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THE FIRST CONJUGATE POINT OF SOLUTION OF THE N-th ORDER ITERATED DIFFERENTIAL EQUATION

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Dedicated to Prof. Miroslav Laitoch on his 60th birthday

Let us consider a *2-nd* order linear homogeneous differential equation having the form

$$y''(t) + q(t)\,y(t) = 0, \tag{1}$$

where the function $q(t) \in \mathbf{C}_{\mathbf{I}}^{(n-2)}$, $\mathbf{I} = (-\infty, +\infty)$, $n \in \mathbf{N}$, $n > 1$, $q(t) > 0$ for $\forall\,(t \in \mathbf{I})$, oscillatory in the sense of [2], i.e. to every $t \in \mathbf{I}$ there exist infinitely many zeros of its arbitrary nontrivial solution lying both to the left and to the right of the point t.

Iterating it n-times (see for instance [1]) leads to an *n-th* order linear homogeneous differential equation (more briefly to an n-*th* order iterated differential equation). Let us write it generally in the form

$$y^{(n)}(t) + \sum_{k=0}^{n-1} a_{k+1}(t)\,y^{(k)}(t) = 0, \tag{n}$$

where $a_{k+1}(t) = a_{k+1}[q(t), q'(t), \dots, q^{(n-2)}(t)]$, whose basis is formed by the ordered n-tuples of functions

$$[u^{n-1}(t), u^{n-2}(t)\,v(t), \dots, u^{n-1-k}(t)\,v^{k}(t), \dots, u(t)\,v^{n-2}(t), v^{n-1}(t)], \tag{B}$$

$k = 0, 1, 2, \dots, n-1$, linearly independent on the interval $\mathbf{I}$. Denoting

$$y_1(t) = u^{n-1}(t), y_2(t) = u^{n-2}(t)\,v(t), \dots, y_k(t) =$$
$$= u^{n-k}(t)\,v^{k-1}(t), \dots, y_{n-1}(t) = u(t)\,v^{n-2}(t), y_n(t) = v^{n-1}(t),$$

i.e. generally

$$y_k(t) = u^{n-k}(t)\,v^{k-1}(t), \qquad k = 0, 1, 2, \dots, n-1,$$

the equation (n) may be written in the form

$$\frac{1}{w[y_1(t), \ldots, y_n(t)]}\begin{vmatrix} y_1(t) & y_2(t) & \ldots & y_k(t) & \ldots & y_n(t) & y(t) \\ y_1'(t) & y_2'(t) & \ldots & y_k'(t) & \ldots & y_n'(t) & y'(t) \\ \ldots & & & & & & \\ y_1^{(n-1)}(t) & y_1^{(n-1)}(t) & \ldots & y_k^{(n-1)}(t) & \ldots & {}_n^{(n-1)}(t) & {}^{(n-1)}(t) \\ y_1^{(n)}(t) & y_2^{(n)}(t) & \ldots & y_k^{(n)}(t) & \ldots & y_n^{(n)}(t) & y^{(n)}(t) \end{vmatrix} = 0,$$

where

$$w[y_1(t), \ldots, y_n(t)] = \begin{vmatrix} y_1(t) & y_2(t) & \ldots & y_k(t) & \ldots & y_n(t) \\ y_1'(t) & y_2'(t) & \ldots & y_k'(t) & \ldots & y_n'(t) \\ \ldots & & & & & \\ y_1^{(n-2)}(t) & y_2^{(n-2)}(t) & \ldots & y_k^{(n-2)}(t) & \ldots & y_n^{(n-2)}(t) \\ y_1^{(n-1)}(t) & y_2^{(n-1)}(t) & \ldots & y_k^{(n-1)}(t) & \ldots & y_n^{(n-1)}(t) \end{vmatrix} \neq 0$$

in the interval **I** is the Wronskian of the basis (B) relating to the differential equation (n).

