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Katedra matematické analýzy a numerické matematiky přírodovědecké fakulty Univerzity Palackého v Olomouci
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SECOND DERIVATIVE LINEAR MULTISTEP FORMULA AND ITS STABILITY ON THE IMAGINARY AXIS

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1. We consider an initial value problem

$$y' = f(x, y), \qquad y(x_0) = y_0$$

and a linear multistep formula with second derivatives of the type

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j y'_{n+j} + h^2 \sum_{j=0}^{k} \gamma_j y''_{n+j} \tag{F}$$

for its numerical solution. Let us denote by

$$\varrho(\zeta) = \sum_{j=0}^{k} \alpha_j \zeta^j, \qquad \sigma(\zeta) = \sum_{j=0}^{k} \beta_j \zeta^j, \qquad \tau(\zeta) = \sum_{j=0}^{k} \gamma_j \zeta^j$$

the characteristic polynomials of the formula (F) and suppose the formula (F) to be

– stable in the Dahlquist's sense ($\varrho(\zeta)$ fulfils the "*root condition*"),
– consistent ($\varrho(1) = 0$, $\varrho'(1) = \sigma(1)$).

Following the known theory (e.g. [7], [10]) the formula (F) is then convergent.

There are another stability concepts for the formula (F) to be found in the numerical analysis area; they are based mostly on applying the formula to the "test equation" $y' = \lambda y$ resulting in the difference equation

$$\sum_{j=0}^{k} (\alpha_j - q\beta_j - q^2\gamma_j)\, y_{k+j} = 0, \qquad q = \lambda h$$

with the characteristic polynomial ("*stability polynomial of* (F)")

$$\pi(\zeta, q) \equiv \varrho(\zeta) - q\sigma(\zeta) - q^2\tau(\zeta).$$

We denote its roots (with respect to ζ) as $\zeta_i = \pi_i(q)$.

The aim of this paper is to give some results concerning the stability on the imaginary axis (see [5], [2]) of the formula (F) for $k = 1, 2$ in addition to those results concerning its A, A_0, A_∞ — stability, given in [8].

Definition (see [5], [9] for the case $\tau(\zeta) \equiv 0$): *A formula* (F) *is stable on the imaginary axis* (A_i-*stable*) *if* $\{iy;\ -\infty < y < +\infty\} \subset A = \{q \in C;\ |\pi_i(q)| \leqq 1,$ *every root with modulus one is simple*$\}$.

Remarks. A is called the absolute stability region. Jeltsch has shown in [5] that the following holds for (ϱ, σ) methods ($\tau(\zeta) \equiv 0$):

– every consistent, A_i-stable method (ϱ, σ) is A-stable,

– the maximal order of the consistent A_i-stable (ϱ, σ) method is $p = 2$ (with the trapezoidal rule having the smallest error constant); this contrasts with Cryer's result for the A_0-stable formula in [1],

– an example of the (ϱ, σ, τ) formula (F) is given, being A_i-stable, but not A-stable.

The imaginary stability boundary of the k-step method (ϱ, σ) of order at least two is shown in [2] with the corollary that for the k-step method of the order $p \geqq 2$ it is at most $\sqrt{3}$.

2. The one-step formula

$$y_{n+1} - y_n = h(\beta_0 y'_n + \beta_1 y'_{n+1}) + h^2(\gamma_0 y''_n + \gamma_1 y''_{n+1}), \tag{F1}$$

has the stability polynomial $\pi(\zeta, q) \equiv \zeta - 1 - q(\beta_0 + \beta_1\zeta) - q^2(\gamma_0 + \gamma_1\zeta)$ with the root $\zeta_1 = (1 + q\beta_0 + q^2\gamma_0)/(1 - q\beta_1 - q^2\gamma_1)$. Following the definition the formula (F1) is A_i stable iff

$$|1 - \gamma_0 y^2 + i\beta_0 y| / |1 + \gamma_1 y^2 - i\beta_1 y| \leqq 1. \tag{1}$$

Theorem 1.

a) *The one-step formula* (F1) *of the maximal order* $p = 4$

$$y_{n+1} - y_n = \frac{h}{2}(y'_n + y'_{n+1}) + \frac{h^2}{12}(y'_n - y''_{n+1}) \tag{F1.4}$$

is A_i-stable.

b) *The one-step formula* (F1) *of the third order* (*parameter* γ_1)

$$y_{n+1} - y_n = h\left[\left(\frac{2}{3} + 2\gamma_1\right)y'_n + \left(\frac{1}{3} - 2\gamma_1\right)y'_{n+1}\right] + h^2\left[\left(\frac{1}{6} + \gamma_1\right)y''_n + \gamma_1 y''_{n+1}\right] \tag{F1.3}$$

s A_i-stable iff $\gamma_1 \leqq -1/12$.

