Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica

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Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica, Vol. 22 (1983), No. 1, 115--130

Persistent URL: http://dml.cz/dmlcz/120127

1983 — ACTA UNIVERSITATIS PALACKIANAE OLOMUCENSIS
FACULTAS RERUM NATURALIUM
VOL. 76, MATHEMATICA XXII

Katedra matematické analýzy a numerické matematiky přírodovědecké fakulty Univerzity Palackého v Olomouci
Vedoucí katedry: Prof. RNDr. Miroslav Laitoch, CSc.

CONJUGATE POINTS OF SOLUTIONS OF AN ITERATED DIFFERENTIAL EQUATION OF THE N-th ORDER

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(*Received March 28, 1982*)

In [1] there was introduced the concept of the so-called first conjugate point of an arbitrary (nontrivial) solution of the n-*th* order linear differential equation

$$y^{(n)}(t) + \sum_{k=0}^{n-1} a_{k+1}(t)\, y^{(k)}(t) = 0, \tag{1}$$

which is directly connected with the iteration of the second order ordinary linear homogeneous differential equation having the form

$$y''(t) + q(t)\, y(t) = 0, \tag{2}$$

where the function $q(t) \in \mathbf{C}_{\mathbf{I}}^{(n-2)}$, $\mathbf{I} = (-\infty, +\infty)$, $n \in \mathbf{N}$, $n > 1$, is understood to be $q(t) > 0$ for all $t \in \mathbf{I}$. Besides, equation (2) is assumed to be oscillatory in the sense of [2], i.e. there exist infinitely many zeros of its arbitrary (nontrivial) solution to every $t \in \mathbf{I}$, lying both to the left and to the right of the point t.

In the interest of brevity, differential equation (1) where (generally)

$$a_{k+1}(t) = a_{k+1}[q(t), q'\langle(t), \ldots, q^{(n-2)}(t)],$$

$k = 0, 1, \ldots, n-1$, will be called the iterated differential equation of the n-th order.

If we denote by $[u(t), v(t)]$ the basis of the space of all solutions of (2), where $u(t)$, $v(t)$ are two arbitrary oscillatory solutions of this equation, being linearly independent in the interval $\mathbf{I}$, then

$$[u^{n-1}(t), u^{n-2}(t)\, v(t), \ldots, u^{n-1-k}(t)\, v^k(t), \ldots, u(t)\, v^{n-2}(t), v^{n-1}(t)],$$

$k = 0, 1, \ldots, n-1$, is a basis of the space of all solutions of (1). Every nontrivial solution of the n-th order iterated differential equation (1) is thus of the form

$$y(t) = \sum_{i=1}^{n} C_i u^{n-i}(t)\, v^{i-1}(t),$$

where $C_i \in \mathbf{R}$, $i = 1, \ldots, n (n \in \mathbf{N}, n > 1)$ are arbitrary independent constants (parameters) whereby $\sum_{i=1}^{n} C_i^2 > 0$. Since (1) is of the n-th order, any arbitrary zero of each of its (nontrivial) oscillatory solution is of multiplicity $v = n - 1$ at most.

In all what follows, we will understand under the solution both of (1) and (2), the nontrivial solution, only.

The definition on the first conjugate point in [1] may be now extended in a natural way:

Definition 1.2: Let $t_0 \in (-\infty, +\infty)$ be an arbitrary firmly chosen point and let $y(t)$ be an arbitrary solution of (1) vanishing at the point t_0 [we will write ${}^{v}t_0$, expressing thus that t_0 is a v-tuple zero of such a solution, $v \in \{1, \ldots, n-1\}$].

Then we call by the m-th ($m \in \mathbf{N}$) conjugate point from the right [or from the left] to the point ${}^{v}t_0$, the m-th zero of this solution $y(t)$ lying from the right [or from the left] to the point ${}^{v}t_0$; we will write it as ${}^{\mu}t_m$ [or ${}^{\mu}t_{-m}$], where $\mu \in \{1, \ldots, n-1\}$ denotes the multiplicity of such a point.

In [1] a theorem was expressed and proved in details guaranteeing the existence of the first conjugate point to the point ${}^{v}t_0$ from the right of all multiplicities $v \in \in \{1, \ldots, n-1\}$. We considered thereby a bundle of all solutions of (1) vanishing together with the solution $u(t)$ of (2) at an arbitrary firmly chosen point $t_0 \in \mathbf{I}$.

In analogy with the remark following Definition 1.1 in [1], it is clear – with respect to the existence of the oscillatory bundles of solutions of (1) – that there always exists the $|m|$-th conjugate point ${}^{\mu}t_m \in \mathbf{I}$ to any arbitrary point ${}^{v}t_0 \in \mathbf{I}$, $v = 1, \ldots, \ldots, n-1$, where $m = 0, \pm 1, \pm 2, \ldots$, of multiplicity convenient $\mu \in \{1, \ldots, n-1\}$ from the right [for $m > 0$] or from the left [for $m < 0$].

Remark to conjugate points

Let $t \in (-\infty, +\infty)$ be an arbitrary firmly chosen point. Then

1. every point $t^* \in (-\infty, +\infty)$ conjugate to t is conjugate to itself (as a zeroth conjugate point), i.e. the property of conjugacy of points is reflexive;

2. if the point $t^* \in (-\infty, +\infty)$ is a conjugate point to t, then the point t is also a conjugate point to t^* (i.e. the property of conjugacy of points is symmetric); thus we speak hereafter of mutually conjugate points;

3. if the point $t^* \in (-\infty, +\infty)$ is a conjugate point to t and $t^{**} \in (-\infty, +\infty)$ is a conjugate point to t, then the point t^{**} is a conjugate point to t (i.e. the property of conjugacy of points is transitive).

With respect to the possible position of the first conjugate points of various solutions of (1) [where instead of "a set of possible positions of the 1st conjugate point" we will hereafter speak of "a set of the 1st conjugate points"] the following theorem holds.

Theorem 1.2: Let $t_0 \in (-\infty, +\infty)$ be an arbitrary firmly chosen point. Let $Y(t)$ be a bundle of all solutions $y(t)$ of (1) vanishing at t_0 of the same multiplicity $\nu \in \in \{1, \ldots, n-1\}, n \in \mathbf{N}, n > 1$, i.e. $Y({}^{\nu}t_0) = 0$.

