrt{X'(t)}\sqrt{-q[X(t)]}} = \frac{z'(t)}{\sqrt{-q(t)}}, \qquad t \in \mathbf{R}. \tag{2}$$

From (1) we obtain $X'(t)\, y'[X(t)] = (\sqrt{X'(t)}\, u(t))'$. Inserting this into (2) gives

$$\frac{(\sqrt{X'(t)}\, u(t))'}{X'(t)\sqrt{X'(t)}\sqrt{-q[X(t)]}} = \frac{z'(t)}{\sqrt{-q(t)}}, \qquad t \in \mathbf{R}. \tag{3}$$

Thus there exists to every solution u of (q) only one solution z of (q) satisfying (3). Relation (3) may be also written in the following form

$$\frac{u'(t)}{X'(t)\sqrt{-q[X(t)]}} + \frac{1}{2} \cdot \frac{X''(t)\, u(t)}{X'^2(t)\sqrt{-q[X(t)]}} = \frac{z'(t)}{\sqrt{-q(t)}}, \qquad t \in \mathbf{R}. \tag{4}$$

Let $u \neq 0$ be a solution of (q), $u'(t_1) = 0$. According to our assumption $X''(t_1) = 0$ and therefore it follows $z'(t_1) = 0$ from (4). Then there exists a number $k \neq 0$: $u(t) = k \,.\, z(t)$ for $t \in \mathbf{R}$ and we obtain

$$2u'(t)\left(\frac{k}{\sqrt{-q(t)}} - \frac{1}{X'(t)\sqrt{-q[X(t)]}}\right) = \frac{X''(t)\,u(t)}{X'^2(t)\sqrt{-q[X(t)]}}, \qquad t \in \mathbf{R}. \qquad (5)$$

Putting $t = t_2$ in (5), gives $X'(t_2)\sqrt{-q[X(t_2)]} = \frac{1}{k}\sqrt{-q(t_2)}$. The equalities $-\{X, t\} + X'^2(t)\,.\,q[X(t)] = q(t)$, $X''(t_2) = X'''(t_2) = 0$ yield $X'^2(t_2)\,.\,q[X(t_2)] = q(t_2)$. Hence $k = 1$ and we have

$$2u'(t)\left(\frac{1}{\sqrt{-q(t)}} - \frac{1}{X'(t)\sqrt{-q[X(t)]}}\right) = \frac{X''(t)\,\mu(t)}{X'^2(t)\sqrt{-q[X(t)]}}, \qquad t \in \mathbf{R}. \qquad (6)$$

Let $v \neq 0$ be a solution of (q), $v(t_2) = 0$. Analogous to the above we can prove

$$2v'(t)\left(\frac{1}{\sqrt{-q(t)}} - \frac{1}{X'(t)\sqrt{-q[X(t)]}}\right) = \frac{X''(t)\,v(t)}{X'^2(t)\sqrt{-q[X(t)]}}, \qquad t \in \mathbf{R}. \qquad (7)$$

The solutions u, v of (q) are independent, $uv' - u'v := w \neq 0$. From (6) and (7) follows

$$\frac{X''(t)\,w}{X'^2(t)\sqrt{-q[X(t)]}} = 0, \qquad t \in \mathbf{R}.$$

Then $X''(t) = 0$ for $t \in \mathbf{R}$, hence $X(t) = bt + a$, where a, b are constants. In consequence of the fact that either $X = \mathrm{id}_{\mathbf{R}}$ or $X(t) \neq t$ for $t \in \mathbf{R}$ (cf. [3]), we see that necessarily $b = 1$.

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Souhrn

STRUKTURA PRŮNIKU GRUP DISPERSÍ ROVNICE $y'' = q(t)y$ A ROVNICE K NÍ PRŮVODNÍ

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Nechť $q \in C^2(\mathbf{R})$, $q(t) < 0$ pro $t \in \mathbf{R}$. Položme $\hat{q}(t) := q(t) + \sqrt{-q(t)} \left(\frac{1}{\sqrt{-q(t)}} \right)''$, $t \in \mathbf{R}$. Rovnice ($\hat{q}$): $y'' = \hat{q}(t)\,y$ se nazývá průvodní rovnice k rovnici (q) : $y'' = q(t)\,y$.

Nechť (q) je oscilatorická rovnice, $q \neq \hat{q}$. V práci je vyšetřována struktura průniku množiny rostoucích řešení dvou Kummerových diferenciálních rovnic

$$-\{X, t\} + X'^2 \,.\, q(X) = q(t),$$
$$-\{X, t\} + X'^2 \,.\, \hat{q}(X) = \hat{q}(t),$$

kde $\{X, t\} = \frac{1}{2} \frac{X'''(t)}{X'(t)} - \frac{3}{4} \left(\frac{X''(t)}{X'(t)} \right)^2$. Je dokázáno, že tento průnik je buď nekonečná cyklická grupa a nebo triviální grupa. Dále jsou uvedeny podmínky, které jsou nutné a postačující k tomu, aby funkce $X(t) = t + a$, $a \in \mathbf{R}$, byla prvkem tohoto průniku.

Резюме

СТРУКТУРА ПЕРЕСЕЧЕНИЙ ГРУПП ДИСПЕРСИЙ УРАВНЕНИЯ $y'' = q(t)y$ И ЕГО СОПРОВОЖДАЮЩЕГО УРАВНЕНИЯ

СВАТОСЛАВ СТАНЕК

Пусть $q \in C^2(\mathbf{R})$, $q(t) < 0$ для $t \in \mathbf{R}$. Положим $\hat{q}(t) := q(t) + \sqrt{-q(t)} \left(\frac{1}{\sqrt{-q(t)}} \right)''$, $t \in \mathbf{R}$. Уравнение ($\hat{q}$): $y'' = \hat{q}(t)y$ называется сопровождающим уравнением относительно уравнения (q): $y = q(t)y$. Пусть (q) колеблющиеся уравнение, $q \neq \hat{q}$. В работе иследуется структура пересечения множеств возрастающих решений двух уравнений Куммера

$$-\{X, t\} + X'^2 \,.\, q(X) = q(t),$$
$$-\{X, t\} + X'^2 \,.\, \hat{q}(X) = \hat{q}(t),$$

где $\{X, t\} = \frac{1}{2}\frac{X'''(t)}{X'(t)} - \frac{3}{4}\left(\frac{X''(t)}{X'(t)}\right)^2$. Доказано что это пересечение или бесконечная циклическая группа или тривиальная группа и указаны необходимые и достаточные условия при выполнении которых функция $X(t) = t + a$, $a \in \mathbf{R}$, элемент этого пересечения.