Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica

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Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica, Vol. 22 (1983), No. 1, 99--105

Persistent URL: http://dml.cz/dmlcz/120125

1983 — ACTA UNIVERSITATIS PALACKIANAE OLOMUCENSIS
FACULTAS RERUM NATURALIUM
VOL. 76, MATHEMATICA XXII

Katedra matematické analýzy a numerické matematiky přírodovědecké fakulty Univerzity Palackého v Olomouci
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ON THE BASIC CENTRAL DISPERSION OF THE DIFFERENTIAL EQUATION $y'' = q(t)\,y$ WITH AN ALMOST PERIODIC COEFFICIENT

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(*Received October 6, 1981*)

1. Introduction

We investigate differential equations having the type

$$y'' = q(t)\,y, \qquad q \in C^0(\mathbf{R}). \tag{q}$$

The distribution of zeros of solutions of (q) may be described by means of the basic central dispersion φ of (q). O. Borůvka proved in [2] that the function $\varphi(t) - t$ is π-periodic if the coeficient q of the oscillatory equation (q) is π-periodic, too. This paper proves

Theorem 1. *Let q be an almost periodic function and let φ be the basic central dispersion of the oscillatory equation* (q). *Then the function $\varphi(t) - t$ is almost periodic.*

2. Fundamental concepts and lemmas

An equation (q) is called oscillatory on $\mathbf{R}$ if $\pm\infty$ are cluster points of zeros of every nontrivial solution of (q). A function $\alpha \in C^0(\mathbf{R})$ is a (first) phase of (q) if there exist independent solutions u, v of (q) such that

$$\operatorname{tg} \alpha(t) = u(t)/v(t) \qquad \text{for } t \in \mathbf{R} - \{t;\, v(t) = 0\}.$$

Every phase α of (q) has a continuous derivative of the third order on $\mathbf{R}$ and $\alpha'(t) \neq \neq 0$ for $t \in \mathbf{R}$. The equation (q) is oscillatory if $\alpha(\mathbf{R}) = \mathbf{R}$. If $\mathfrak{E}$ denotes the set of phases of $y'' = -y$ and α is a phase of (q), then $\mathfrak{E}\alpha := \{\varepsilon\alpha, \varepsilon \in \mathfrak{E}\}$ is the set of phases of (q) and $\varepsilon(t + \pi) = \varepsilon(t) + \operatorname{sign} \varepsilon'$ for $\varepsilon \in \mathfrak{E}$.

Let α be a phase of (q) and let us put $\varphi(t) := \alpha^{-1}[\alpha(t) + \pi \operatorname{sign} \alpha']$ for $t \in \mathbf{R}$. The function φ is called the basic central dispersion (of the first kind) of (q) (see [1], [2]).

Definition 1. ([4]). *The continuous function f is called almost periodic (on $\mathbf{R}$) if there exists a number $L(= L(\varepsilon))$ to every $\varepsilon > 0$, such that at least one number τ exists in $[x, x + L)$ for every $x \in \mathbf{R}$ so that*

$$|f(t + \tau) - f(t)| < \varepsilon \qquad \textit{for } t \in \mathbf{R}.$$

It was proved in [5] that every equation (q) with an almost periodic coefficient q is either oscillatory or disconjugate (i.e. every (nontrivial) solution of (q) has at most one zero on $\mathbf{R}$).

Lemma 1. *Let $q_n \in C^0(\mathbf{R})$ and $\lim_{n\to\infty} q_n(t) = q(t)$ uniformly on $\mathbf{R}$. Then there exist phases α_n of (q_n) and α of (q) such that*

$$\lim_{n\to\infty} \alpha_n(t) = \alpha(t), \qquad \lim_{n\to\infty} \alpha_n'(t) = \alpha'(t) \tag{1}$$

uniformly on every compact interval.

Proof. Let u_n, v_n be solutions of (q_n) and u, v be solutions of (q) satisfying the initial conditions: $u_n(0) = u(0) = v_n'(0) = v'(0) = 0$, $u_n'(0) = u'(0) = v_n(0) = v(0) = 1$. Let us put

$$\beta_n(t) := 1/(u_n^2(t) + v_n^2(t)), \qquad \beta(t) := 1/(u^2(t) + v^2(t)), \qquad t \in \mathbf{R}.$$

Then $\lim_{n\to\infty} (u_n^2(t) + v_n^2(t)) = u^2(t) + v^2(t)$ uniformly on every compact interval ([3], Theorem 2.4.) and thus also $\lim_{n\to\infty} \beta_n(t) = \beta(t)$ uniformly on every compact interval. Let us put

$$\alpha_n(t) := \int_0^t \beta_n(s)\, ds, \qquad \alpha(t) := \int_0^t \beta(s)\, ds, \qquad t \in \mathbf{R}.$$

Then α_n is a phase of (q_n) and α is a phase of (q) having properties given in the Lemma.

