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Katedra matematické analýzy a numerické matematiky přírodovědecké fakulty Univerzity Palackého v Olomouci
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ON A CERTAIN TRANSFORMATION OF THE SOLUTION SET OF TWO LINEAR SECOND ORDER DIFFERENTIAL EQUATIONS

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1. Introduction

In [1] – [3] there were investigated transformations of the type

$$A(t)\,u + B(t)\,v \tag{1}$$

with such a property that under certain assumptions the function $A(t)\,y(t) + B(t)\,y'(t)$ is a solution of equation (Q) : $Y'' = Q(t)\,Y$ for every solution y of equation (q) : $y'' = q(t)\,y$ and also conversely: there exists one and only one solution y of (q) such that $Y(t) = A(t)\,y(t) + B(t)\,y'(t)$ to every solution Y of (Q). The present paper gives conditions necessary and sufficient for the transformation of the solution set of (q) onto the solution set of (Q) to be of the form (1).

2. Auxiliary lemmas

We consider differential equations of the type

$$y'' = p(t)\,y, \qquad p \in C^{0}(\mathbf{j}), \tag{p}$$

where $\mathbf{j} = (a, b)$, $-\infty \leqq a < b \leqq \infty$. If necessary further restrictive assumptions will be imposed on the coefficient p of (p).

Definition 1. *Let* $q \in C^1(\mathbf{j})$, $Q \in C^\circ(\mathbf{j})$. *Let a function* $F = F(t, u, v)$ *be defined on* $\mathbf{j} \times \mathbf{R} \times \mathbf{R}$ *having here continuous partial derivatives* $\frac{\partial^k F(t, u, v)}{\partial t^i \partial u^j \partial v^l}$ *for* $0 \leqq k \leqq 4$, $0 \leqq i \leqq 2$, $0 \leqq j \leqq 4$, $0 \leqq l \leqq 4$, $i + j + l = k$. *Say, the function* F *maps the solution set of* (q) *onto the solution set of* (Q) (*or more briefly: equation* (q) *onto equation* (Q)) *if the function* $Y(t) := F(t, y(t), y'(t))$, $t \in \mathbf{j}$, *is a solution of* (Q) *for every solution* $y = y(t)$ *of* (q) *and also conversely, there exists one and only one solution* $y = y(t)$ *of* (q) *to every solution* $Y = Y(t)$ *of* (Q) *such that the equality* $Y(t) = F(t, y(t), y'(t))$ *holds on* $\mathbf{j}$.

Lemma 1. *Let* $q \in C^1(\mathbf{j})$, $Q \in C^\circ(\mathbf{j})$. *The function*

$$F(t, u, v) := A(t)\, u + B(t)\, v, \qquad (t, u, v) \in \mathbf{j} \times \mathbf{R} \times \mathbf{R}, \tag{2}$$

maps equation (q) *onto equation* (Q) *iff* A, B *is a solution of the system of differential equations*

$$\begin{gathered} A'' + (q(t) - Q(t))\, A + 2q(t)\, B' + q'(t)\, B = 0, \\ 2A' + B'' + (q(t) - Q(t))\, B = 0, \end{gathered} \tag{3}$$

and

$$q(t)\, B^2(t) - A^2(t) + A'(t)\, B(t) - B'(t)\, A(t) = a \text{ constant } (\neq 0). \tag{4}$$

Proof. ($\Rightarrow$) If the function F defined by (2) maps (q) onto (Q) then the function $Y := Ay + By'$ is the solution of (Q) for every solution y of (q). It follows from the equalities

$$\begin{aligned} A''y + 2A'y' + qAy + B''y' + 2qB'y + B(qy' + q'y) = \\ = Q(Ay + By') \end{aligned}$$

holding for every solution y of (q) that A, B is a solution of (3). Let y_1, y_2 be independent solutions of (q) and put $Y_i := Ay_i + By'_i$, $i = 1, 2$. Then Y_1, Y_2 are necessarily independent solutions of (Q) and

$$Y_1 Y'_2 - Y'_1 Y_2 = (qB^2 - A^2 + A'B - AB')\,(y'_1 y_2 - y_1 y'_2)$$

holds for their Wronskian, whence (4) follows.

($\Leftarrow$) Let A, B be a solution of (3) and (4) be valid. Let y be a solution of (q) and put $Y := Ay + By'$. By an easy calculation it can be verified that Y is a solution of (Q). Let y_1, y_2 be independent solutions of (q) and put $Y_i := Ay_i + By'_i$, $i = 1, 2$. Then it follows from (4) and (5) that Y_1 and Y_2 are independent solutions of (Q) and thus the function F defined by (2) maps (q) onto (Q).