Then the set of all 1st conjugate points ${}^{\mu}t_1$, $\mu \in \{1, \ldots, n-1\}$ from the right to the point ${}^{\nu}t_0$ in case of

1. $\nu = n-1$ is one-point and exactly ${}^{n-1}t_1$,
2. $\nu = n-2$ is an open interval $({}^{n-2}t_0, {}^{n-2}t_2)$, when exactly

$$ {}^{1}t_1 \in ({}^{n-2}t_0, {}^{n-2}t_2), $$

etc.

⋮

$n-1$) $\nu = 1$

a) n even: is one-point and exactly ${}^{1}t_1$ or an open interval, whereby $\mu \in \{1, \ldots, \ldots, n-2\}, n > 2$,

b) n odd: is always an open interval, whereby $\mu \in \{1, \ldots, n-2\}$.

Generally:

If $\nu = n - k$, $1 < k \leqq n - 1$, let us distinguish the following two possibilities:

1. The number n is even

a) if ν is an even number (k even), then the set of first conjugate points ${}^{\mu}t_1$ from the right to the point ${}^{\nu}t_0$ is always an open interval (the position of the first conjugate point is not uniquely determined),

b) if ν is an odd number (k odd), then the set of first conjugate points is either one-point [the first conjugate point ${}^{\mu}t_1$ from the right to the point ${}^{\nu}t_0$ is uniquely determined, whereby $\mu = \nu$], or an open interval [when the position of the first conjugate point ${}^{\mu}t_1$ from the right to the point ${}^{\nu}t_0$ is not uniquely determined, whereby $\mu \in \{1, \ldots, n-2\}$].

2. The number n is odd

a) if ν is an odd number (k even), then the set of first conjugate points ${}^{\mu}t_1$ from the right to the point ${}^{\nu}t_0$ is always an open interval [the position of the first conjugate points is not uniquely determined, whereby $\mu \in \{1, \ldots, n-2\}$],

b) if ν is even number (k odd), then the set of first conjugate points is either one-point [the first conjugate point ${}^{\mu}t_1$ from the right to the point ${}^{\nu}t_0$ is uniquely determined, whereby $\mu = \nu$] or an open interval [the position of the first conjugate point ${}^{\mu}t_1$ from the right to the point ${}^{\nu}t_0$ is not uniquely determined, whereby $\mu \in \{1, \ldots, n-2\}$].

This theorem—together with the concept of the n-th conjugate point from the right to the point t_0 introduced in the preceding Definition 1.2—is a consequence of Theorem 1.1 [1]. From the statement of Theorem 1.2 it turns out that the character (type) of the set of first conjugate points ${}^{\mu}t_1$ from the right to the point ${}^{\nu}t_0$

is the same as in case when the order n of (1) and the multiplicity $\nu \in \{1, \dots, n-1\}$ of t_0 are of the same parity (i.e. either ν, μ are even at the same time or they are odd at the same time), or they are of a different parity (i.e either n even and ν odd or conversely).

Theorem 1.3: Let ${}^{\nu}t_0 \in (-\infty, +\infty)$ be an arbitrary firmly chosen point wherein the bundle $Y(t)$ of all solutions $y(t)$ of (1) of multiplicity $\nu \in \{1, \dots, n-2\}$, $n \in \mathbf{N}$, $n > 2$ is vanishing and let the second conjugate point ${}^{\mu}t_2$ from the right to the point ${}^{\nu}t_0$ be uniquely determined. Then $\mu = \nu$ and it holds for multiplicity σ of the first conjugate point ${}^{\sigma}t_1$ not uniquely given because ${}^{\sigma}t_1 \in ({}^{\nu}t_0, {}^{\nu}t_2)$ that:
1. if $\nu = n - 2$, then $\sigma = 1$,
2. if $\nu = n - 3$, then $\sigma = 2$,
3. if $\nu = n - 4$, then either $\sigma = 1$ or $\sigma = 3$,
4. if $\nu = n - 5$, then either $\sigma = 2$ or $\sigma = 4$,

$\vdots$

$n - 2$) if $\nu = 1$, then
a) for n even: either $\sigma = 2$ or $\sigma = 4$ or … or $\sigma = n - 2$,
b) for n odd: either $\sigma = 1$ or $\sigma = 3$ or … or $\sigma = n - 2$.

The position of the $|k|$-th ($k = \pm 1, \pm 2, \dots$) conjugate point from the right or from the left to an arbitrary point $t_0 \in (-\infty, +\infty)$ is displayed in the following generalized

Theorem 1.4: Let ${}^{\nu}t_0 \in (-\infty, +\infty)$, $\nu \in \{1, \dots, n-1\}$, $n \in \mathbf{N}$, $n > 1$, be an arbitrary firmly chosen point and let $Y(t)$ be the bundle of all solutions $y(t)$ of (1) vanishing at the point ${}^{\nu}t_0$ of multiplicity ν.

Then

1. every $|k|$-th conjugate point ${}^{\mu}t_k$, $k = \pm 1, \pm 2, \dots$, is uniquely given to the point ${}^{n-1}t_0$, whereby $\mu = n - 1$; at the same time there holds the inequality

$$ {}^{n-1}t_k < {}^{n-1}t_{k+1}. $$

2. every $2|k|$-th conjugate point ${}^{\mu}t_{2k}$, $k = \pm 1, \pm 2, \dots$, is uniquely given to the point ${}^{n-2}t_0$, whereby $\mu = n - 2$ and the set of all $|2k + 1|$-st conjugate points ${}^{\mu}t_{2k+1}$, $k = \pm 1, \pm 2, \dots$, to the point ${}^{n-2}t_0$ forms an open interval $({}^{n-2}t_{2k}, {}^{n-2}t_{2k+2})$, whereby $\mu = 1$; at the same time there hold the inequalities

$$ {}^{n-2}t_{2k} < {}^{1}t_{2k+1} < {}^{n-2}t_{2k+2}. $$

3. a) If the first conjugate point ${}^{\mu}t_1$ from the right to the point ${}^{n-3}t_0$ is uniquely given, then the arbitrary $|k|$-*th* conjugate point ${}^{\mu}t_k$, $k = \pm 1, \pm 2, \dots$, is also uniquely given, whereby $\mu = n - 3$; at the same time there holds the inequality

$$ {}^{n-3}t_k < {}^{n-3}t_{k+1}. $$

b) If the set of all first conjugate points ${}^{\mu}t_1$ from the right to the point ${}^{n-3}t_0$ forms an open interval, whereby $\mu = 2$, then the arbitrary $2|k|$-th conjugate point ${}^{\sigma}t_{2k}$, $k = \pm 1, \pm 2, \ldots$, to the point ${}^{n-3}t_0$ is uniquely given, where $\sigma = n - 3$. The set of all $| 2k + 1 |$-st conjugate points ${}^{\sigma}t_{2k+1}$, $k = \pm 1, \pm 2, \ldots$, to the point ${}^{n-3}t_0$ forms an open interval $({}^{n-3}t_{2k}, {}^{n-3}t_{2k+2})$, where $\sigma = 2$; at the same time there hold the inequalities