Thus we have for the coefficients $a_{k+1}(t)$, $k = 0, 1, 2, \ldots, n-1$; $n \in \mathbf{N}$, $n > 1$, occuring in the equation (n):

$$a_n(t) = -\frac{1}{w[y_1(t), \ldots, y_n(t)]}\begin{vmatrix} y_1(t) & y_2(t) & \ldots & y_k(t) & \ldots & y_n(t) \\ y_1'(t) & y_2'(t) & \ldots & y_k'(t) & \ldots & y_n'(t) \\ \ldots & & & & & \\ y_1^{(n-2)}(t) & y_2^{(n-2)}(t) & \ldots & y_k^{(n-2)}(t) & \ldots & y_n^{(n-2)}(t) \\ y_1^{(n)}(t) & y_2^{(n)}(t) & \ldots & y_k^{(n)}(t) & \ldots & y_n^{(n)}(t) \end{vmatrix} =$$

$$= -\frac{w'[y_1(t), \ldots, y_n(t)]}{w[y_1(t), \ldots, y_n(t)]}, \quad \ldots, \quad a_1(t) =$$

$$= \frac{(-1)^n}{w[y_1(t), \ldots, y_n(t)]}\begin{vmatrix} y_1'(t) & y_2'(t) & \ldots & y_k'(t) & \ldots & y_n'(t) \\ y_1''(t) & y_2''(t) & \ldots & y_k''(t) & \ldots & y_n''(t) \\ \ldots & & & & & \\ y_1^{(n-1)}(t) & y_2^{(n-1)}(t) & \ldots & y_k^{(n-1)}(t) & \ldots & y_n^{(n-1)}(t) \\ y_1^{(n)}(t) & y_2^{(n)}(t) & \ldots & y_k^{(n)}(t) & \ldots & y_n^{(n)}(t) \end{vmatrix} =$$

$$= (-1)^n \frac{w[y_1'(t), \ldots, y_n'(t)]}{w[y_1(t), \ldots, y_n(t)]}.$$

Because of (1) it is necessary to write throughout our discussion $-q(t)\,u(t)$ or $-q(t)\,v(t)$ instead of $u''(t)$ or $v''(t)$ respectively.

Hence, every nontrivial solution of the iterated differential equation of the n-*th* order (n) is of the form

$$y(t) = C_1 u^{n-1}(t) + C_2 u^{n-2}(t)\,v(t) + \ldots + C_i u^{n-i}(t)\,v^{i-1}(t) + \ldots$$

$$\dots + C_{n-1}u(t)v^{n-2}(t) + C_n v^{n-1}(t) = \sum_{i=1}^{n} C_i u^{n-i}(t) v^{i-1}(t), \tag{2}$$

where $C_i \in \mathbf{R}$, $i = 1, 2, \dots, n (n \in N, n > 1)$ and $\sum_{i=1}^{n} C_i^2 > 0$.

Since the differential equation (n) is of the n-*th* order, any arbitrary zero of its nontrivial oscillatory solution $y(t)$ is of multiplicity $v = n - 1$ at most. In what follows we will understand under a solution both of the differential equation (1) and the differential equation (n) nontrivial solution only.

Oscillatory solutions of the differential equation (n)

For the oscillatority of the solution $y(t)$ of the differential equation (n) of an arbitrary order $n \in \mathbf{N}$, $n > 1$, among whose zeros $t^* \in \mathbf{I}$ are such that $y(t^*) = u(t^*)$ or $v(t^*)$, the sufficient condition is that it should be either of the form

$$^1y(t) = C_1 u^{n-1}(t) + C_2 u^{n-2}(t) v(t) + \dots + C_{n-1} u(t) v^{n-2}(t) = \\ = \sum_{i=1}^{n-1} C_i u^{n-i}(t) v^{i-1}(t),$$

$\sum_{i=1}^{n-1} C_i^2 > 0, n > 1,$

or

$$^2y(t) = C_2 u^{n-2}(t) v(t) + C_3 u^{n-3}(t) v^2(t) + \dots + C_{n-1} u(t) v^{n-2}(t) + \\ + C_n v^{n-1}(t) = \sum_{i=2}^{n} C_i u^{n-i}(t) v^{i-1}(t),$$