Example 1. Let $q \in C^2(\mathbf{j})$, $q(t) \neq 0$ for $t \in \mathbf{j}$. Let $F(t, u, v) := B(t)\, v$ for $(t, u, v) \in \mathbf{j} \times \mathbf{R} \times \mathbf{R}$ map (q) onto (Q). Then it follows from Lemma 1 that B is a solution of the system

$$\begin{gathered} 2q(t)\, B' + q'(t)\, B = 0, \\ B'' + (q(t) - Q(t))\, B = 0 \end{gathered}$$

and $q(t)\,B^2(t) = k$, where $k \neq 0$ is a constant. Then $B(t) = \dfrac{\sqrt{|k|}}{\sqrt{|q(t)|}}$ and $Q(t) = q(t) + \sqrt{|q(t)|}\left(\dfrac{1}{\sqrt{|q(t)|}}\right)''$. The transformation of the form $F = B(t)\,v$ was investigated in [1].

Example 2. Let $q \in C^2(\mathbf{j})$, $\alpha, \beta \in \mathbf{R}$, $\alpha^2 + \beta^2 > 0$, $\alpha^2 - \beta^2 q(t) \neq 0$ for $t \in \mathbf{j}$. Let $F(t, u, v) := \alpha A(t)\, u + \beta A(t)\, v$, $(t, u, v) \in \mathbf{j} \times \mathbf{R} \times \mathbf{R}$, map (q) onto (Q). Then it follows from Lemma 1 that A is a solution of the system

$$\alpha A'' + \alpha(q(t) - Q(t))\, A + 2\beta q(t)\, A' + \beta q'(t)\, A = 0,$$
$$2\alpha A' + \beta A'' + \beta(q(t) - Q(t))\, A = 0,$$

and $\beta^2 q(t)\, A^2(t) - \alpha^2 A^2(t) - \alpha^2 A^2(t) = k$, where $k \neq 0$ is a constant. Then $A(t) = \dfrac{\sqrt{|k|}}{\sqrt{|\alpha^2 - \beta^2 q(t)|}}$, the function F may be written in the form $F = \dfrac{\alpha u + \beta v}{\sqrt{|\alpha^2 - \beta^2 q(t)|}}$ and $Q = q + \dfrac{\alpha\beta q'}{\alpha^2 - \beta^2 q} + \sqrt{|\alpha^2 - \beta^2 q|}\left(\dfrac{1}{\sqrt{|\alpha^2 - \beta^2 q|}}\right)''$.

The transformation of the form $F = \dfrac{\alpha u + \beta v}{\sqrt{|\alpha^2 - \beta^2 q|}}$ was investigated in [3].

Example 3. Let $q \in C^2(\mathbf{j})$, $\alpha \in C^3(\mathbf{j})$, $\beta \in C^3(\mathbf{j})$, $\alpha^2(t) + \alpha(t)\,\beta'(t) - \alpha'(t)\,\beta(t) - \beta^2(t)\, q(t) \neq 0$ for $t \in \mathbf{j}$. The transformation of form (2) mapping (q) onto (Q) was investigated in [2], where

$$A = \frac{\alpha}{\sqrt{|\alpha^2 + \alpha\beta' - \alpha'\beta - \beta^2 q|}}, \qquad B = \frac{\beta}{\sqrt{|\alpha^2 + \alpha\beta' - \alpha'\beta - \beta^2 q|}}.$$

Then
$$Q = q + (\alpha\alpha'' + 2\alpha\beta' q + \alpha\beta q' + \alpha''\beta' + 2\beta'^2 q + \beta\beta' q' - \alpha'\beta'' - 2\alpha'^2 - 2\alpha'\beta q - \beta\beta' q)(\alpha^2 + \alpha\beta' - \alpha'\beta - \beta^2 q)^{-1} + \sqrt{|\alpha^2 + \alpha\beta' - \alpha'\beta - \beta^2 q|} \times \left(\frac{1}{\sqrt{|\alpha^2 + \alpha\beta' - \alpha'\beta - \beta^2 q|}}\right)''.$$

Lemma 2. *Let $q \in C^1(\mathbf{j})$, $Q \in C^\circ(\mathbf{j})$. Let y_1, y_2 be independent solutions of* (q) *and Y_1, Y_2 be independent solutions of* (Q). *Then*

$$A = c_1 y_1' Y_1 + c_2 y_1' Y_2 + c_3 y_2' Y_1 + c_4 y_2' Y_2,$$
$$B = -c_1 y_1 Y_1 - c_2 y_1 Y_2 - c_3 y_2 Y_1 - c_4 y_2 Y_2, \tag{6}$$

with c_1, c_2, c_3, c_4 being arbitrary constants, is the general solution of (3).