$$ {}^{n-3}t_{2k} < {}^{2}t_{2k+1} < {}^{n-3}t_{2k+2} . $$

c) If the set of all first conjugate points ${}^{\mu}t_1$ from the right to the point ${}^{n-3}t_0$ forms an open interval whereby $\mu = 1$, then

c$_1$) the arbitrary $3 | k |$-th conjugate point ${}^{\sigma}t_{3k}$, $k = \pm 1, \pm 2, \ldots$, to the point ${}^{n-3}t_0$ is uniquely determined, where $\sigma = n - 3$,

c$_2$) the set of all $| 3k + 1 |$-st conjugate points ${}^{\sigma}t_{3k+1}$, $k = \pm 1, \pm 2, \ldots$, to the point ${}^{n-3}t_0$ forms an open interval $({}^{n-3}t_{3k}, {}^{1}t_{3k+2})$, where $\sigma = 1$,

c$_3$) the set of all $| 3k + 2 |$-nd conjugate points ${}^{\sigma}t_{3k+2}$, $k = \pm 1, \pm 2, \ldots$, to the point ${}^{n-3}t_0$ forms an open interval $({}^{1}t_{3k+1}, {}^{n-3}t_{3k+3})$, where $\sigma = 1$ and there hold the following inequalities

$$ {}^{n-3}t_{3k} < {}^{1}t_{3k+1} < {}^{1}t_{3k+2} < {}^{n-3}t_{3k+3} $$

etc.

Two kinds of points occur among the conjugate points. There are such whose position is given by choosing the point $t_0 \in (-\infty, +\infty)$ uniquely. On the other side there are such whose position is determined by an open interval (whose boundary points are conjugate points again) only. Conjugate points of the latter kind are "movable" owing to the arbitrariness (choice) of the parameters occurring in the form of the relative bundle $Y(t)$ of all solutions $y(t)$ of (1) vanishing at ${}^{\nu}t_0$ of multiplicity $\nu \in \{1, \ldots, n - 1\}$.

This leads us to their mutual distinguishing in the following

Definition 1.3: Let the points ${}^{\nu}t_0$, ${}^{\mu}t_k \in (-\infty, +\infty)$, where $\nu, \mu \in \{1, \ldots, n - 1\}$, $n \in \mathbf{N}$, $n > 1$, $k = \pm 1, \pm 2, \ldots$, are conjugate points of a solution $y(t)$ from the bundle $Y(t)$ of solutions of (1) vanishing at ${}^{\nu}t_0$.

Let us say that ${}^{\mu}t_k$ is a strongly conjugate point to the point ${}^{\nu}t_0$ exactly if all solutions $y(t)$ from the bundle $Y(t)$ of solutions of (1), ν-times vanishing at ${}^{\nu}t_0$ are vanishing at ${}^{\mu}t_k$ as well.

Every conjugate point to ${}^{\nu}t_0$ being not strongly conjugate point to ${}^{\nu}t_0$ will be called a weakly conjugate point to ${}^{\nu}t_0$.

Thus, it holds by the above definition: the point $t_k \in (-\infty, +\infty)$, where $k = \pm 1, \pm 2, \ldots$, is a weakly conjugate point to ${}^{\nu}t_0 \in (-\infty, +\infty)$, $\nu \in \{1, \ldots, n - 1\}$, if among all solutions $y(t)$ of (1) from the bundle $Y(t)$ ν-times vanishing at ${}^{\nu}t_0$, there exist at least two such solutions one of which is vanishing at t_k while the other does not.

The following two remarks give fuller account of the strongly and weakly conjugate points belonging to an arbitrary bundle $Y(t)$ of oscillatory solutions $y(t)$ of (1) vanishing at ${}^{\nu}t_0 \in (-\infty, +\infty)$ being of multiplicity $\nu \in \{1, \ldots, n - 1\}$.

Remark 1. Let $t_0 \in (-\infty, +\infty)$ be an arbitrary firmly chosen point.

1. Every point $T^* \in (-\infty, +\infty)$ strongly conjugate to t_0 is strongly conjugate to itself (the property of the strong conjugacy of points is reflexive),

2. If $T^* \in (-\infty, +\infty)$ is a strongly conjugate point to t_0, then the point t_0 is also a strongly conjugate point to T^* (the property of the strong conjugacy of points is symmetric); hereafter, these will be referred to as "mutually strongly conjugate points",

3. If $T^* \in (-\infty, +\infty)$ is a strongly conjugate point to t_0 and $T^{**} \in (-\infty, +\infty)$ is a strongly conjugate point to T^* (preserving the same multiplicity of the point T^* possessed in case of the pair of points t_0 and T^*), then the point T^{**} is a strongly conjugate point to t_0 (the property of the strong conjugacy of points is transitive).

Remark 2. Let $t_0 \in (-\infty, +\infty)$ be an arbitrary firmly chosen point and let $t^* \in$ $\in (-\infty, +\infty)$ be a weakly conjugate point to t_0. Then

1. t_0 is a weakly conjugate point to t^* (the property of the weak conjugacy of two conjugate points is symmetric),

2. if $t^{**} \in (-\infty, +\infty)$ is a strongly conjugate point to t^*, then t^{**} is a weakly conjugate point to t_0.

3. The property of the transitivity of the weak conjugacy for two pairs t_0, t^* and t^*, t^{**} of the mutually weakly conjugate points generally fails (i.e. it does not generally follow from the weak conjugacy of t_0, t^* and t^*, t^{**} that t_0 and t^{**} are weakly conjugate points).

In the following Theorem will be shown that the multiplicities ν, μ of the point ${}^{\nu}t_0 \in (-\infty, +\infty)$ and to it strongly conjugate point ${}^{\mu}t_k \in (-\infty, +\infty)$ coincide for every value $\nu, \mu \in \{1, \ldots, n-1\}$ and for all $k = \pm 1, \pm 2, \ldots$

Theorem 1.5: Let ${}^{\nu}t_0, {}^{\mu}t_k \in (-\infty, +\infty)$, where ν, $\mu \in \{1, \ldots, n-1\}$, $n \in \mathbf{N}$, $n > 1$ and $k = \pm 1, \pm 2, \ldots$, be two conjugate points of a solution $y(t)$ from the bundle $Y(t)$ of all solutions of (1) vanishing at ${}^{\nu}t_0$.