$\sum_{i=2}^{n} C_i^2 > 0, n > 1.$

Thereby between the zeros of any oscillatory solution of the form $^1y(t)$ or $^2y(t)$ of the differential equation (n) there always belong the zeros of the function $u(t)$ or $v(t)$. Because of the symmetry of the both functions $u(t)$, $v(t)$ occuring in the general solution (2) of the differential equation (n) we shall restrict our study to zeros of oscillatory solutions of (n) being of the form

$$y(t) = \sum_{i=1}^{n-1} C_i u^{n-i}(t) v^{i-1}(t),$$

$\sum_{i=1}^{n=1} C_i^2 > 0, n > 1,$

i.e. to such solutions among whose zeros there always belong all zeros of the function $u(t)$.

Remark: Let the function $f(t) \in \mathbf{C}_{\mathbf{I}}^{(n)}$, $\mathbf{I} = (-\infty, +\infty)$, $n \in \mathbf{N}$; if $f(t_0) = 0$ holds for the point $t_0 \in \mathbf{I}$, and t_0 is a simple zero of the function $f(t)$, then it holds also $f^n(t_0) = 0$ for every $n \in \mathbf{N}$, and the point t_0 is an n-tuple zero of the function $f^n(t)$, i.e. it holds

$$f'(t_0) = f''(t_0) = \dots = f^{(n-1)}(t_0) = 0, \qquad f^{(n)}(t_0) \neq 0.$$

In particular it holds: If the solution $y(t)$ of (n) is expressible in the form $y(t) = u^k(t)\, y^*(t)$, $1 \leqq k \leqq n - 1$; $k, n \in \mathbf{N}$, $n > 1$, where $u(t)$ is a nontrivial oscillatory solution of (1) having all zeros simple, and all functions $u(t)$, $y^*(t)$ are k-fold continuous differentiable in $\mathbf{I} = (-\infty, +\infty)$ and if it holds that $u(t_0) = 0$, $y^*(t_0) \neq 0$ at a point $t_0 \in \mathbf{I}$, then t_0 is a k-tuple zero of the oscillatory solution $y(t)$ of (n).

Lemma 1. Let $t_0 \in \mathbf{I} = (-\infty, +\infty)$ be an arbitrary firmly chosen point. Then any oscillatory solution $y(t)$ of the differential equation (n) vanishing at t_0 has the form

$$y(t) = \sum_{i=1}^{n-1} C_i u^{n-i}(t)\, v^{i-1}(t), \qquad C_{n-1} \neq 0,$$

exactly if t_0 is a simple zero of the solution $y(t)$;

$$y(t) = \sum_{i=1}^{n-2} C_i u^{n-i}(t)\, v^{i-1}(t), \qquad C_{n-2} \neq 0,$$

exactly if t_0 is a double zero of the solution $y(t)$;

$$\vdots$$

$$y(t) = C_1 u^{n-1}(t), \qquad C_1 \neq 0$$

exactly if t_0 is an $(n - 1)$-tuple zero of the solution $y(t)$, i.e. generally: for $\forall\, [k \in N$ $1 \leqq k \leqq n - 1;\ n \in N,\ n > 1]$

$$y(t) = \sum_{i=1}^{n-k} C_i u^{n-i}(t)\, v^{i-1}(t), \qquad C_{n-k} \neq 0,$$

exactly if t_0 is a k-tuple zero of the solution $y(t)$, where $[u(t), v(t)]$ is such a basis of the differential equation (1) that $u(t_0) = 0$.

Proof: According to what was said in the introduction, every solution $y(t)$ of (n) has the form (2), where $C_i \in \mathbf{R}$, $i = 1, 2, \dots, n$; $n \in \mathbf{N}$, $\sum_{n=1}^{n} C_i^2 > 0$ and $[u(t), v(t)]$ denotes a basis of an oscillatory differential equation (1). Let $t_0 \in \mathbf{I} = (-\infty, +\infty)$ be an arbitrary zero of the solution $y(t)$ of (n) and $[u(t), v(t)]$ be a basis of an oscillatory differential equation (1) such that

$$u(t_0) = v'(t_0) = 0, \tag{P}$$

[so that $u'(t_0) \neq 0$, $v(t_0) \neq 0$; hence the point t_0 is a simple zero of the function $u(t)$].