Remark 1. It appears from the following example that (1) generally does not uniformly hold on $\mathbf{R}$.

Example 1. Let $q_n(t) := -((n + 1)/n)^2$, $q(t) := -1$ for $t \in \mathbf{R}$. Then $\lim_{n\to\infty} q_n(t) = q(t)$ uniformly on $\mathbf{R}$. The function $t(n + 1)/n$ is a phase of (q_n) and the function t

is a phase of (q). Assume the existence of phases α_n of (q_n) and α of (q) such that (1) uniformly holds on **R**. Then there exist phases ε_n, $\varepsilon \in \mathfrak{E}$, such that

$$\lim_{n\to\infty} \varepsilon_n(t(n+1)/n) = \varepsilon(t) \tag{2}$$

uniformly on **R**. Hence, there is an index N_1 such that $|\varepsilon_n(t(n+1)/n) - \varepsilon(t)| < 1$ for $t \in \mathbf{R}$ and every $n > N_1$. Specially (we put $t = n\pi$)

$$|\varepsilon_n((n+1)\pi) - \varepsilon(n\pi)| < 1 \qquad \text{for } n > N_1. \tag{3}$$

From (2) we get $\lim_{n\to\infty} \varepsilon_n(0) = \varepsilon(0)$ and further $\operatorname{sign} \varepsilon_n' = \operatorname{sign} \varepsilon'$, $\varepsilon_n((n+1)\pi) = \varepsilon_n(0) + (n+1)\pi \,.\, \operatorname{sign} \varepsilon'$ for $n > N_2$, $\varepsilon(n\pi) = \varepsilon(0) + n\pi \operatorname{sign} \varepsilon'$. Then we have for $n > N_2$

$$|\varepsilon_n((n+1)\pi) - \varepsilon(n\pi)| \geqq \pi - |\varepsilon_n(0) - \varepsilon(0)|,$$

which leads, because of $\lim_{n\to\infty} \varepsilon_n(0) = \varepsilon(0)$, to a contradiction of (3).

Lemma 2. *Let φ_n and φ be the basic central dispersions of the oscillatory equations* (q_n) and (q), *respectively. Let $\lim_{n\to\infty} q_n(t) = q(t)$ uniformly on* **R**. *Then*

$$\lim_{n\to\infty} \varphi_n(t) = \varphi(t)$$

uniformly on every compact interval.

Proof. Let α_n and α be, respectively, phases of (q_n) and (q) having the properties given in Lemma 1. Then $\alpha_n(\mathbf{R}) = \mathbf{R}$, $\alpha(\mathbf{R}) = \mathbf{R}$ and $\operatorname{sign} \alpha_n' = \operatorname{sign} \alpha'$ for $n > N$. By Lemma 1 $\lim_{n\to\infty} \alpha_n^{(i)}(t) = \alpha^{(i)}(t)$ uniformly on every compact interval, $i = 0,1$. It follows from this that $\lim_{n\to\infty} \alpha_n^{-1}(t) = \alpha^{-1}(t)$ uniformly on every compact interval. From the above properties of sequences $\{\alpha_n(t)\}$ and $\{\alpha_n^{-1}(t)\}$ and from the equalities $\varphi_n(t) = \alpha_n^{-1}[\alpha_n(t) + \pi \operatorname{sign} \alpha_n']$, $\varphi(t) = \alpha^{-1}[\alpha(t) + \pi \operatorname{sign} \alpha']$ then follows the assertion of the Lemma.

Lemma 3. *Let q be an almost periodic function and φ be the basic central dispersion of the oscillatory equation* (q). *Then there exists a number $K > 0$ such that $\varphi(t) - t < K$ for $t \in \mathbf{R}$.*

Proof. Let $s := \varphi(0)$. By the Theorem on continuous dependence of solutions on parameters there exists an $\varepsilon > 0$ such that for $|\lambda| < \varepsilon$ any (nontrivial) solution z_λ of $z'' = (q(t) + \lambda) z$, $z_\lambda(0) = 0$, has a zero in $(s/2, 3s/2)$. Then it follows from the Sturm comparison theorem that any (nontrivial) solution z of (p), $z(0) = 0$, has a zero in $(s/2, 3s/2)$ for every $p \in C^0([0, 3s/2])$ for which $|q(t) - p(t)| < \varepsilon$ for $t \in [0, 3s/2]$. By assumption q is an almost periodic function i.e. there is a number L with such a property that there exists a τ: $|q(t) - q(t+\tau)| < \varepsilon$ for $t \in \mathbf{R}$ on every interval of type $[x, x+L)$ ($x \in \mathbf{R}$). If $x_0 \in \mathbf{R}$, then there exists such a $\tau_0 \in [x_0, x_0 + L)$ that $|q(t) - q(t+\tau_0)| < \varepsilon$ for $t \in \mathbf{R}$.