Proof: The formula (F1.3) was presented and its A, A_0-stability for $\gamma_1 \leqq -1/12$ shown in [8].

a) A_i-stability condition (1) for the formula (F1.4)

$$|\pi_1(iy)| = \left|1 - \frac{1}{12}y^2 + \frac{1}{2}iy\right| \bigg/ \left|1 - \frac{1}{12}y^2 - \frac{1}{2}iy\right| \leqq 1$$

is fulfilled for any $y \in \mathbf{R}$ because the numerator and denominator are complex conjugate numbers.

b) For the formula (F1.3), the A_i-stability condition (1)

$$|\pi_1(iy)| = \left|1 - \left(\frac{1}{6} + \gamma_1\right)y^2 + 2\left(\frac{1}{3} + \gamma_1\right)iy\right| \bigg/ \left|1 + \gamma_1 y^2 + \left(-\frac{1}{3} + 2\gamma_1\right)iy\right| \leqq 1$$

is equivalent to the simple condition $(1/6 + 2\gamma_1)\, y^2 \leqq 0$ which holds for all $y \in \mathbf{R}$ iff $\gamma_1 \leqq -1/12$.

The one-step formula of the second order (parameters β_0, γ_1)

$$y_{n+1} - y_n = h[\beta_0 y'_n + (1 - \beta_0)\, y'_{n+1}] + h^2\left[\left(-\frac{1}{2} + \beta_0 - \gamma_1\right)y''_n + \gamma_1 y''_{n+1}\right] \quad \text{(F1.2)}$$

was studied in [8]; its stability polynomial

$$\pi(\zeta, q) \equiv \varrho(\zeta) - q\sigma(\zeta) - q^2\tau(\zeta)$$

has the root

$$\zeta_1 = \pi_1(q) = \left[1 + \beta_0 q + \left(-\frac{1}{2} + \beta_0 - \gamma_1\right)q^2\right] \bigg/ [1 - (1 - \beta_0)\, q - \gamma_1 q^2].$$

Theorem 2. *The formula* (F1.2) *is A_i-stable if and only if*

$$(\beta_0, \gamma_1) \in \left\{\beta_0 \leqq \frac{1}{2},\ \gamma_1 \leqq \frac{1}{2}\beta_0 - \frac{1}{4}\right\} \cup \left\{\beta_0 \geqq \frac{1}{2},\ \gamma_1 \geqq \frac{1}{2}\beta_0 - \frac{1}{4}\right\}.$$

Proof: The A_i-stability condition (1) for the formula (F1.2) can be written as follows

$$\left|1 - \left(-\frac{1}{2} + \beta_0 - \gamma_1\right)y^2 + i\beta_0 y\right|^2 \leqq |1 + \gamma_1 y^2 - i(1 - \beta_0)\, y\,|^2;$$

after some algebraic modification we can write it as $\left(\frac{1}{2} - \beta_0\right)\left(\frac{1}{2} - \beta_0 + 2\gamma_1\right) \leqq 0$. This condition is fulfilled exactly in the above mentioned region of the (β_0, γ_1)-plane, pictured in g. Fi1.

Remarks.

– We have the formula (F1.3) for (β_0, γ_1) on the line $\gamma_1 = \frac{1}{2}\beta_0 - \frac{1}{4}$ in Fig. 1.

– The point $(\beta_0, \gamma_1) = (1/2, -1/12)$ corresponds to formula (F1.4).

– Jeltsch's example from [5] with $(\beta_0, \gamma_1) = (1/2, 1)$ corresponds to the boundary point of the region.
– The upper part of the stability region corresponds to formulas, which are A_i-stable but not A_0, A-stable (see [8]).

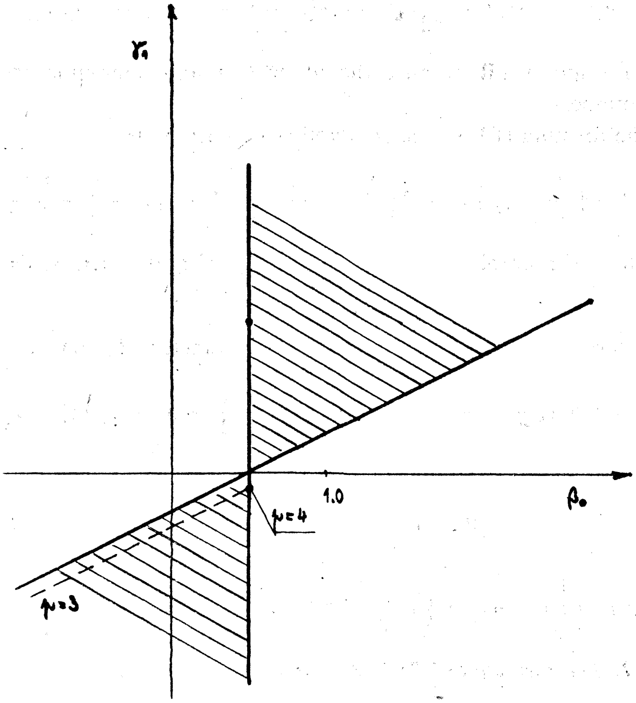

Fig. 1

3. Some stability criteria for second degree polynomials

3.1. Let the polynomial $f(z) = a_2z^2 + a_1z + a_0$, $a_j \in C, a_j = \alpha_j + i\beta_j, j = 0, 1, 2$ not to possess the roots on the imaginary axis. Then the following criterion may be written from the generalized Hurwitz's criterion (see [4]):