Then the system of all solutions $y(t)$ of (n) vanishing at t_0 are exactly of the form

$$y(t) = \sum_{i=1}^{n-1} C_i u^{n-i}(t)\, v^{i-1}(t),$$

where $C_i \in \mathbf{R}$, $i = 1, 2, \ldots, n-1$ ($n \in \mathbf{N}$, $n > 1$), $\sum_{i=1}^{n-1} C_i^2 > 0$, are arbitrary constants.

1. Let $C_{n-1} \neq 0$, so that

$$\begin{aligned} y(t) &= C_1 u^{n-1}(t) + C_2 u^{n-2}(t)\, v(t) + C_3 u^{n-3}(t)\, v^2(t) + \ldots + \\ &+ C_i u^{n-i}(t)\, v^{i-1}(t) + \ldots + C_{n-2} u^2(t)\, v^{n-3}(t) + \\ &+ C_{n-1} u(t)\, v^{n-2}(t) = u(t) \sum_{i=1}^{n-1} C_i u^{n-i-1}(t)\, v^{i-1}(t). \end{aligned}$$

Since

$$\begin{aligned} y'(t) &= C_1(n-1)\, u^{n-2}(t)\, u'(t) + C_2[(n-2)\, u^{n-3}(t)\, u'(t)\, v(t) + u^{n-2}(t)\, v'(t)] + \\ &+ C_3[(n-3)\, u^{n-4}(t)\, u'(t)\, v^2(t) + 2u^{n-3}(t)\, v(t)\, v'(t)] + \\ &+ \ldots + C_i[(n-i)\, u^{n-i-1}(t)\, u'(t)\, v^{i-1}(t) + u^{n-i}(t)\,(i-1)\, v^{i-2}(t)\, v'(t)] + \\ &\ldots + C_{n-2}[2u(t)\, u'(t)\, v^{n-3}(t) + u^2(t)\,(n-3)\, v^{n-4}(t)\, v'(t)] + \\ &+ C_{n-1}[u'(t)\, v^{n-1}(t) + u(t)\,(n-2)\, v^{n-3}(t)\, v'(t)] = \\ &= u(t)\{C_1(n-1) u^{n-3}(t) u'(t) + C_2[(n-2) u^{n-4}(t) u'(t) v(t) + u^{n-3}(t) v'(t)] + \\ &+ \ldots + C_i[(n-i)\, u^{n-i-2}(t)\, u'(t)\, v^{i-1}(t) + u^{n-i-1}(t)\,(i-1)\, v^{i-2}(t)\, v'(t)] + \\ &+ \ldots + C_{n-2}[2u'(t)\, v^{n-3}(t) + u(t)\,(n-3)\, v^{n-4}(t)\, v'(t)] + \\ &+ C_{n-1}(n-2)\, v^{n-3}(t)\, v'(t)\} + C_{n-1} u'(t)\, v^{n-2}(t) \end{aligned}$$

and by assumption (P) $y(t_0) = 0$ holds, whereby $y'(t_0) = C_{n-1} u'(t_0)\, v^{n-2}(t_0) \neq 0$, the point t_0 is a simple zero of the solution $y(t)$ of (n).

2. Let $C_{n-1} = 0$, $C_{n-2} \neq 0$, so that

$$\begin{aligned} y(t) &= C_1 u^{n-1}(t) + C_2 u^{n-2}(t)\, v(t) + \ldots + C_i u^{n-i}(t)\, v^{i-1}(t) + \ldots \\ &\ldots + C_{n-3} u^3(t)\, v^{n-4}(t) + C_{n-2} u^2(t)\, v^{n-3}(t) = \\ &= u^2(t) \sum_{i=1}^{n-2} C_i u^{n-i-2}(t)\, v^{i-1}(t). \end{aligned}$$