If u is a (nontrivial) solution of $y'' = q(t + \tau_0)\, y$, $u(0) = 0$, then there exists such a solution y_0 of (q) that $u(t) = y_0(t + \tau_0)$ for $t \in \mathbf{R}$. Let $u(\xi) = 0$, $u(t) \neq 0$ for $t \in (0, \xi)$. Evidently $\xi \in (s/2,\ 3s/2)$, $y_0(\tau_0) = y_0(\tau_0 + \xi) = 0$ and using the Sturm comparison theorem we obtain

$$\varphi(x_0) \leqq \varphi(\tau_0) = \tau_0 + \xi < x_0 + L + \xi < x_0 + L + 3s/2.$$

It then follows that

$$\varphi(x_0) - x_0 < L + 3s/2.$$

The assertion of the Lemma can be found by setting $K := L + 3s/2$.

Lemma 4. *Let q be an almost periodic function,* (q) *be an oscillation equation and*

$$\lim_{n \to \infty} q(t + h_n) = p(t)$$

uniformly on $\mathbf{R}$, *where* $\{h_n\}$ *is a sequence of numbers. Then* (p) *is an oscillatory equation.*

Proof. Let α be a phase of (q), sign $\alpha' = 1$. Then evidently $\alpha(t + h_n)$ is a phase of (q_n), where $q_n(t) := q(t + h_n)$ for $t \in \mathbf{R}$. By Lemma 1 there exist $\varepsilon_n \in \mathfrak{E}$, sign $\varepsilon_n' = 1$ and a phase β of (p), sign $\beta' = 1$, such that

$$\lim_{n \to \infty} \varepsilon_n(\alpha(t + h_n)) = \beta(t) \tag{4}$$

uniformly on every compact interval. Let φ be the basic central dispersion of (q). By Lemma 3 there exists a number $K > 0$:

$$\varphi(t) - t < K \qquad \text{for } t \in \mathbf{R},$$

whence $\alpha(t + h_n + K) > \alpha\varphi(t + h_n) = \alpha(t + h_n) + \pi$. Hence $\alpha(h_n + K) > \alpha(h_n) + \pi$. If (p) is not oscillatory equation, then it is a disconjugate eqation so that

$$|\, \beta(t) - \beta(0)\, | < \pi, \qquad \text{for } t \in \mathbf{R}, \tag{5}$$

(see [1], [2]). Putting $t = 0$ or $t = K$ in (4), we find

$$\beta(K) - \beta(0) = \lim_{n \to \infty} [\varepsilon_n(\alpha(K + h_n)) - \varepsilon_n(\alpha(h_n))] \geqq$$
$$\geqq \lim_{n \to \infty} [\varepsilon_n(\alpha(h_n) + \pi) - \varepsilon_n(\alpha(h_n))] = \lim_{n \to \infty} \pi = \pi.$$

But this contradicts (5) above.

3. Proof of Theorem 1

By the Bohr–Bochner theorem a function $g \in C^0(\mathbf{R})$ is almost periodic if and only if there exist for any sequence $\{h_n\}$ a sequence of functions selected from $\{g(t + h_n)\}$ uniformly converging on $\mathbf{R}$ (see [4], Theorem 4).

To prove that the function $\varphi(t) - t$ is almost periodic, it suffices to show that for any sequence $\{h_n\}$ a subsequence uniformly convergent on **R** may be selected from the sequence $\{\varphi(t + h_n) - t - h_n\}$. By our assumption, q is an almost periodic function. Thus, we can select a subsequence uniformly convergent on **R** from $\{q(t + h_n)\}$. Without any loss of generality we may assume, $\lim_{n\to\infty} q(t + h_n) = p(t)$ uniformly on **R**. Let $\alpha(t)$ be a phase of (q). Then $\alpha(t + h_n)$ is a phase of

$$y'' = q(t + h_n)\, y. \tag{6}$$

Let ψ be the basic central dispersion of (6). It then follows from $\alpha\varphi(t) = \alpha(t) + \pi \operatorname{sign} \alpha'$, $\alpha(\psi(t) + h_n) = \alpha(t + h_n) + \pi \operatorname{sign} \alpha'$ that $\psi(t) = \varphi(t + h_n) - h_n$. Following Lemma 2 $\{\varphi(t + h_n) - t - h_n\}$ converges uniformly on every compact interval. Let us assume $\{\varphi(t + h_n) - t - h_n\}$ not to converge uniformly on **R**. Then there exists a number a (> 0) and increasing sequences of positive integers $\{k_n\}$, $\{r_n\}$ and $\{t_n\}$ ($|t_n| \to \infty$) such that

$$|\varphi(t_n + h_{k_n}) - h_{k_n} - \varphi(t_n + h_{r_n}) - h_{r_n}| \geqq a, \qquad n = 1, 2, 3, \ldots \tag{7}$$