$f(z)$ has all roots in the open left half-plane iff

a) $\alpha_1\alpha_2 + \beta_1\beta_2 > 0,$ (H)

b) $(\alpha_1\alpha_2 + \beta_1\beta_2)(\alpha_0\alpha_1 + \beta_0\beta_1) - (\alpha_0\beta_2 - \alpha_2\beta_0)^2 > 0.$

3.2. Using another stability criterion by Schur–Cohen (see [3], [6]) leads to *$f(z)$ has all roots inside the unit disc iff*

a) $D2 \equiv |a_2| - |a_0| > 0,$ (SC)

b) $D4 \equiv |a_2|^2 - |a_0|^2 - |a_0\bar{a}_1 - a_1\bar{a}_2| > 0,$

(*$\bar{a}$ denotes complex conjugate to a*).

Remark: In this special case the condition a) is involved in the condition b) so that it suffices to consider the condition b) only.

3.3. The Möbius transformation $z = (w + 1)/(w - 1)$ maps one-to-one the left half-plane and the unit disc of the complex plane. The polynomial $f(z) = a_2z^2 + a_1z + a_0$ is transformed into

$$f((w + 1)/(w - 1)) = [a_2(w + 1)^2 + a_1(w^2 - 1) + a_0(w - 1)^2]/(w - 1)^2 = \\ = (b_2w^2 + b_1w + b_0)/(w - 1)^2,$$

with $b_2 = a_2 + a_1 + a_0$, $b_1 = 2(a_2 - a_0)$, $b_0 = a_2 - a_1 + a_0$.

Using the criterion (H) we get

$f(z)$ possesses all roots inside the unit disc if and only if

a) $(\alpha_0 + \alpha_1 + \alpha_2)(\alpha_2 - \alpha_0) + (\beta_0 + \beta_1 + \beta_2)(\beta_2 - \beta_0) > 0,$ (HT)

b) $[(\alpha_2 - \alpha_0)(\alpha_0 + \alpha_1 + \alpha_2) + (\beta_2 - \beta_0)(\beta_0 + \beta_1 + \beta_2)]\,[(\alpha_2 - \alpha_0) \times (\alpha_0 - \alpha_1 + \alpha_2) + (\beta_2 - \beta_0)(\beta_0 - \beta_1 + \beta_2)] > [(\alpha_0 - \alpha_1 + \alpha_2)(\beta_0 + \beta_1 + \beta_2) - (\alpha_0 + \alpha_1 + \alpha_2)(\beta_0 - \beta_1 + \beta_2)]^2.$

3.4. Respecting the possible root of the stability polynomial with modulus one, we need to express this case not involved in the criteria (H), (SC).

Lemma 1. *Let* $a_0\bar{a}_1 - a_1\bar{a}_2 = 0$ *hold for the polynomial* $f(z) = a_2z^2 + a_1z + a_0$, $a_j \in C$, $a_2 \neq 0$ *and let us write* $D = a_1\bar{a}_1(a_1\bar{a}_1 - 4a_0\bar{a}_0) \in \mathbf{R}$.

Then

a) if $a_1 = 0$, $|a_2| = |a_0|$ *or* $a_1 \neq 0$, $D < 0$, *the roots of* $f(z)$ *are lying on the unit circle and are simple,*

b) if $a_1 \neq 0$, $D = 0$, *then* $f(z)$ *has double root* $z = -a_0/a_1$ *on the unit circle,*

c) if $a_1 = 0$, $|a_2| \neq |a_0|$, *then the roots of* $f(z)$ *are simple and are lying on the circle* $|z| = |a_0/a_2|^{1/2}$,

d) if $a_1 \neq 0$, $D > 0$ *so the roots of* $f(z)$ *are simple and are lying symmetrically with respect to the unit circle* ($z_1\bar{z}_2 = 1$).

Proof: Let $a_0\bar{a}_1 - a_1\bar{a}_2 = 0$ be valid. If $a_1 = 0$, then it holds $|z_j| = |a_0/a_2|^{1/2}$ for the roots of $f(z)$. If $a_0a_1 \neq 0$, let us put $a_2 = \bar{a}_0a_1/\bar{a}_1$; then the roots of $f(z)$ are also a solution of the quadratic equation $\bar{a}_0a_1z^2 + a_1\bar{a}_1z + a_0\bar{a}_1 = 0$ with the discriminant $D = a_1\bar{a}_1(a_1\bar{a}_1 - 4a_0\bar{a}_0) \in \mathbf{R}$. In this case $|z_1z_2| = |a_0\bar{a}_1/\bar{a}_0a_1| = 1$. If $|a_1| = 2|a_0|$, then $D = 0$, $|z_j| = |-a_1\bar{a}_1/2\bar{a}_0a_1| = 1$; $D < 0$ implies $z_j = (-a_1\bar{a}_1 + (-1)^j i\sqrt{-D})/(2\bar{a}_0a_1)$, $j = 1, 2$.