Since

$$\begin{aligned} y'(t) &= C_1(n-1)\, u^{n-2}(t)\, u'(t) + C_2[(n-2)\, u^{n-3}(t)\, u'(t)\, v(t) + u^{n-2}(t)\, v'(t)] + \\ &+ \ldots + C_i[(n-i)\, u^{n-i-1}(t)\, u'(t)\, v^{i-1}(t) + u_n{}^{-i}(t)\,(i-1)\, v^{i-2}(t)\, v'(t)] + \\ &+ \ldots + C_{n-3}[3u^2(t)\, u'(t)\, v^{n-4}(t) + u^3(t)\,(n-4)\, v^{n-5}(t)\, v'(t)] + \\ &+ C_{n-2}[2u(t)\, u'(t)\, v^{n-3}(t) + u^2(t)\,(n-3)\, v^{n-4}(t)\, v'(t)] = \\ &= u(t)\{C_1(n-1) u^{n-3}(t) u'(t) + C_2[(n-2) u^{n-4}(t) u'(t) v(t) + u^{n-3}(t) v'(t)] + \end{aligned}$$

$$+ \dots + C_i[(n - i) u^{n-i-2}(t) u'(t) v^{i-1}(t) + u^{n-i-1}(t) (i - 1) v^{i-2}(t) v'(t)] +$$
$$+ \dots + C_{n-3}[3u(t) u'(t) v^{n-4}(t) + u^2(t) (n - 4) v^{n-5}(t) v'(t)] +$$
$$+ C_{n-2}[2u'(t) v^{n-3}(t) + u(t) (n - 3) v^{n-4}(t) v'(t)]\},$$

$$y''(t) = C_1(n - 1) [(n - 2) u^{n-3}(t) u'^2(t) + u^{n-2}(t) u''(t)] +$$
$$+ C_2\{(n - 2) [(n - 3) u^{n-4}(t) u'^2(t) v(t) + u^{n-3}(t) u''(t) v(t) +$$
$$+ u^{n-3}(t) u'(t) v'(t)] + (n - 2) u^{n-3}(t) u'(t) v'(t) + u^{n-2}(t) v''(t)\} +$$
$$+ \dots + C_i\{(n - i) [(n - i - 1) u^{n-i-2}(t) u'^2(t) v^{i-1}(t) + u^{n-i-1}(t) \times$$
$$\times u''(t) v^{i-1}(t) + u^{n-i-1}(t) u'(t) (i - 1) v^{i-2}(t) v'(t)] +$$
$$+ (i - 1) [(n - i) u^{n-i-1}(t) u'(t) v^{i-2}(t) v'(t) + u^{n-i}(t) (i - 2) v^{i-3}(t) \times$$
$$\times v'^2(t) + u^{n-i}(t) v^{i-2}(t) v''(t)]\} + \dots + C_{n-2}\{2[u'^2(t) v^{n-3}(t) +$$
$$+ u(t) u''(t) v^{n-3}(t) + u(t) u'(t) (n - 3) v^{n-4}(t) v'(t)] +$$
$$+ (n - 3) [2u(t) u'(t) v^{n-4}(t) v'(t) + u^2(t) (n - 4) v^{n-5}(t) v'^2(t) +$$
$$+ u^2(t) v^{n-4}(t) v''(t)]\} =$$
$$= u(t) \{C_1(n - 1) [(n - 2) u^{n-4}(t) u'^2(t) + u^{n-3}(t) u''(t)] +$$
$$+ C_2\{(n - 2) [(n - 3) u^{n-5}(t) u'^2(t) v(t) + u^{n-4}(t) u''(t) v(t) +$$
$$+ u^{n-4}(t) u'(t) v'(t)] + (n - 2) u^{n-4}(t) u'(t) v'(t) + u^{n-3}(t) v''(t)\} +$$
$$+ \dots + C_i\{(n - i) [(n - i - 1) u^{n-i-3}(t) u'^2(t) v^{i-1}(t) +$$
$$+ u^{n-i-2}(t) u''(t) v^{i-1}(t) + u^{n-i-2}(t) u'(t) (i - 1) v^{i-2}(t) v'(t)] +$$
$$+ (i - 1) [(n - i) u^{n-i-2}(t) u'(t) v^{i-2}(t) v'(t) + u^{n-i-1}(t) (i - 2) v^{i-3}(t) \times$$
$$\times v'^2(t) + u^{n-i-1}(t) v^{i-2}(t) v''(t)]\} + \dots +$$
$$+ C_{n-2}\{2[u''(t) v^{n-3}(t) + u'(t) (n - 3) v^{n-4}(t) v'(t)] +$$
$$+ (n - 3) [2u'(t) v^{n-4}(t) v'(t) + u(t) (n - 4) v^{n-5}(t) v'^2(t) +$$
$$+ u(t) v^{n-4}(t) v''(t)]\}\} + 2C_{n-2} u'^2(t) v^{n-3}(t),$$