Proof. Put $x_1 := A$, $x_3 := B$. System (3) may then be written in the following equivalent form

$$x_1' = x_2,$$
$$x_2' = (Q(t) - q(t))\, x_1 - q'(t)\, x_3 - 2q(t)\, x_4,$$

$$x_3' = x_4,$$
$$x_4' = (Q(t) - q(t))\,x_3 - 2x_2.$$

Let y and Y be solutions of (q) and (Q), respectively. Putting $x_1 := y'Y$, $x_2 := x_1'$, $x_3 := -yY$, $x_4 := x_3'$ yields

$$\begin{aligned} x_2' = x_1'' &= (q'(t)\,y + q(t)'y')\,Y + 2q(t)\,yY' + Q(t)\,y'Y = \\ &= (Q(t) - q(t))\,y'Y + q'(t)\,yY + 2q(t)\,(y'Y + yY') = \\ &= (Q(t) - q(t))\,x_1 - q'(t)\,x_3 - 2q(t)\,x_3' = \\ &= (Q(t) - q(t))\,x_1 - q'(t)\,x_3 - 2q(t)\,x_4, \end{aligned}$$

$$\begin{aligned} x_4' = x_3'' &= -q(t)\,yY - 2y'Y' - Q(t)\,yY = \\ &= (q(t) - Q(t))\,yY - 2(q(t)\,yY + y'Y') = \\ &= (Q(t) - q(t))\,x_3 - 2x_1' = (Q(t) - q(t))\,x_3 - 2x_2. \end{aligned}$$

From this we find that $A := y'Y$, $B := -yY$ is a solution of (3). Let us put $w := y_1y_2' - y_1'y_2$, $W := Y_1Y_2' - Y_1'Y_2$. Then a brief calculation verifies

$$\begin{vmatrix} y_1'Y_1 & y_2'Y_1 & y_1'Y_2 & y_2'Y_2 \\ qy_1Y_1 + y_1'Y_1' & qy_2Y_1 + y_2'Y_1' & qy_1Y_2 + y_1'Y_2' & qy_2Y_2 + y_2'Y_2' \\ -y_1Y_1 & -y_2Y_1 & -y_1Y_2 & -y_2Y_2 \\ -y_1'Y_1 - y_1Y_1' & -y_2'Y_1 - y_2Y_1' & -y_1'Y_2 - y_1Y_2' & -y_2'Y_2 - y_2Y_2' \end{vmatrix} = w^2W^2 \neq 0.$$

This proves the assertion of the Theorem above.

Corollary 1. *Let the assumptions of Lemma 2 be satisfied. Then all solutions A, B of (3) satisfying (4) are of the form (6), where c_1, c_2, c_3, c_4 are arbitrary constant, $c_1c_4 - c_2c_3 \neq 0$.*

Proof. Let us put $w := y_1y_2' - y_1'y_2$, $W := Y_1Y_2' - Y_1'Y_2$ and A, B be defined by (6). Following Lemma 2 A, B is a solution of (3) and a simple calculation verifies that

$$qB^2 - A^2 + A'B - B'A = wW(c_2c_3 - c_1c_4).$$

From this immediately follows the assertion of Corollary 1.

Remark 1. Let $q \in C^1(j)$, $Q \in C^\circ(j)$. It appears from the examples below that the form of the mapping F of (q) onto (Q) is not generally of form (1), where A, B are suitable functions.

Example 4. Let us put $q(t) := 1$, $Q(t) := 9$ for $t \in \mathbf{R}$, $F(t, u, v) := (u + v)^3 + (u - v)^3$ for $(t, u, v) \in \mathbf{R} \times \mathbf{R} \times \mathbf{R}$. The functions e^t, e^{-t} are independent solutions of (q). It holds

$$\begin{aligned} F(t, c_1e^t + c_2e^{-t}, c_1e^t - c_2e^{-t}) &= (c_1e^t + c_1e^t)^3 + (c_2e^{-t} + c_2e^{-t})^3 = \\ &= 8c_1^3e^{2t} + 8c_2^3e^{-3t} \end{aligned}$$

for every $c_1, c_2 \in \mathbf{R}$. Let Y be a solution of (Q), $Y(0) = \alpha$, $Y'(0) = \beta$. It follows $c_1 = \frac{1}{2}\sqrt[3]{\frac{3\alpha+\beta}{6}}$, $c_2 = \frac{1}{2}\sqrt[3]{\frac{3\alpha-\beta}{6}}$ from the equalities $8c_1^3 + 8c_2^3 = \alpha$, $24c_1^3 - 24c_2^3 = \beta$. We see that F maps (q) onto (Q).