$4|\bar{a}_0a_1|^2\,|z_j|^2 = |-a_1\bar{a}_1 + (-1)^j i\sqrt{-D}|^2 = (a_1\bar{a}_1)^2 - D = 4a_0\bar{a}_0a_1\bar{a}_1$

from which $|z_j| = 1$ follows. $D > 0$ implies

$$z_1\bar{z}_2 = \frac{-a_1\bar{a}_1 + \sqrt{D}}{2\bar{a}_0a_1} \cdot \frac{-a_1\bar{a}_1 - \sqrt{D}}{2a_0\bar{a}_1} = \frac{(-a_1\bar{a}_1)^2 - D}{4a_0\bar{a}_0a_1\bar{a}_1} = 1.$$

Corollaries: 1. $a_1a_2 \neq 0$, $a_0\bar{a}_1 - a_1\bar{a}_2 = 0$ implies $|a_0| = |a_2|$ under the condition a), b) in (SC),

2. $f(z) = a_2z^2 + a_1z + a_0$ under the condition $a_0\bar{a}_1 - a_1\bar{a}_2 = 0$ has at least one root outside or double root on the unit circle exactly if one of the following conditions holds:

a) $a_1 = 0, |a_2| < |a_0|$ (both roots outside),
b) $a_1 \neq 0, D = 0$ (double root on the unit circle),
c) $a_1 \neq 0, D > 0$ (one root inside and the other outside).

4. The two-step formula of the type considered is

$$y_{n+2} - (1 + a)\, y_{n+1} + ay_n = \\ = h(\beta_0 y'_n + \beta_1 y'_{n+1} + \beta_2 y'_{n+2}) + h^2(\gamma_0 y''_n + \gamma_1 y''_{n+1} + \gamma_2 y''_{n+2}) \qquad \text{(F2)}$$

with the stability polynomial

$$\pi(\zeta, q) \equiv \varrho(\zeta) - q\sigma(\zeta) - q^2\tau(\zeta) = \\ = (1 - \beta_2 q - \gamma_2 q^2)\, \zeta^2 - (1 + a + \beta_1 q + \gamma_1 q^2)\, \zeta + (a - \beta_0 q - \gamma_0 q^2).$$

Lemma 2. *If the formula* (F2) *is A_i-stable, then*
a) $\gamma_2 \geqq 0$ *or* $\beta_2 \neq 0$, *b)* $|\gamma_1| < |\gamma_0 + \gamma_2|$, $|\gamma_0| \leqq |\gamma_2|$.

Proof:

a) The leading coefficient in

$$\pi(\zeta, iy) = (1 + \gamma_2 y^2 - i\beta_2 y)\, \zeta^2 - (1 + a - \gamma_1 y^2 + i\beta_1 y)\, \zeta + (a + \gamma_0 y^2 - i\beta_0 y)$$

vanishes if $y^2 = -1/\gamma_2$ and $\beta_2 = 0$; the roots of $\pi(\zeta, iy)$ are lying outside the unit circle in the neighborhood of these values.

b) The roots of $\pi(\zeta, iy)$ are lying inside the unit disc iff (following the (SC) criterion, the condition b))

$$|1 + \gamma_2 y^2 - i\beta_2 y|^2 - |a + \gamma_0 y^2 - i\beta_0 y|^2 > \\ > |-(a + \gamma_0 y^2 - i\beta_0 y)(1 + a - \gamma_1 y^2 - i\beta_1 y) + \\ + (1 + a - \gamma_1 y^2 + i\beta_1 y)(1 + \gamma_2 y^2 + i\beta_2 y)|.$$

We can write this condition after putting in proper form

$$1 - a^2 + [2(\gamma_2 - a\gamma_0) + \beta_2^2 - \beta_0^2]\, y^2 + (\gamma_2^2 - \gamma_0^2)\, y^4 > \\ > |(1 + a - \gamma_1 y^2)[1 - a + (\gamma_2 - \gamma_1)\, y^2 + \beta_1(\beta_0 - \beta_2)\, y^2 + \\ + iy\{(1 + a - \gamma_1 y^2)(\beta_0 + \beta_2) + \beta_1[1 + a + (\gamma_0 + \gamma_2)\, y^2]\}|.$$

This condition may be further modified to the form $y^2 P_6(y) > 0$, where the polynomial of the sixth order $P_6(y)$ possess the leading coefficient $(\gamma_2^2 - \gamma_0^2)^2 - \gamma_1^2(\gamma_0 - \gamma_2)^2$; from the condition of its nonnegativity results the necessity of the condition b).