Then ${}^{\mu}t_k$ is a strongly conjugate point to ${}^{\nu}t_0$ exactly if

1. $\mu = \nu = n - 1, k = \pm 1, \pm 2, \ldots,$

2. $\mu = \nu = n - 2$, whereby $k = 2m$, $m = \pm 1, \pm 2, \ldots,$

3. $\mu = \nu = n - 3$, whereby

3.1. $k = 3m$, $m = \pm 1, \pm 2, \ldots$, if there exist simple weakly conjugate points to ${}^{\nu}t_0, {}^{\mu}t_k$,

3.2. $k = 2m$, $m = \pm 1, \pm 2, \ldots$, if there exist double weakly conjugate points to ${}^{\nu}t_0, {}^{\mu}t_k$,

3.3. $k = m$, $m = \pm 1, \pm 2, \ldots$, if there does not exist any weakly conjugate points to ${}^{\nu}t_0, {}^{\mu}t_k$,

4. $\mu = \nu = n - 4$, whereby

4.1. $k = 2m$ or $4m$, $m = \pm 1, \pm 2, \ldots$, if there exist simple weakly conjugate points to ${}^{\nu}t_0, {}^{\mu}t_k$, only

4.2. $k = 2m$, $m = \pm 1, \pm 2, \ldots$, if there exist triple weakly conjugate points to ${}^{\nu}t_0, {}^{\mu}t_k$,

4.3. $k = 3m$, $m = \pm 1, \pm 2, \ldots$, if there exist both simple and double weakly conjugate points at the same time to ${}^{\nu}t_0, {}^{\mu}t_k$,

$\vdots$

etc.

Proof: Let $t_0 \in (-\infty, +\infty)$ be an arbitrary firmly chosen point and let $[u(t), v(t)]$ be a basis of the oscillatory differential equation (2) such that both functions $u(t)$, $v(t)$ together with their first derivatives $u'(t)$, $v'(t)$ satisfy the condition

$$u(t_0) = v'(t_0) = 0 \tag{P}$$

at t_0 [so that $v(t_0) \neq 0$, $u'(t_0) \neq 0$]. Let $y(t)$ be such a solution of (1) from the oscillatory bundle $Y(t)$ ν-times, $\nu \in \{1, \ldots, n-1\}$, vanishing at t_0 (so that ${}^{\nu}t_0$). Then it follows from statement of Theorem 1.4, applying Lemma 1 from [1] and by Definition 1.3 on strongly conjugate points that

1. if $\nu = n - 1$, then every zero of the solution $y(t)$ of (1) from the bundle

$$Y(t) = C_1 u^{n-1}(t), \qquad C_1 \neq 0$$

is a zero of the function $u^{n-1}(t)$ at the same time and more precisely of multiplicity $(n-1)$. Thus, every $|k|$-th ($k = \pm 1, \pm 2, \ldots$) conjugate point ${}^{\mu}t_k$ to ${}^{n-1}t_0$ is a strongly conjugate point to it at the same time, whereby $\mu = n - 1$

2. if $\nu = n - 2$, then all zeros of multiplicity $(n-2)$ of the solution $y(t)$ of (1) from the bundle

$$Y(t) = u^{n-2}(t)\,[C_1 u(t) + C_2 v(t)],$$

$C_i \in \mathbf{R}$, $i = 1, 2$, $C_2 \neq 0$, are zeros of multiplicity $(n-2)$ of the function $u^{n-2}(t)$ at the same time, whereby there always lies exactly one simple zero of the two-parametric system of functions

$$Y^*(t) = C_1 u(t) + C_2 v(t)$$

between every two neighbouring zeros of multiplicity $(n-2)$ of the solution $y(t)$ of (1).

Thus, every $|k|$-th (where $k = 2m$, $m = \pm 1, \pm 2, \ldots$) conjugate point ${}^{\mu}t_k$ to ${}^{n-2}t_0$ is at the same time a strongly conjugate point to it, whereby $\mu = n - 2$.

3. If $\nu = n - 3$, then it holds for the point ${}^{n-3}t_0$, wherein every solution $y(t)$ of (1) from the bundle

$$Y(t) = u^{n-3}(t)\,[C_1 u^2(t) + C_2 u(t)\, v(t) + C_3 v^2(t)],$$

$C_i \in \mathbf{R}$, i $= 1, 2, 3$, $C_3 \neq 0$, vanishes at the same time with the function $u^{n-3}(t)$ that every $|k|$-th [where $k = 3m$ or $k = 2m$ or $k = m$, $m = \pm 1, \pm 2, \ldots$] conjugate point ${}^{\mu}t_k$ to ${}^{n-3}t_0$ is to it a strongly conjugate point at the same time, whereby $\mu = n - 3$. This situation occurs exactly if there exist two simple zeros different from each other [or one double zero or there does not exist any zero] between two arbitrary neighbouring zeros of multiplicity $(n - 3)$ of the function $u^{n-3}(t)$ of the three-parametric system of functions

$$Y^{**}(t) = C_1 u^2(t) + C_2 u(t)\, v(t) + C_3 v^2(t).$$

4. If $v = n - 4$, then it holds for the point ${}^{n-4}t_0$, wherein every solution $y(t)$ of (1) from the bundle

$$Y(t) = u^{n-4}(t)\,[C_1 u^3(t) + C_2 u^2(t)\, v(t) + C_3 u(t)\, v^2(t) + C_4 v^3(t)],$$

$C_i \in \mathbf{R}$, i $= 1, \ldots, 4$, $C_4 \neq 0$, together with the function $u^{n-4}(t)(t)$ vanishes at the same time that every $|\,k\,|$-th conjugate point ${}^{\mu}t_k$ [where $k = 4m$ or $k = 3m$ or $k = 2m$, $m = \pm 1, \pm 2, \ldots$] to the point ${}^{n-4}t_0$ is at the same time a strongly conjugate point to it, whereby $\mu = n - 4$. This case occurs exactly if there exist three simple zeros different from one another [or one simple and one double or exactly one simple or exactly one triple zero] between two arbitrary neighbouring zeros of multiplicity $(n - 4)$ of the function $u^{n-4}(t)$ of the four-parametric system of functions

$$Y^{***}(t) = \sum_{i=1}^{4} C_i u^{4-i}(t)\, v^{i-1}(t)$$

etc.