Example 5. Let $q(t) := 0$, $Q(t) := 0$ for $t \in \mathbf{R}$, $F(t, u, v) := u + v^2$ for $(t, u, v) \in \mathbf{R}\times\mathbf{R}\times\mathbf{R}$. The function $c_1 + c_2 t$, where c_1, c_2 are arbitrary constants, is the general solution of (q). It holds for every $c_1, c_2 \in \mathbf{R}$ and $t \in \mathbf{R}$ that

$$F(t, c_1 + c_2 t, c_2) = (c_1 + c_2^2) + c_2 t.$$

Let $t_0 \in \mathbf{R}$ and Y be a solution of (Q), $Y(t_0) = \alpha$, $Y'(t_0) = \beta$. From equations $c_1 + c_2^2 + c_2 t_0 = \alpha$, $c_2 = \beta$ it follows $c_1 = \alpha - \beta^2 - \beta t_0$, $c_2 = \beta$. We see that the function F maps (q) onto (Q).

Example 6. Let $q(t) := 1$, $Q(t) := 1$ for $t \in \mathbf{R}$, $F(t, u, v) := e^{-2t}(u + v)^3 + u - v$ for $(t, u, v) \in \mathbf{R}\times\mathbf{R}\times\mathbf{R}$. Then $c_1 e^t + c_2 e^{-t}$, where c_1, c_2 are arbitrary constants, is the general solution of (q). Then we have for every c_1, c_2 and $t \in \mathbf{R}$

$$F(t, c_1 e^t + c_2 e^{-t}, c_1 e^t - c_2 e^{-t}) = e^{-2t}(2c_1 e^t)^3 + 2c_2 e^{-t} = 8c_1^3 e^t + 2c_2 e^{-t}.$$

Let Y be a solution of (Q) satisfying the initial conditions $Y(0) = \alpha$, $Y'(0) = \beta$. From equations $8c_1^3 + 2c_2 = \alpha$, $8c_1^3 - 2c_2 = \beta$ we obtain $c_1 = \frac{1}{2}\sqrt[3]{\frac{\alpha+\beta}{2}}$, $c_2 = \frac{\alpha - \beta}{4}$. Thus it appears that F maps (q) onto (Q).

Lemma 3. *Let $q \in C^1(\mathbf{j})$, $Q \in C^0(\mathbf{j})$ and let F map (q) onto (Q). Furthermore, let x be a solution of (q) and write*

$$a(t) := \frac{\partial^2 F}{\partial u^2}(t, x(t), x'(t)), \qquad b(t) := \frac{\partial^2 F}{\partial u\, \partial v}(t, x(t), x'(t)),$$

$$c(t) := \frac{\partial^2 F}{\partial v^2}(t, x(t), x'(t)), \qquad t \in \mathbf{j}. \tag{7}$$

Then

$$(ay + by')\, z + (by + cy')\, z'$$

is a solution of (Q) for every solution y or z of (q).

Proof. Let x_1, x_2 be independent solutions of (q). Putting $Y(t, \alpha, \beta) := F(t, \alpha x_1(t) + \beta x_2(t), \alpha x_1'(t) + \beta x_2'(t))$, $(t, \alpha, \beta) \in \mathbf{j}\times\mathbf{R}\times\mathbf{R}$, then $Y(t, \alpha, \beta)$ is a solution of (Q) for every $\alpha, \beta \in \mathbf{R}$. But also the functions

$$\frac{\partial^2 Y}{\partial \alpha^2}, \quad \frac{\partial^2 Y}{\partial \alpha\, \partial \beta}, \quad \frac{\partial^2 Y}{\partial \beta^2},$$