By Lemma 3, the sequences $\{\varphi(t_n + h_{k_n}) - t_n - h_{k_n}\}$, $\{\varphi(t_n + h_{r_n}) - t_n - h_{r_n}\}$ are bounded and q is an almost periodic function. Possing to appropriate subsequences, under the same notation for simplification, we find that

$$\lim_{n\to\infty} (\varphi(t_n + h_{k_n}) - t_n - h_{k_n}) = b,$$

$$\lim_{n\to\infty} (\varphi(t_n + h_{r_n}) - t_n - h_{r_n}) = c$$

and

$$\lim_{n\to\infty} q(t + t_n + h_{k_n}) = p_1(t), \qquad \lim_{n\to\infty} q(t + t_n + h_{r_n}) = p_2(t) \tag{8}$$

uniformly on **R**.

With respect to (7)

$$|b - c| \geqq a. \tag{9}$$

We now prove that $p_1 = p_2$. Let $\varepsilon > 0$. Since $\lim_{n\to\infty} q(t + h_n) = p(t)$ uniformly on **R** and (8) holds, there exists an index N such that:

(i) $|q(t + h_{k_n}) - q(t + h_{r_n})| < \varepsilon/3$ for $t \in$ **R** and all $n > N$. Thus, also $|q(t + t_n + h_{k_n}) - q(t + t_n + h_{r_n})| < \varepsilon/3$ for $t \in$ **R** and all $n > N$,

(ii) $|q(t + t_n - h_{k_n}) - p_1(t)| < \varepsilon/3$, $|q(t + t_n + h_{r_n}) - p_2(t)| < \varepsilon/3$ for $t \in$ **R** and all $n > N$.

It then follows from (i) and (ii) $|p_1(t) - p_2(t)| < \varepsilon$ for $t \in$ **R** and since ε is an arbitrary positive number, we get $p_1 = p_2$.

Since $\varphi(t + t_n + h_{k_n}) - t_n - h_{k_n}$ is the basic central dispersion of $y'' = q(t + t_n + h_{k_n})\, y$ and $\varphi(t + t_n + h_{r_n}) - t_n - h_{r_n}$ is the basic central dispersion of $y'' = q(t + t_n + h_{r_n})\, y$, we obtain from $p_1 = p_2$ and by Lemma 3

$$\lim_{n\to\infty} [\varphi(t_n + h_{k_n}) - t_n - h_{k_n}] = \lim_{n\to\infty} [\varphi(t_n + h_{r_n}) - t_n - h_{r_n}]$$

contrary to (9).

ОСНОВНАЯ ЦЕНТРАЛЬНАЯ ДИСПЕРСИЯ ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ $y'' = q(t)\, y$ С ПОЧТИ-ПЕРИОДИЧЕСКИМ КОЭФФИЦИЕНТОМ q

Резюме

Пусть

(q) $$y'' = q(t)\, y, \qquad q \in C^0(\mathbf{R}),$$

колеблющееся уравнение, $t_0 \in \mathbf{R}$ и $y(\neq 0)$ — решение уравнения (q), $y(t_0) = 0$. Пусть $\varphi(t_0)$ — первое справа сопряженное число с t_0. Тогда функция φ определена на $\mathbf{R}$ и называется основная центральная дисперсия уравнения (q). В работе показана теорема:

Пусть φ — основная центральная дисперсия колеблющего уравнения (q) с почти-периодическим коэффициентом q. Тогда $\varphi(t) - t$ почти-периодическая функция.

ZÁKLADNÍ CENTRÁLNÍ DISPERSE DIFERENCIÁLNÍ ROVNICE $y'' = q(t)\, y$ SE SKOROPERIODICKÝM KOEFICIENTEM q

Souhrn

Nechť

(q) $$y'' = q(t)\, y, \qquad q \in C^0(\mathbf{R}),$$

je oscilatorická rovnice, $t_0 \in \mathbf{R}$ a y je netriviální řešení rovnice (q), $y(t_0) = 0$. Nechť $\varphi(t_0)$ je první zprava od bodu t_0 ležící nulový bod řešení y. Pak funkce φ je definovaná na $\mathbf{R}$ a nazývá se základní centrální disperse rovnice (q). V práci je dokázána věta:

Nechť φ je základní centrální disperse oscilatorické rovnice (q) se skoroceriodickým koeficientem q. Pak $\varphi(t) - t$ je skoroperiodická funkce.

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AUPO, Fac. rerum nat. Vol. 76, Math. XXII (1983)