$\vdots$

$n - 1$) If $v = 1$, then every solution $y(t)$ of (1) from the $(n - 1)$-parametric bundle

$$Y(t) = u(t)\,[C_1 u^{n-2}(t) + C_2 u^{n-3}(t)\, v(t) + \ldots + C_{n-2} ut)\, v^{n-3}(t) + C_{n-1} v^{n-2}(t)],$$

$C_i \in \mathbf{R}$, $i = 1, 2, \ldots, n - 2$, $C_{n-1} \neq 0$, together with the function $u(t)$ vanishes at the point ${}^{1}t_0$. All zeros of the function $u(t)$ which are altogether simple, are the single strongly conjugate points of every solution $y(t)$ of (1) from this bundle at the same time. The existence of zeros of the $(n - 1)$-parametric system of functions

$$Y^{*}(t) = \sum_{i=1}^{n-1} C_i u^{n-i-1}(t)\, v^{i-1}(t)$$

acting in the bundle $Y(t)$ decides about the existence of further zeros of the solution $y(t)$ of (1).

In so far as such zeros exist, they are altogether weakly conjugate points to the point ${}^{1}t_0$ [and to all zeros of the function $u(t)$ at the same time] and may have multiples from 1 to $(n - 2)$ inclusive.

Thus in case of $n \in \mathbf{N}$ being odd there always exists at least one zero t^* of the system $Y^*(t)$ on an open interval $({}^{1}t_0, T_1)$, where T_1 is a neighbouring zero of the function $u(t)$ lying to the right of the point ${}^{1}t_0$.

If such a point t^* exists exactly one on the interval $({}^{1}t_0, T_1)$, then the system $Y^*(t)$ is of the form

$$Y^{*}(t) = [c_1 u(t) + c_2 v(t)] \sum_{i=1}^{n-2} C_i u^{n-i-2}(t)\, v^{i-1}(t),$$

where $c_j \in \mathbf{R}, j = 1, 2, c_2 \neq 0$ and $\sum_{i=1}^{n-2} C_i u^{n-i-2}(t)\, v^{i-1}(t) > 0$ or < 0 on $(-\infty, +\infty)$.

Let us remark that the system $Y^*(t)$ – for the reason of oscillatority of both functions $u(t)$ and $c_1 u(t) + c_2 v(t)$, linearly independent on $(-\infty, +\infty)$ – has by one zero of multiplicity 1 also at each open interval (T_k, T_{k+1}), $k = \pm 1, \pm 2, \ldots$, where T_k, T_{k+1} are two arbitrary neighbouring zeros of the function $u(t)$.

Thus, in this case, all strongly conjugate points of any solution $y(t)$ of (1) from the bundle $Y(t)$ of the form

$$Y(t) = u(t)\,[c_1 u(t) + c_2 v(t)] \sum_{i=1}^{n-2} C_i u^{n-i-2}(t)\, v^{i-1}(t)$$

are ${}^{\mu}t_k$, where $k = 2m$. $m = 0, \pm 1, \pm 2, \ldots$, whereby $\mu = 1$.

Consider a case, where the system $Y^*(t)$ in the bundle $Y(t)$ of the solutions $y(t)$ of (1) has exactly $(n - 2)$ simple mutually different zeros $t_1^*, \ldots, t_{n-2}^*$ on $({}^1t_0, T_1)$ – generally: on every open interval (T_k, T_{k+1}), where $k = 0, \pm 1, \pm 2, \ldots$

Then it is [up to an arbitrary multiplicative constant $C \in \mathbf{R} - \{0\}$] of the form

$$Y^*(t) = \prod_{j=1}^{n-2} [c_{1j} u(t) + c_{2j} v(t)],$$

where $c_{1j}, c_{2j} \in \mathbf{R}$, $j = 1, \ldots, n - 2$; $n \in \mathbf{N}$, $n > 2$, $c_{2j} \neq 0$ and where any two of the functions $c_{1j} u(t) + c_{2j} v(t)$, $j = 1, \ldots, n - 2$, are linearly independent on the interval $(-\infty, +\infty)$.

Then every $|k|$-th conjugate point ${}^{\mu}t_k$ [where $k = (n - 1)\, m$, $m = \pm 1, \pm 2, \ldots$] to the point ${}^1t_0$ is a strongly conjugate point to it at the same time, whereby $\mu = 1$.

If, for instance, the system $Y^*(t)$ has all its zeros exactly of multiplicity $(n - 2)$, then it is [up to an arbitrary multiplicative constant $C \in \mathbf{R} - \{0\}$] of the form

$$Y^*(t) = [c_1 u(t) + c_2 v(t)]^{n-2},$$

where $c_j \in \mathbf{R}, j = 1, 2, c_2 \neq 0$.

It then holds for all these zeros ${}^{\sigma}t_k^*$, $k = \pm 1, \pm 2, \ldots, \sigma = n - 2$, that on every open interval (T_k, T_{k+1}), where T_k, T_{k+1} are two arbitrary neighbouring zeros of the function $u(t)$ again, there always lies by one of them.

Hence it holds in this case that $\mu = 1$ for an arbitrary $|k|$-th strongly conjugate point ${}^{\mu}t_k$, $k = 2m$, $m = \pm 1, \pm 2, \ldots$, to the point ${}^1t_0$ of any solution $y(t)$ of (1) from the bundle $Y(t) = u(t)\, Y^*(t)$.

In this way we could continue considering the occurence of all possible zeros of the system $Y^*(t)$ from the $(n - 1)$-parametric bundle

$$Y(t) = u(t) \sum_{i=1}^{n-1} C_i u^{n-i-1}(t)\, v^{i-1}(t),$$

$C_i \in \mathbf{R}$, $C_{n-1} \neq 0$, $i = 1, \ldots, n - 1$; $n \in \mathbf{N}$, $n > 1$, of all solutions $y(t)$ of (1) vanishing together with the function $u(t)$ at the point ${}^1t_0$. Thereby we could consider these zeros of corresponding multiplicities (and all their admissible combinations at the given even or odd $n \in \mathbf{N}$) from 1 to $(n - 2)$, $n > 2$.

But, in all such cases, we are led to the same conclusion, namely that the single strongly conjugate points ${}^{\mu}t_k$, $k = 0, \pm 1, \pm 2, \ldots$, of the corresponding solutions $y(t)$ of (1) from the bundle $Y(t)$ are simple zeros of the function $u(t)$, whereby $\mu = 1$.

If the $(n - 1)$-parametric system of functions $Y^*(t) = \sum_{i=1}^{n-1} C_i u^{n-i-1}(t)\, v^{i-1}(t)$ has no zeros on the interval $(-\infty, +\infty)$ [which may occur at the given even $n \in \mathbf{N}$, $n > 2$], then the single zeros of any solution $y(t)$ of (1) from the bundle $Y(t) = u(t)\, Y^*(t)$ are exactly all simple zeros of the function $u(t)$, only, being mutually strongly conjugate.