are solutions of (Q). It holds

$$\begin{aligned}
\frac{\partial^2 Y}{\partial \alpha^2} &= \frac{\partial^2 F}{\partial u^2} x_1^2 + 2 \frac{\partial^2 F}{\partial u\, \partial v} x_1 x_1' + \frac{\partial^2 F}{\partial v^2} x_1'^2, \\
\frac{\partial^2 Y}{\partial \alpha\, \partial \beta} &= \frac{\partial^2 F}{\partial u^2} x_1 x_2 + \frac{\partial^2 F}{\partial u\, \partial v} (x_1 x_2' + x_1' x_2) + \frac{\partial^2 F}{\partial v^2} x_1' x_2', \\
\frac{\partial^2 Y}{\partial \beta^2} &= \frac{\partial^2 F}{\partial u^2} x_2^2 + 2 \frac{\partial^2 F}{\partial u\, \partial v} x_2 x_2' + \frac{\partial^2 F}{\partial v^2} x_2'^2,
\end{aligned} \tag{8}$$

where the values of the partial derivatives of the function F are taken at the point $(t, \alpha x_1(t) + \beta x_2(t), \alpha x_1'(t) + \beta x_2'(t))$. Let $x = \alpha_1 x_1 + \beta_1 x_2$, where α_1, β_1 are appropriate numbers. Writing α_1 and β_1 for α and β, respectively, in (8), we obtain

$$\begin{aligned}
Y_1 &:= a x_1^2 + 2b x_1 x_1' + c x_1'^2, \\
Y_2 &:= a x_1 x_2 + b(x_1 x_2' + x_1' x_2) + c x_1' x_2', \\
Y_3 &:= y x_2^2 + 2b x_2 x_2' + c x_2'^2,
\end{aligned}$$

as solutions of (Q). The assertion of the Lemma follows from the equalities

$$\begin{aligned}
\alpha Y_1 + \beta Y_2 &= [a(\alpha x_1 + \beta x_2) + b(\alpha x_1 + \beta x_2)'] x_1 + \\
&\quad + [b(\alpha x_1 + \beta x_2) + c(\alpha x_1 + \beta x_2)'] x_1', \\
\alpha Y_2 + \beta Y_3 &= [a(\alpha x_1 + \beta x_2) + b(\alpha x_1 + \beta x_2)'] x_2 + \\
&\quad + [b(\alpha x_1 + \beta x_2) + c(\alpha x_1 + \beta x_2)'] x_2'.
\end{aligned}$$

3. Main results

Theorem 1. *Let $q \in C^1(\mathbf{j})$, $Q \in C^0(\mathbf{j})$ and let q not be equal to a nonnegative constant in any interval. Let a functional F map* (q) *onto* (Q). *It then follows that*

$$F(t, u, v) = A(t)\, u + B(t)\, v, \qquad (t, u, v) \in \mathbf{j} \times \mathbf{R} \times \mathbf{R}, \tag{9}$$

iff

$$\frac{\partial^2 F}{\partial u^2}(t, u, v) \cdot \frac{\partial^2 F}{\partial v^2}(t, u, v) - \left(\frac{\partial^2 F}{\partial u\, \partial v}(t, u, v)\right)^2 = 0, \quad \text{for } (t, u, v) \in \mathbf{j}_0 \times \mathbf{R} \times \mathbf{R} \tag{10}$$

where $\mathbf{j}_0 \subset \mathbf{j}$ is an appropriate interval and $F(t, 0, 0) = 0$ for $t \in \mathbf{j}$.

Proof. ($\Rightarrow$) Let the function F, defined by (9), map (q) onto (Q). Then, by Lemma 1, A, B is a solution of (3) and (4) is true. Since $\dfrac{\partial^2 F}{\partial u^2} = \dfrac{\partial^2 F}{\partial u\, \partial v} = \dfrac{\partial^2 F}{\partial v^2} = 0$, condition (10) is fulfilled even on the set $\mathbf{j} \times \mathbf{R} \times \mathbf{R}$ and $F(t, 0, 0) = 0$ for $t \in \mathbf{j}$.