4.1 The two-step sixth-order formula with the parameter a

$$y_{n+2} + (1 + a)\, y_{n+1} + ay_n =$$

$$= h[(101 - 11a)\, y'_{n+2} + 128(1 - a)\, y'_{n+1} + (11 - 101a)\, y'_n]/240 +$$
$$+ h^2[(-13 + 3a)\, y''_{n+2} + 40(1 + a)\, y''_{n+1} + (3 - 13a)\, y''_n]/240 \qquad \text{(F2.6)}$$

was derived in [8]; it is stable for $a \in [-1, 1)$ and possesses the stability polynomial

$$\pi(\zeta, iy) = \left[1 + \frac{-13 + 3a}{240} y^2 - i\frac{101 - 11a}{240} y\right]\zeta^2 +$$
$$+ \left[\frac{1}{6}(1 + a)\, y^2 - 1 - a - iy\frac{8}{15}(1 - a)\right]\zeta + \left[a + \frac{3 - 13a}{240} y^2 - iy\frac{11 - 101a}{240}\right].$$

Theorem 3. *The formula* (F2.6) *is not A_i-stable for any $a \in (-1, 1)$.*

Proof: Applying the Möbius transformation the roots of $\pi(\zeta, iy)$ are mapped into the roots of $F(z) = a_2z^2 + a_1z + a_0$, $a_j = \alpha_j + i\beta_j$, $j = 0, 1, 2$ where

$$\alpha_2 = (1 + a)\, y^2/8 \qquad \beta_2 = (a - 1)\, y,$$
$$\alpha_1 = 2(1 - a)(1 - y^2/15), \quad \beta_1 = -3(1 + a)\, y/4,$$
$$\alpha_0 = (1 + a)(2 - y^2/6), \qquad \beta_0 = (1 - a)/15.$$

The first from the conditions of (H) turns to be $y^2(1 - a^2)(4 - y^2/15)/4 > 0$; it is fulfilled with $a \in (-1, 1)$ for $y^2 < 60$ only.

Remarks.

1. With $y = 0$, $a \in [-1, 1)$ we have $\pi(\zeta, 0) = \zeta^2 - (1 + a)\,\zeta + a$ and $D4(a, 0) \equiv \equiv 0$ in the (SC) criterion; following Lemma 1, we can write $D(a, 0) = = (1 + a)^2 (1 + 3a)(1 - a)$. Thus the polynomial $\pi(\zeta, 0)$ has simple roots on the unit circle for $a \in [-1, -1/3)$, double root on the unit circle for $a = -1/3$ and one root outside the unit disc for $a \in (-1/3, 1)$.

2. For $a = -1$ we have

$$\pi(\zeta, iy) = \left(1 - \frac{1}{15} y^2 - \frac{7}{15} iy\right)\zeta^2 - \frac{16}{15} iy\zeta + \left(-1 + \frac{1}{15} y^2 + \frac{7}{15} y\right),$$
$$D4(-1, y) \equiv 0, \qquad D(-1, y) = -ky^2(y^4 - 45y^2 + 225), \qquad k > 0,$$
$$D(-1, y) < 0 \qquad \text{for } y \in (-\alpha_1, \alpha_1) \cup (\alpha_2, \infty) \cup (-\infty, -\alpha_2)$$

with $\alpha_j = [(45 \pm \sqrt{1125})/2]^{1/2} \doteq 2.39;\ 6.27$. The A_i-stability conditions are not satisfied for $a = -1$ and $y \in (-\alpha_2, \alpha_1) \cup (\alpha_1, \alpha_2)$.

3. $a = 1$ implies

$$\pi(\zeta, iy) = \left(1 - \frac{1}{24} y^2 - \frac{3}{8} iy\right)\zeta^2 + \left(\frac{1}{3} y^2 - 2\right)\zeta + \left(1 - \frac{1}{24} y^2 + \frac{3}{8} iy\right)$$
$$D4(1, y) \equiv 0,$$
$$D(1, y) = \frac{5}{8} y^2\left(\frac{1}{6} y^2 - \frac{5}{2}\right) < 0 \qquad \text{for } y^2 < 15 \text{ only.}$$

4. Using the computer it was numerically found that $D4(a, y) < 0$ holds for

$y \neq 0$, $y \in \mathbf{R}$, $a \in (-1, 1)$ (it is sufficient to undergo the search for $y \geqq 0$ only). The author didn't succeed in proving this fact analytically.