It becomes apparent from the proof of Theorem 1.5 that all types of bundles $Y(t)$ of solutions $y(t)$ of (1) – with respect to how they were selected, see condition (P) – are mutually vanishing at the zeros of the function $u(t)$ and namely of multiplicity determined by the corresponding exponent of the function $u(t)$. The strongly conjugate points of all these bundles coincide, whereby the multiplicity of all strongly conjugate points of the same bundle is the same.

All the bundles $Y(t)$ of the oscillatory solutions $y(t)$ of (1) introduced, mutually differ by the multiplicities of their strongly conjugate points. Moreover, this difference is given by the possible existence of further zeros (weakly conjugate points) of the bundles $Y(t)$ of solutions of (1), lying always on the open intervals between two neighbouring strongly conjugate points.

The occurence of weakly conjugate points of the bundles $Y(t)$ of the oscillatory solutions $y(t)$ of (1) is summarized in

Theorem 1.6: Let ${}^{\nu}t_0 \in (-\infty, +\infty)$ be an arbitrary firmly chosen point of multiplicity $\nu \in \{1, \ldots, n - 1\}$, $n \in \mathbf{N}$, $n > 1$, wherein all bundles $Y(t)$ of oscillatory solutions $y(t)$ of (1) are vanishing. Then

1. no weakly conjugate points exist to the point ${}^{n-1}t_0$ (hence for $k = \pm 1, \pm 2, \ldots$ every point ${}^{\mu}t_k$ conjugate to ${}^{n-1}t_0$ is strongly conjugate),

2. there always exist weakly conjugate points to the point ${}^{n-2}t_0$, every exactly of multiplicity $\nu = 1$,

3. either no weakly conjugate points exist to the point ${}^{n-3}t_0$ or they exist either all of multiplicity $\mu = 1$, or all of multiplicity $\mu = 2$,

4. there always exist weakly conjugate points to the point ${}^{n-4}t_0$ either all exactly of multiplicity $\mu = 1$,
or all exactly of multiplicity $\mu = 3$,
or they are of multiplicity $\mu = 1$ or $\mu = 2$,

⋮

$n - 1$) a) In case of an even $n \in \mathbf{N}$: weakly conjugate point to the
point ${}^{1}t_0$ do not exist,
or all are of multiplicity $\mu = 1$,

or are of multiplicity $\mu = 1$ or $\mu = 2$,
or are of multiplicity $\mu = 1$ or $\mu = 3$,
$\vdots$
or all are of multiplicity $\mu = n - 2$,

b) In case of an odd $n \in \mathbf{N}$: weakly conjugate points always

exist. They are either
all of multiplicity $\mu = 1$,
or they are of multiplicity $\mu = 1$ or $\mu = 2$,
or all are of multiplicity $\mu = 3$,
or they are of multiplicity $\mu = 2$ or $\mu = 3$,
$\vdots$
or all are of multiplicity $\mu = n - 2$.

A survey on the existence and multiplicities of weakly conjugate points on an open interval between two arbitrary strongly conjugate points of the bundle $Y(t)$ of solutions $y(t)$s of the differential equation (1) is given in the following

Theorem 1.7: Let ${}^{\nu}T^*$, ${}^{\nu}T^{**} \in (-\infty, +\infty)$ be two arbitrary neighbouring strongly conjugate points of the bundle $Y(t)$ of all solutions $y(t)$ of the differential equation (1) vanishing at these points of multiplicities $\nu \in \{1, \ldots, n-1\}$, $n \in \mathbf{N}$, $n > 1$.

1. If $\nu = n - 1$, then no weakly conjugate point of the bundle $Y(t)$ lies between two points ${}^{n-1}T^*, {}^{n-1}T^{**}$.

2. If $\nu = n - 2$, then there lies exactly one, more precisely simple weakly conjugate point t_1 between two points ${}^{n-2}T^*$, ${}^{n-2}T^{**}$ such that

$$^{1}t_1 \in ({}^{n-2}T^*, {}^{n-2}T^{**}).$$

3. If $\nu = n - 3$, then there lies no weakly conjugate point between two points ${}^{n-3}T^*, {}^{n-3}T^{**}$, or there lies exactly one between them, and namely, a double weakly conjugate point t_1, such that

$$^{2}t_1 \in ({}^{n-3}T^*, {}^{n-3}T^{**}),$$

or there lie between them exactly two, and namely, simple weakly conjugate points t_1, t_2 different from each other, such that

$$^{1}t_1, {}^{1}t_2 \in ({}^{n-3}T^*, {}^{n-3}T^{**}).$$

4. If $\nu = n - 4$, then there lies either exactly one simple weakly conjugate point t_1 between two points ${}^{n-4}T^*$, ${}^{n-4}T^{**}$, i.e.

$$^{1}t_1 \in ({}^{n-4}T^*, {}^{n-4}T^{**}),$$

or there lie between them exactly two weakly conjugate points t_1, t_2 different from each other, one of which is of multiplicity $\mu = 1$ and the other of $\mu = 2$, such that

$$^{1}t_1, {}^{2}t_2 \in ({}^{n-4}T^*, {}^{n-4}T^{**}),$$

or there lie exactly three between them, and namely, simple weakly conjugate points t_1, t_2, t_3 different from one another, such that

$$^1t_1, {}^1t_2, {}^1t_3 \in ({}^{n-4}T^*, {}^{n-4}T^{**}),$$

or there lies exactly one between them, and namely, a threefold weakly conjugate point t_1, such that

$$^3t_1 \in ({}^{n-4}T^*, {}^{n-4}T^{**})$$

etc.