so that by assumption (P) $y(t_0) = y'(t_0) = 0$ holds, while $y''(t_0) = = 2C_{n-2} u'^2(t_0) v^{n-3}(t_0) \neq 0$, the point t_0 is a double zero of the solution $y(t)$ of (n).

⋮

(n − 1) Let $C_{n-1} = C_{n-2} = \dots = C_3 = C_2 = 0$, $C_1 \neq 0$, so that $y(t) = = C_1 u^{n-1}(t)$.

Since

$$y'(t) = C_1(n - 1) u^{n-2}(t) u'(t),$$
$$y''(t) = C_1(n - 1) u^{n-3}(t) [(n - 2) u'^2(t) + u(t) u''(t)],$$
$$y'''t) = C_1(n - 1) u^{n-4}(t) \{(n - 2) (n - 3) u'^3(t) + 3(n - 2) u(t) u'(t) \times$$
$$\times u''(t) + u^2(t) u'''(t)\},$$

$$y^{IV}(t) = C_1(n-1)\, u^{n-5}(t)\,\{(n-2)(n-3)(n-4)\, u'^4(t) + $$
$$+6(n-2)(n-3)\, u(t)\, u'^2(t)\, u''(t) + 4(n-2)\, u^2(t)\, u'(t)\, u'''(t) +$$
$$+ 3(n-2)\, u^2(t)\, u''^2(t) + u^3(t)\, u^{IV}(t)\}$$

$\vdots$

$$y^{(n-2)}(t) = C_1 u(t)\,\{(n-1)(n-2)(n-3)\dots 2u'^{n-2}(t) + (n-1)(n-2)\times$$
$$\times (n-3)\dots 3[n-3) + (n-2) + \dots + 3]\, u(t)\, u'^{n-4}(t)\, u''(t) +$$
$$+ \dots u^{n-2}(t)\, u^{(n-2)}(t)\},$$

$$y^{(n-1)}(t) = C_1\{(n-1)!\, u'^{n-1}(t) + u(t)\,\{(n-1)!\,[(n-2) + (n-3) +$$
$$+ \dots + 2]\, u'^{n-3}(t)\, u''(t) + \dots + u^{n-2}(t)\, u^{(n-1)}(t)\}\},$$

so that by assumption (P) $y(t_0) = y'(t_0) = \dots = y^{(n-2)}(t_0) = 0$, while $y^{(n-1)}(t_0) = C_1(n-1)!\, u'^{n-1}(t_0) \neq 0$, the point t_0 is an $(n-1)$-tuple zero of the solution $y(t)$ of (n).

The necessity of the assumption $C_{n-k} \neq 0$ for the oscillatory solution $y(t)$ of (n) being of the form

$$y(t) = \sum_{i=1}^{n-k} C_i u^{n-i}(t)\, v^{i-1}(t),$$

$\sum_{i=1}^{n-k} C_i^2 > 0$, $1 \leqq k \leqq n-1$, $n \in \mathbf{N}$, $n > 1$, to be vanishing with the function $u^k(t)$ at a k-tuple zero $t_0 \in \mathbf{I}$ is obvious [for, according to assumption that the point $t_0 \in \mathbf{I}$ is a k-tuple zero of $y(t)$ and $C_{n-k} = 0$, by the assumption

$$\sum_{i=1}^{n-k} C_i^2 > 0,$$

and with respect to the above part of the proof, we should be led to a contradiction to the fact that the multiplicity of $t_0 \in \mathbf{I}$ of $y(t)$ of (n) is k].