4.2 The two-step fifth-order formula with parameters a, γ_2

$$y_{n+2} - (1 + a)\, y_{n+1} + a y_n =$$

$$= h\left[\left(\frac{5}{24} - \frac{11}{24} a + 3\gamma_2\right) y'_n + \frac{8}{15}(1 - a)\, y'_{n+1} + \left(\frac{31}{120} - \frac{1}{120} a - 3\gamma_2\right) y'_{n+2}\right] +$$

$$+ h^2\left[\left(\frac{1}{15} - \frac{1}{15} a + \gamma_2\right) y''_n + \left(\frac{23}{60} + \frac{7}{60} a + 4\gamma_2\right) y''_{n+1} + \gamma_2 y''_{n+2}\right] \quad \text{(F2.5)}$$

—see [8]—is stable for $a \in [-1, 1)$; the stability polynomial (with $q = iy$)

$$\pi(\zeta, iy) = \left[1 + \gamma_2 y^2 - i\left(\frac{31}{120} - \frac{1}{120} a - 3\gamma_2\right) y\right] \zeta^2 -$$

$$- \left[1 + a - \left(\frac{23}{60} + \frac{7}{60} a + 4\gamma_2\right) y^2 + iy\frac{8}{15}(1 - a)\right] \zeta +$$

$$+ a + \left(\frac{1}{15} - \frac{1}{15} a + \gamma_2\right) y^2 - iy\left(\frac{5}{24} - \frac{11}{24} a + 3\gamma_2\right). \quad (2)$$

Theorem 4. *The formula* (F2.5) *is not A_i-stable for any* (a, γ_2), $a \in [-1, 1)$, $\gamma_2 \in \mathbf{R}$.

Proof: The condition a) from the criterion (HT) applied to (2) results in $(1 - a) \times \times [-40y^2(360\gamma_2 + 27 + 3a) + 21240\gamma_2 + 207a + 1648] > 0$. It can be fulfilled with $a \in [-1, 1)$ for all $y \in \mathbf{R}$ iff any of the conditions

a) $\gamma_2 = -(a + 9)/120$,

b) $360\gamma_2 + 27 + 3a < 0$ and $21240\gamma_2 + 207a + 1648 > 0$ is valid. Both of them are satisfied in the narrow strip around $\gamma_2 = -0.075$ in (a, γ_2)-plane where the condition b) of (SC) fails to hold (as can be verified by a lenghty calculation—see also remark 3 and Fig. 2).

1. By a direct calculation we can establish $D4(a, \gamma_2, 0) = 0$ for all a, γ_2, $D(a, \gamma_2, 0) = (1 + a)^2 (1 + 3a)(1 - a)$ with the same conclusions as for (F2.6), $D4(1, \gamma_2, y) \equiv 0$.

2. The condition $|\gamma_1| < |\gamma_0 + \gamma_2|$ from Lemma 2 now states

$$|23/60 + 7a/60 + 4\gamma_2| \leqq |(1 - a)/15 + 2\gamma_2|.$$

With $a \in [-1,1)$ it is fulfilled just for $(a, \gamma_2) = (-1, -1/15)$; we have $|\gamma_0| = |\gamma_2| = 1/15$, but $D2 = D4 = 0$ in (SC) criterion for this case. Using Lemma 1 we find that $D > 0$ for approximatelly $y \in (2{,}39; 6{,}27)$ and so, following statement d) of this Lemma, our formula is not A_i-stable.

3. Using the computer the first root of the equation $D4(a, \gamma_2, y) = 0$ has been calculated for various values of (a, γ_2). Special care has been devoted to the region $(a, \gamma_2) \in \{-1 \leqq a \leqq 1, -0.5 \leqq \gamma_2 \leqq 0\}$, where A_0-stable formulas can be found

(see [8]). The numerical results has shown that $D4(a, \gamma_2, y)$ assumes positive values for (a, γ_2) from the region shown in Fig. 2, however for small values of y only. (The numbers give the level of the first root in y, where $D4$ turns to be negative or positive).