⋮

$n - 1$) If $\nu = 1$, then

a) in case of an even $n \in \mathbf{N}$, $n > 1$, there lies no weakly conjugate point between both points $^1T^*$, $^1T^{**}$, or for every $k = 2m$, $m = 1, 2, \dots, \frac{n-2}{2}$, there lie k weakly conjugate points $t_1, \dots, t_k$ there inclusive their multiplicities $\mu_1, \dots, \mu_k \in \{1, \dots, n - 2\}$, whereby $\sum_{j=1}^{k} \mu_j = k$. Thus

$$^{\mu_1}t_1, \dots, {}^{\mu_k}t_k \in ({}^1T^*, {}^1T^{**}),$$

b) in case of an odd $n \in \mathbf{N}$, $n > 1$, there always lies at least one weakly conjugate point. Generally: for every $k = 2m - 1$, $m = 1, 2, \dots, \frac{n-1}{2}$, there lie k weakly conjugate points $t_1, \dots, t_k$ inclusive their multiplicities $\mu_1, \dots, \mu_k \in \{1, \dots, n - 2\}$, whereby $\sum_{j=1}^{k} \mu_j = k$. Thus again

$$^{\mu_1}t_1, \dots, {}^{\mu_k}t_k \in ({}^1T^*, {}^1T^{**}).$$

The position of the $|k|$-th, $k = \pm 1, \pm 2, \dots$, weakly conjugate point $^{\mu}t_k$ of multiplicity $\mu \in \{1, \dots, n - 2\}$ to the point $^{\nu}t_0 \in (-\infty, +\infty)$, at which the bundles $Y(t)$ of all solutions $y(t)$ of (1) of multiplicity $\nu \in \{1 \dots n - 2\}$ are vanishing, is discussed into detail in the following

Theorem 1.8: Let $k = 0 \pm 1 \pm 2 \dots$

1. Let $Y(t)$ be a bundle of all solutions $y(t)$ of the differential equation (1) which are vanishing at the strongly conjugate points of multiplicity $(n - 2)$. Then it holds for an arbitrary weakly conjugate point, at which this bundle vanishes that

$$^{n-2}T_k < {}^1t_{k+1} < {}^{n-2}T_{k+2},$$

where $^{n-2}T_k$, $^{n-2}T_{k+2}$ are two arbitrary neighbouring strongly conjugate points of the bundle $Y(t)$,

2. Let $Y(t)$ be a bundle of all solutions $y(t)$ of the differential equation (1) which are vanishing at the strongly conjugate points of multiplicity $(n - 3)$. Then

a) it holds for an arbitrary double weakly conjugate point, at which this bundle vanishes that

$$^{n-3}T_k < {}^2t_{k+1} < {}^{n-3}T_{k+2},$$

where ${}^{n-3}T_k$, ${}^{n-3}T_{k+2}$ are two arbitrary neighbouring strongly conjugate points of the bundle $Y(t)$,

b) it holds for arbitrary simple weakly conjugate points at which this bundle vanishes that

$$ {}^{n-3}T_k < {}^{1}t_{k+1} < {}^{1}t_{k+2} < {}^{n-3}T_{k+3}, $$

where ${}^{n-3}T_k$, ${}^{n-3}T_{k+3}$ are two arbitrary neighbouring strongly conjugate points of the bundle $Y(t)$.

3. Let $Y(t)$ be a bundle of all solutions $y(t)$ of the differential equation (1) which are vanishing at the strongly conjugate points of multiplicity $(n - 4)$. Then

a) it holds for an arbitrary threefold weakly conjugate point, at which this bundle vanishes that

$$ {}^{n-4}T_k < {}^{3}t_{k+1} < {}^{n-4}T_{k+2}, $$

where ${}^{n-4}T_k$, ${}^{n-4}T_{k+2}$ are two arbitrary neighbouring strongly conjugate points of the bundle $Y(t)$

b) if there exist together with the simple weakly conjugate points also twofold weakly conjugate points at which this bundle vanishes, then either

$$ {}^{n-4}T_k < {}^{1}t_{k+1} < {}^{2}t_{k+2} < {}^{n-4}T_{k+3} $$

or

$$ {}^{n-4}T_k < {}^{2}t_{k+1} < {}^{1}t_{k+2} < {}^{n-4}T_{k+3}, $$

where ${}^{n-4}T_k$, ${}^{n-4}T_{k+3}$ are two arbitrary neighbouring strongly conjugate points of the bundle $Y(t)$,

c) if there exist only simple weakly conjugate points at which this bundle vanishes, then either

$$ {}^{n-4}T_k < {}^{1}t_{k+1} < {}^{n-4}T_{k+2} $$

or

$$ {}^{n-4}T_k < {}^{1}t_{k+1} < {}^{1}t_{k+2} < {}^{1}t_{k+3} < {}^{n-4}T_{k+4}, $$

where ${}^{n-4}T_k$, ${}^{n-4}T_{k+2}$ or ${}^{n-4}T_k$, ${}^{n-4}T_{k+4}$ are two arbitrary neighbouring strongly conjugate points of the bundle $Y(t)$

etc.

⋮

$n - 2$) Let $Y(t)$ be a bundle of all solutions $y(t)$ of the differential equation (1) which are vanishing at the simple strongly conjugate points; let us denote by ${}^{1}T_k$, ${}^{1}T_{k+m}$, $m \in \{2, 3, \dots, n - 1\}$, two arbitrary neighbouring of them.

I. Let $n \in \mathbf{N}$ be even, $n > 2$. Then it holds for arbitrary weakly conjugate points of the bundle $Y(t)$ whose sum of multiplicities is $\sigma \in \{2, 4, \dots, n - 2\}$ that

1. if $\sigma = 2$, then either

$$ {}^{1}T_k < {}^{1}t_{k+1} < {}^{1}t_{k+2} < {}^{1}T_{k+3} $$

or

$$ {}^{1}T_k < {}^{2}t_{k+1} < {}^{1}T_{k+2}, $$

where ${}^1T_k$, ${}^1T_{k+3}$ or ${}^1T_k$, ${}^1T_{k+2}$ are two arbitrary neighbouring strongly conjugate points of the bundle $Y(t)$.

2. if $\sigma = 4$, then

either

$${}^1T_k < {}^1T_{k+1} < {}^1T_{k+2} < {}^1t_{k+3} < {}^1t_{k+4} < {}^1T_{k+5}$$

or

$${}^1T_k < {}^1t_{k+1} < {}^1t_{k+2} < {}^2t_{k+3} < {}^1T_{k+4}$$

or

$${}^1T_k < {}^1t_{k+1} < {}^2t_{k+2} < {}^1t_{k+3} < {}^1T_{k+4}$$

or

$${}^1T_k < {}^2t_{k+1} < {}^1t_{k+2} < {}^1t_{k+3} < {}^1T_{k+4}$$

or

$${}^1T_k < {}^2t_{k+1} < {}^2t_{k+2} < {}^1T_{k+3}$$

or

$${}^1T_k < {}^1t_{k+1} < {}^3t_{k+2} < {}^1T_{k+3}$$

or

$${}^1T_k < {}^3t_{k+1} < {}^1t_{k+2} < {}^1T_{k+3}$$

or

$${}^1T_k < {}^4t_{k+1} < {}^1T_{k+2},$$

where ${}^1T_k$, ${}^1T_{k+4}$ or ${}^1T_k$, ${}^1T_{k+4}$ or ${}^1T_k$, ${}^1T_{k+3}$ or ${}^1T_k$, ${}^1T_{k+2}$ are two arbitrary neighbouring strongly conjugate points of the bundle $Y(t)$

etc.