($\Leftarrow$) Let there exist an interval $\mathbf{j}_0 \subset \mathbf{j}$ such that (10) is true. Let x be a solution of (q) and let the functions a, b, c be defined by (7). Then $a(t)\, c(t) - b^2(t) = 0$

for $t \in \mathbf{j}_0$. It follows from Lemma 3 that $ay^2 + 2byy' + cy'^2$ is a solution of (Q) for every solution y of (q). Let $a(t) = 0$ for $t \in \mathbf{j}_1$, where $\mathbf{j}_1 \subset \mathbf{j}_0$ is a subinterval of $\mathbf{j}_0$. Then also $b(t) = 0$ for $t \in \mathbf{j}_1$. Let $c(t) \not\equiv 0$ for $t \in \mathbf{j}_1$. Then $c(t)\, y'^2(t)$ is a solution of (Q) on $\mathbf{j}_1$ for every solution y of (q). Let $t_0 \in \mathbf{j}_1$ and y_1 be a nontrivial solution of (q), $y_1'(t_0) = 0$. Then $y_1'(t)$ does not vanish in any subinterval of $\mathbf{j}$. Put $Y_1(t) := c(t)\, y_1'^2(t)$, $t \in \mathbf{j}_1$. Then $Y_1(t_0) = Y_1'(t_0) = 0$ and therefore $Y_1(t) = 0$ for $t \in \mathbf{j}_1$, hence $c(t) = 0$ for $t \in \mathbf{j}_1$. So we have proved: $a(t) = b(t) = c(t) = 0$, $t \in \mathbf{j}_1$. In like manner we can prove that $a(t) = b(t) = c(t) = 0$ for $t \in \mathbf{j}_2$ follows from the equality $c(t) = 0$ or $b(t) = 0$ for $t \in \mathbf{j}_2 \subset \mathbf{j}_0$. So, let $a(t) = b(t) = c(t) = 0$ for $t \in \mathbf{j}_3$, where $\mathbf{j}_3$ means a subinterval of $\mathbf{j}_0$. It then follows from Lemma 3 that

$$(a(t)\, y + b(t)\, y')\, z + (b(t)\, y + c(t)\, y')\, z' \tag{11}$$

is a solution of (Q) for any solution y or z of (q). Since for any solution y or z of (q) the function defined by (11) vanishes for $t \in \mathbf{j}_3$, we obtain from this even

$$a(t) = b(t) = c(t) = 0 \qquad \text{for } t \in \mathbf{j}. \tag{12}$$

Assume $a(t) \neq 0$, $b(t) \neq 0$ for $t \in \mathbf{j}_4 \subset \mathbf{j}_0$. Then $a(t)\, y^2 + 2b(t)\, yy' + c(t)\, y'^2 = (1/a(t))\, (a(t)\, y + b(t)\, y')^2$ is a solution of (Q) on $\mathbf{j}_4$ for every solution y of (q). We may assume without loss of generality that (Q) is disconjugate on $\mathbf{j}_4$. The remaining part of the proof can be splitted into two parts:

(i) let there exist $t_1, t_2 \in \mathbf{j}_4$ such that

$$\begin{vmatrix} a(t_1) & b(t_1) \\ a(t_2) & b(t_2) \end{vmatrix} \neq 0. \tag{13}$$

Let y_i, $i = 1, 2$, be such nontrivial solutions of (q) that $a(t_1)\, y_1(t_1) + b(t_1)\, y_1'(t_1) = 0$, $a(t_2)\, y_2(t_2) + b(t_2)\, y_2'(t_2) = 0$. It then follows from (13) that y_1, y_2 are independent solutions of (q). Let us set $Y_i(t) := (1/a(t))\, (a(t)\, y_i(t) + b(t)\, y_i'(t))^2$ for $t \in \mathbf{j}_4$, $i = 1, 2$. Then Y_i are solutions of (Q) on $\mathbf{j}_4$, $Y_i(t_i) = Y_i'(t_i) = 0$. Since $a(t)\, y_i(t) + b(t)\, y_i'(t) = 0$ for $t \in \mathbf{j}_4$. Then, of course, $a(t)\, y(t) + b(t)\, y'(t) = 0$ for every solution y of (q) whence it follows that $a(t) = b(t) = 0$ for $t \in \mathbf{j}_4$ which is a contradiction;

(ii) let

$$\begin{vmatrix} a(t_1) & b(t_1) \\ a(t_2) & b(t_2) \end{vmatrix} = 0 \tag{14}$$