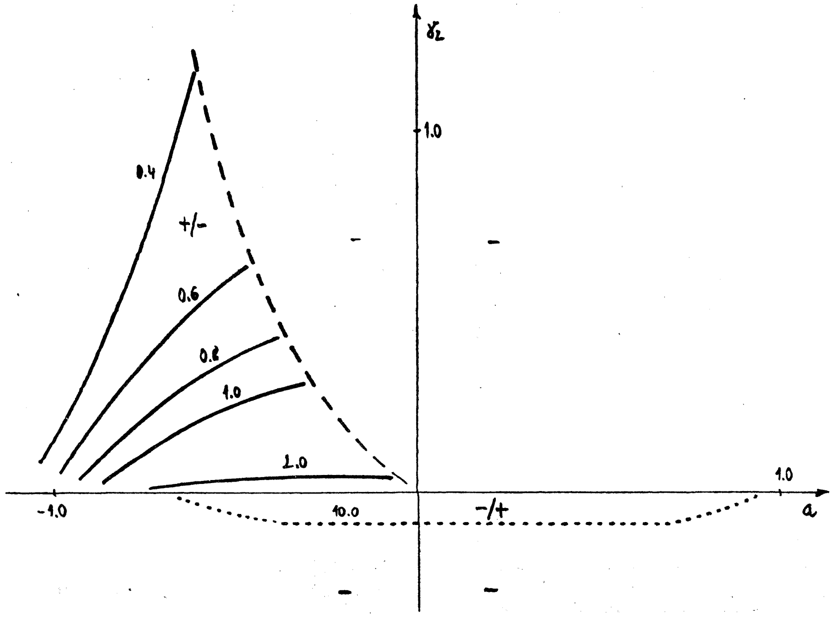

Fig. 2

4.3 The two-step fourth-order formula with the parameters a, β_1, γ_1

$$y_{n+2} - (1 + a)\, y_{n+1} + a y_n = h\left[\left(\frac{9}{48} - \frac{39}{48}a - \frac{1}{2}\beta_1 + \frac{3}{4}\gamma_1\right) y'_n + \right.$$
$$\left. + \beta_1 y'_{n+1} + \left(\frac{39}{48} - \frac{9}{48}a - \frac{1}{2}\beta_1 - \frac{3}{4}\gamma_1\right) y'_{n+2}\right] +$$
$$+ h^2\left[\left(\frac{5}{48} - \frac{11}{48}a - \frac{1}{4}\beta_1 + \frac{1}{4}\gamma_1\right) y''_n + \gamma_1 y''_{n+1} + \right.$$
$$\left. + \left(-\frac{11}{48} + \frac{5}{48}a + \frac{1}{4}\beta_1 + \frac{1}{4}\gamma_1\right) y''_{n+2}\right] \qquad \text{(F2.4)}$$

derived in [8] possesses a stability polynomial

$\pi(\zeta, iy) = a_2\zeta^2 + a_1\zeta + a_0,\ a_j = \bar{\alpha}_j + i\bar{\beta}_j,\ j = 0, 1, 2$ with

$$\bar{\alpha}_2 = 1 + y^2\left(-\frac{11}{48} + \frac{5}{48}a + \frac{1}{4}\beta_1 + \frac{1}{4}\gamma_1\right)$$

$$
\begin{aligned}
\bar{\alpha}_1 &= -(1 + a) + \gamma_1 y^2 \\
\bar{\alpha}_0 &= a + y^2\left(\frac{5}{48} - \frac{11}{48}a - \frac{1}{4}\beta_1 + \frac{1}{4}\gamma_1\right), \\
\bar{\beta}_2 &= -y\left(\frac{39}{48} - \frac{9}{48}a - \frac{1}{2}\beta_1 - \frac{3}{4}\gamma_1\right) \\
\bar{\beta}_1 &= -\beta_1 y \\
\bar{\beta}_0 &= -y\left(\frac{9}{48} - \frac{39}{48}a - \frac{1}{2}\beta_1 + \frac{3}{4}\gamma_1\right).
\end{aligned}
$$

Applying the stability criterion (HT), its condition a) can be written as

$$
y^2\left[\frac{}{2}(1 - a^2) + y^2\left(\frac{a - 1}{3} + \frac{\beta_1}{2}\right)\left(-\frac{1 + a}{8} + \frac{3}{2}\gamma_1\right)\right] > 0.
$$

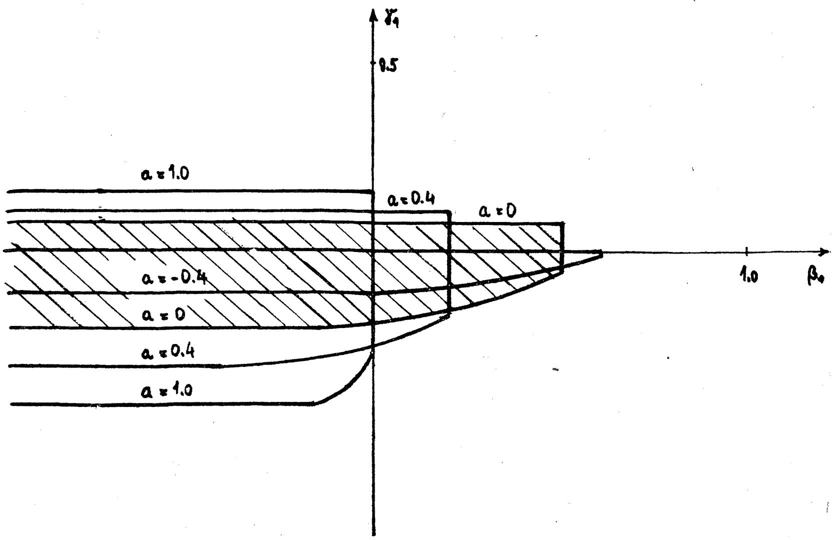

Fig. 3

It can be satisfied for any $y \in \mathbf{R}$, when we have

$$
\left[\beta_1 > \frac{2}{3}(1 - a) \quad \text{and} \quad \gamma_1 > (1 + a)/12\right]
$$

or

$$
\left[\beta_1 < \frac{2}{3}(1 - a) \quad \text{and} \quad \gamma_1 < (1 + a)/12\right].
$$