⋮

$\frac{n-2}{2}\Big/$ if $\sigma = n - 2$, then

either

$${}^1T_k < {}^1t_{k+1} < \ldots < {}^1t_{k+n-2} < {}^1T_{k+n-1}$$

or

⋮

or

$${}^1T_k < {}^{n-2}t_{k+1} < {}^1T_{k+2},$$

where ${}^1T_k$, ${}^1T_{k+n-1}$ or ${}^1T_k$, ${}^1T_{k+2}$ are two arbitrary neighbouring strongly conjugate points of the bundle $Y(t)$.

II. Let $n \in \mathrm{N}$ be odd, $n > 1$. Then it holds for arbitrary weakly conjugate points of the bundle $Y(t)$ whose sum of multiplicities is $\sigma \in \{1, 3, \ldots, n-2\}$ that

1. if $\sigma = 1$, then

$${}^1T_k < {}^1t_{k+1} < {}^1T_{k+2},$$

where ${}^1T_k$, ${}^1T_{k+2}$ are two arbitrary neighbouring strongly conjugate points of the bundle $Y(t)$

2. if $\sigma = 3$, then

either

$$^1T_k < {}^3t_{k+1} < {}^1T_{k+2}$$

or

$$^1T_k < {}^1t_{k+1} < {}^2t_{k+2} < {}^1T_{k+3}$$

or

$$^1T_k < {}^2t_{k+1} < {}^1t_{k+2} < {}^1T_{k+3}$$

or

$$^1T_k < {}^1t_{k+1} < {}^1t_{k+2} < {}^1t_{k+3} < {}^1T_{k+4},$$

where ^1T_k, $^1T_{k+2}$ or ^1T_k, $^1T_{k+3}$ or ^1T_k, $^1T_{k+4}$ are two arbitrary neighbouring strongly conjugate points of the bundle $Y(t)$

etc.

⋮

$\left.\frac{n-1}{2}\right/$ if $\sigma = n-2$, then

either

$$^1T_k < {}^1t_{k+1} < \ldots < {}^1t_{k+n-2} < {}^1T_{k+n-1}$$

or

⋮

or

$$^1T_k < {}^{n-2}t_{k+1} < {}^1T_{k+2},$$

where $^1T_k, {}^1T_{k+n-1}$ or ^1T_k, $^1T_{k+2}$ are two arbitrary neighbouring strongly conjugate points of the bundle $Y(t)$.

СОПРЯЖЕННЫЕ ТОЧКИ РЕШЕНИЙ ИТЕРИРОВАННОГО ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ n-ГО ПОРЯДКА

Резюме

В работе решается вопрос разложения нулевых точек пучков колеблющихся решений итерированного линейного дифференциального уравнения n-го порядка взглядом на их насобности. Изучаются нулевые точки пучков постепенно с насобностьями ν от 1 до $n-1$.

Понятие так называемой первой сопряженной точки справа взглядом к произвольной закрепленно выбранной нулевой точке $t_0 \in (-\infty, +\infty)$ решения изучаемого дифференциального уравнения расширено на понятие $|\kappa|$-ой ($\kappa = \pm 1, \pm 2, \ldots$) сопряженной точки справа или слева к точке t_0.

Между всеми сопряженными точками любой насобности $\nu \in \{1, \ldots, n-1\}$ различаются так называемые сильно или слабо сопряженные точки. В нескольких теоремах показано появление, номер и насобности одинаковых слабо сопряженных точек однако между двумя произвольными соседними сильно сопряженными точками, однако изучается вопрос существования оби двух типа сопряженных точек в целом на интервале $(-\infty, +\infty)$ и соотношений между их насобностьями включительно.

Показывается, что все нулевые точки с насобностью $\nu = n-1$ сильно сопряженные; для каждой нисшей насобности существуют рядом с сильно сопрященными точками тоже

слабо сопряженные точки. К сильно сопряженным точкам с насобностью $\nu = 1$ могут существовать слабо сопряженные точки с насобностьями от $\mu = 1$ до $\mu = n - 2$ включительно (для $n \in N$ счетного не должны существовать ни какие слабо сопряженные точки — тогда единственными нулевыми точками таких пучков решений появляются только сильно сопряженные точки).

KONJUGOVANÉ BODY ŘEŠENÍ ITEROVANÉ DIFERENCIÁLNÍ ROVNICE N-TÉHO ŘÁDU

Souhrn

V práci je řešena otázka rozložení nulových bodů jednotlivých svazků oscilatorických řešení iterované lineární diferenciální rovnice n-tého řádu s ohledem na jejich násobnosti. Jsou vyšetřovány nulové body svazků postupně s násobnostmi od 1 do $n - 1$.

Pojem tzv. prvního konjugovaného bodu zprava vzhledem k libovolnému pevně zvolenému nulovému bodu $t_0 \in (-\infty, +\infty)$ řešení uvažované diferenciální rovnice je rozšířen na pojem $|k|$-tého $(k = \pm 1, \pm 2, \ldots)$ konjugovaného bodu zprava resp. zleva k bodu t_0.

Mezi všemi konjugovanými body libovolné násobnosti $\nu \in \{1, \ldots, n - 1\}$ jsou rozlišeny tzv. silně resp. slabě konjugované body. V několika větách je ukázáno na výskyt, počet a násobnosti jednotlivých slabě konjugovaných bodů jednak mezi dvěma libovolnými sousedními silně konjugovanými body, jednak jsou studovány otázky koexistence obou druhů konjugovaných bodů globálně na celém intervalu $(-\infty, +\infty)$ včetně vztahů mezi jejich násobnostmi.

Ukazuje se, že všechny nulové body násobnosti $\nu = n - 1$ jsou silně konjugované; pro každou nižší násobnost existují vedle silně konjugovaných bodů též slabě konjugované body. K silně konjugovaným bodům násobnosti $\nu = 1$ mohou existovat slabě konjugované body s násobnostmi $\mu = 1$ až do $\mu = n - 2$ včetně (při sudém $n \in N$ nemusí existovat žádné slabě konjugované body, kdy jedinými nulovými body takových svazků řešení jsou jen silně konjugované body).

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AUPO Fac. rerum nat. Vol. 76, Math. XXII (1983)