be valid for all $t_1, t_2 \in \mathbf{j}_4$. Then there is a function $k(t) \neq 0$ defined on $\mathbf{j}_4$ such that $a(t) = \alpha \,.\, k(t)$, $b(t) = \beta \,.\, k(t)$, where $\alpha, \beta \in \mathbf{R}$, $\alpha \neq 0$, $\beta \neq 0$. Then $(k(t)/\alpha) \times (\alpha y(t) + \beta y'(t))^2$ is a solution of (Q) on $\mathbf{j}_4$ for every solution y of (q). Let y_1, y_2 be two independent solutions of (q) such that $\alpha y_i(t_i) + \beta y_i'(t_i) = 0$ with $t_1 \neq t_2$, $t_1, t_2 \in \mathbf{j}_4$, $i = 1, 2$. Such solutions exist, for in the contrary case there would exist a nontrivial solution y of (q) such that $\alpha y(t) + \beta y'(t) = 0$ for $t \in \mathbf{j}_4$, i.e. $y(t) = ce^{-(\alpha t/\beta)}$, where $c \neq 0$ is a constant and furthermore $q(t) = (\alpha/\beta)^2$ for $t \in \mathbf{j}_4$, which, however contradicts our assumption of the Theorem above. Let us put $Y_i(t) := (k(t)/\alpha)(\alpha y_i(t) + \beta y_i'(t))^2$, $t \in \mathbf{j}_4$, $i = 1, 2$. Then Y_1, Y_2 are solutions of (Q) on $\mathbf{j}_4$, $Y_i(t_i) = Y_i'(t_i) = 0$, hence $\alpha y_i(t) + \beta y_i'(t) = 0$, $i = 1, 2$. Consequently, the equality $\alpha z(t) + \beta z'(t) = 0$ holds for every solution z of (q) which leads to $\alpha = \beta = 0$, i.e. a contradiction.

This proves $a(t) = b(t) = c(t) = 0$ for $t \in \mathbf{j}_0$. Completely analogous we can prove the validity of (12).

In view of the fact that x is an arbitrary solution of (q) in the definition of functions a, b, c, we find from (12)

$$\frac{\partial^2 F}{\partial u^2}(t, u, v) = \frac{\partial^2 F}{\partial v^2}(t, u, v) = \frac{\partial^2 F}{\partial u\, \partial v}(t, u, v) = 0, \qquad (t, u, v) \in \mathbf{j} \times \mathbf{R} \times \mathbf{R}. \quad (15)$$

From (15) and from the assumption $F(t, 0, 0) = 0$ for $t \in \mathbf{j}$, we see that F is of form (9).

Remark 2. We find from the examples below that it is impossible to delete the assumption of Theorem 1 saying that q is not equal to a nonnegative constant in any subinterval of $\mathbf{j}$.

Example 7. Let us put $q(t) := 0$, $Q(t) := 0$ for $t \in \mathbf{R}$. We know from Example 5 that the function $F(t, u, v) := u + v^2$, $(t, u, v) \in \mathbf{R} \times \mathbf{R} \times \mathbf{R}$, maps (q) onto (Q). It holds $\frac{\partial^2 F}{\partial u^2} = 0$, $\frac{\partial^2 F}{\partial v^2} = 2$, $\frac{\partial^2 F}{\partial u\, \partial v} = 0$, whereby the function F is not of form (9).

Example 8. Let $q(t) := 1$, $Q(t) := 1$ for $t \in \mathbf{R}$. We know from Example 6 that the function $F(t, u, v) := e^{-2t}(u + v)^3 + u - v$, $(t, u, v) \in \mathbf{R} \times \mathbf{R} \times \mathbf{R}$, maps (q) onto (Q). It holds $\frac{\partial^2 F}{\partial u^2} = 6e^{-2t}(u + v)$, $\frac{\partial^2 F}{\partial v^2} = 6e^{-2t}(u + v)$, $\frac{\partial^2 F}{\partial u\, \partial v} = 6e^{-2t}(u+v)$, hence $\frac{\partial^2 F}{\partial u^2} \cdot \frac{\partial^2 F}{\partial v^2} - \left(\frac{\partial^2 F}{\partial u\, \partial v}\right)^2 = 0$, whereby the function F is not of form (9).

Theorem 2. *Let $q \in C^1(\mathbf{j})$, $Q \in C^\circ(\mathbf{j})$ and let* (q) *not be a disconjugate equation on* $\mathbf{j}$. *Let a function F map* (q) *onto* (Q) *and $t_1, t_2 \in \mathbf{j}$, $t_1 \neq t_2$, be not conjugate points of* (Q). *Then* (9) *is valid iff*

$$\frac{\partial^2 F}{\partial u^2}(t_i, u, v) = \frac{\partial^2 F}{\partial v^2}(t_i, u, v) = \frac{\partial^2 F}{\partial u\, \partial v}(t_i, u, v) = 0 \quad (16)$$

$$\textit{for } (u, v) \in \mathbf{R} \times \mathbf{R} \quad \textit{and} \quad F(t, 0, 0) = 0 \quad \textit{for } t \in \mathbf{j}.$$

Proof. ($\Rightarrow$) Letting a function F, written in the form (9), map (q) onto (Q), yields $\frac{\partial^2 F}{\partial u^2}(t, u, v) = \frac{\partial^2 F}{\partial v^2}(t, u, v) = \frac{\partial^2 F}{\partial u \partial v}(t, u, v) = 0$ for $(t, u, v) \in \mathbf{j} \times \mathbf{R} \times \mathbf{R}$. Thus (16) is true even for all $t_1, t_2 \in \mathbf{j}$.