The condition $|\gamma_1| < |\gamma_0 + \gamma_2|$ from Lemma 2 takes here the form $|\gamma_1| <$

$< \left| \frac{1}{8}(1 + a) + \frac{1}{2}\gamma_1 \right|$ (independently on β_1!); it is fulfilled just for $(a, \gamma_1) \in \in \{-(1 + a)/4 \leqq \gamma_1 \leqq (1 + a)/12\}$. The condition $|\gamma_0| \leqq |\gamma_2|$ can be written in this case as

$$\beta_1 \geqq 2(1 - a)/3 \quad \text{for } \gamma_0 \geqq 0, \gamma_2 \geqq 0; \qquad \gamma_1 \geqq 2(1 - a)/3 \quad \text{for } \gamma_0 < 0, \gamma_2 > 0,$$
$$\beta_1 \leqq 2(1 - a)/3 \quad \text{for } \gamma_0 < 0, \gamma_2 < 0; \qquad \gamma_1 \leqq (1 + a)/4 \quad \text{for } \gamma_0 > 0, \gamma_2 < 0.$$

When we substitute for γ_2, γ_0 the expressions with the parameters a, b_1, γ_1, then the lines $\gamma_2 = 0$, $\gamma_0 = 0$ (depending on a) in the (b_1, γ_1)-plane divide this plane into four quarters in which the conditions written above take place.

Theorem 5. *There exist A_i-stable formulas* (F2.4).

The proof follows from Theorem 1, for the formula (F2.4) with $a = 0$, $\beta_1 = 1/2$, $\gamma_1 = 1/12$ corresponds to A_i-stable formula (F1.4). Another example of the A_i-stable formula (F2.4) is the choice $(a, \beta_1, \gamma_1) = (0, 0, 0)$ as can be proved using criterion (SC) directly.

Using the (SC) criterion and computing facilities, the search was undertaken to find the A_i-stability region in the (β_1, γ_1)-plane for various values of $a \in [-1, 1)$.

Results obtained are pictured in Fig. 3. The stability region is represented by the convex part of the plane cut by the marked curve. It is interesting to compare this result with the similar giving the A_0-stability region of this formula in [8].

УСТОЙЧИВОСТЬ НА МНИМОЙ ОСИ ЛИНЕЙНЫХ МНОГОШАГОВЫХ ФОРМУЛ С ВТОРОЙ ПРОИЗВОДНОЙ

Резюме

В работе изучается A_i-устойчивость (устойчивость на мнимой оси) одношаговых и двухшаговых формул (F1), (F2) численного интергирования обыкновенных дифференциальных уравнений. В теоремах 1—5 показывается

— одношаговая формула (F1.4) максимального порядка $p = 4$ не является A_i-устойчивой

— одношаговая формула (F1.3) порядка $p = 3$ является A_i-устойчивой только тогда, когда $\gamma_1 \leqq -1/12$

— область A_i-устойчивости формулы (F1.2) в плоскости параметров (β_0, γ_1) совпадает с областью изображенной на рис. 1

— двухшаговая формула (F2.6) не является A_i-устойчивой для любого $a \in [-1, 1)$

— двухшаговая формула (F2.5) не является A_i-устойчивой для любых значений (a, γ_2)

— существуют A_i-устойчивые формулы (F2.4) порядка $p = 4$; на рисунке 3 показаны результаты численной проверки области A_i-устойчивости в плоскости параметров (β_1, γ_1) для некоторых значений параметра a.

STABILITA NA IMAGINÁRNÍ OSE LINEÁRNÍCH MNOHOKROKOVÝCH FORMULÍ S DRUHÝMI DERIVACEMI

Shrnutí

V práci se vyšetřuje A_i-stabilita (stabilita na imaginární ose) lineárních jedno- a dvoukrokových formulí (F1), (F2) pro numerické řešení počáteční úlohy u obyčejných diferenciálních rovnic. Ve větách 1—5 se dokazuje, že

— jednokroková formule (F1.4) maximálního řádu $p = 4$ je A_i-stabilní

— jednokroková formule (F1.3) řádu $p = 3$ je A_i-stabilní právě pro $\gamma_1 \leqq -1/12$

— oblast A_i-stability formule (F1.2) řádu $p = 2$ v rovině parametrů (β_0, γ_1) je dána oblastí na obr. 1

— dvoukroková formule (F2.6) řádu $p = 6$ není A_i-stabilní pro žádné $a \in [-1, 1)$

— dvoukroková formule (F2.5) řádu $p = 5$ není A_i-stabilní pro žádné hodnoty parametrů (a, γ_2)

— existují A_i-stabilní formule řádu $p = 4$; na obr. 3 jsou ukázány výsledky numerického vyšetřování oblastí A_i-stability formule (F2.4) v rovině parametrů (β_1, γ_1) pro některé hodnoty parametru a.

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