($\Leftarrow$) Let $t_1, t_2 (\in \mathbf{j})$, $t_1 \neq t_2$, not be conjugate numbers of (Q) and let (16) hold. If x is a solution of (q) and the functions a, b, c are defined by (7), then $a(t_i) = b(t_i) = c(t_i) = 0$ for $i = 1, 2$. By Lemma 3 the function $(ay + by')z + (by + cy')z'$ is a solution of (Q) for every solution y or z of (q). All these solutions have a zero at the points t_1, t_2. Hence we have $(ay + by')z + (by + cy')z' = 0$ for every solution y or z of (q), whence (12) follows. The proof of (15) is similar to that of Theorem 1, from which and from the assumption $F(t, 0, 0) = 0$ for $t \in \mathbf{j}$, it follows that the function F is of form (9).

Theorem 3. *Let* $q \in C^1(\mathbf{j})$, $Q \in C^0(\mathbf{j})$. *Let a function* F *map* (q) *onto* (Q). *Then* (9) *is valid iff*

$$F(t, ku, kv) = k \,.\, F(t, u, v) \qquad \text{for } (t, u, v) \in \mathbf{j} \times \mathbf{R} \times \mathbf{R} \qquad \text{and} \qquad k \in \mathbf{R}. \tag{17}$$

Proof. If F satisfies (9), then it satisfies (17), too. Let for a function F the relation (17) hold. By differentiating (17) with respect to k we obtain

$$\frac{\partial F}{\partial u}(t, ku, kv) \,.\, u + \frac{\partial F}{\partial v}(t, ku, kv) \,.\, v = F(t, u, v) \tag{18}$$

and writing $k = 0$ in (18) we obtain

$$F(t, u, v) = A(t)\, u + B(t)\, v,$$

where $A(t) := \frac{\partial F}{\partial u}(t, 0, 0)$, $B(t) := \frac{\partial F}{\partial v}(t, 0, 0)$, $t \in \mathbf{j}$.

О НЕКОТОРОМ ВИДЕ ПРЕОБРАЗОВАНИЯ РЕШЕНИЙ ДВУХ ЛИНЕЙНЫХ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ 2-ГО ПОРЯДКА

Резюме

Говорят, что функция $F = F(t, u, v)$ отображает множество решений уравнения (q): $y'' = q(t)\, y$ на множество решений уравнения (Q): $Y'' = Q(t) Y$, если для каждого решения $y = y(t)$ уравнения (q) функция $Y(t) := F(t, y(t), y'(t))$ является решением уравнения (Q) и тоже обратно: для каждого решения $Y = Y(t)$ уравнения (Q) существует только одно решение $y = y(t)$ уравнения (q) такое, что имеет место равенство $Y(t) = F(t, y(t), y'(t))$.

В работе приводятся условия, которые необходимые и достатишные для того, чтобы функция F имела вид $F(t, u, v) = A(t)\, u + B(t)\, v$.

O JISTÉM TVARU TRANSFORMACE ŘEŠENÍ DVOU LINEÁRNÍCH DIFERENCIÁLNÍCH ROVNIC 2. ŘÁDU

Souhrn

Řekneme, že funkce $F = F(t, u, v)$ zobrazuje množinu řešení rovnice (q): $y'' = q(t)y$ na množinu řešení rovnice (Q): $Y'' = Q(t)Y$, jestliže pro každé řešení $y = y(t)$ rovnice (q) je funkce $Y(t) := F(t, y(t), y'(t))$ řešením rovnice (Q) a také opačně, ke každému řešení $Y = Y(t)$ rovnice (Q) existuje jediné řešení $y = y(t)$ rovnice (q) takové, že platí rovnost $Y(t) = F(t, y(t), y'(t))$.

V práci jsou uvedeny podmínky, které jsou nutné a postačující k tomu, aby funkce F byla tvaru $F(t, u, v) = A(t)u + B(t)v$.

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