Corollary of Lemma 1.: Any oscillatory solution $y(t) = \sum_{i=1}^{k} C_i u^{n-i}(t)\, v^{i-1}(t)$ of the iterated differential equation of the n-*th* order (n), $n \in \mathbf{N}$, $n > 1$, possessing a zero of multiplicity $v = n - k$, $k = 1, 2, \dots, n-1$, at $t_0 \in \mathbf{I}$, where $u(t_0) = 0$, may be just written in the form

$$y(t) = u^{n-k}(t)\,[C_1 u^{k-1}(t) + C_2 u^{k-2}(t)\, v(t) + \dots + C_{k-1} u(t)\, v^{k-2}(t) + C_k v^{k-1}(t)] =$$
$$= u^{n-k}(t) \sum_{i=1}^{k} C_i u^{k-i}(t)\, v^{i-1}(t) = u^{n-k}(t)\, y^*(t),$$

where $C_i \in \mathbf{R}$, $i = 1, 2, \dots, k$, $C_k \neq 0$, i.e. for $k > 1$ in the form of a product of the $(n-k)^{th}$ power of the function $u(t)$ with the solution $y^*(t)$ of the iterated differential equation of the k^{th} order generating the k-parametric system of functions in the form

$$y^*(t) = C_1 u^{k-1}(t) + C_2 u^{k-2}(t)\, v(t) + \ldots + C_{k-1} u(t)\, v^{k-2}(t) + + C_k v^{k-1}(t) = \sum_{i=1}^{k} C_i u^{k-i}(t)\, v^{i-1}(t),$$

wherein $C_k \neq 0$ [so that $y^*(t_0) \neq 0$].

Conjugate points

Definition 1.1: Let $t_0 \in \mathbf{I} = (-\infty, +\infty)$ be an arbitrary point and $y(t)$ be an arbitrary solution of the differential equation (n) vanishing at it [we will write ${}^{\nu}t_0$, where $\nu = 1, 2, \ldots, n - 1$; $n \in \mathbf{N}$, $n > 1$, denotes the multiplicity of the point t_0]. Then the first zero of the solution $y(t)$ lying to the right of ${}^{\nu}t_0$ will be called the first conjugate point on the right to the point ${}^{\nu}t_0$ [we indicate it by writing ${}^{\mu}t_1$, where $\mu \in \{1, 2, \ldots, n - 1\}$ denotes the multiplicity].

Since, by the assumption of oscillatority of the differential equation (1), every (nontrivial) solution of the form

$$y(t) = \sum_{i=1}^{n-1} C_i u^{n-i}(t)\, v^{i-1}(t),$$

$\sum_{i=1}^{n-1} C_i^2 > 0$, $n > 1$, of the differential equation (n) is oscillatory (in the sense of [2]), we see that the first conjugate point ${}^{\mu}t_1$ on the right to the point ${}^{\nu}t_0$ with an appropriate multiplicity $\mu \in \{1, 2, \ldots, n - 1\}$ always exists to an arbitrary point ${}^{\nu}t_0 \in \mathbf{I} = (-\infty, +\infty)$, $\nu = 1, 2, \ldots, n - 1$; $n \in \mathbf{N}$, $n > 1$, at which the solution $y(t)$ ν-times vanishes.

Theorem 1.1. Let ${}^{\mu}t_1$ denote the first conjugate point from the right to ${}^{\nu}t_0$, where $\nu, \mu \in \{1, 2, \ldots, n - 1\}$, $n \in \mathbf{N}$, $n > 1$. If:

1. $\nu = n - 1$ then $\mu = n - 1$,
2. $\nu = n - 2$ then $\mu = 1$,
3. $\nu = n - 3$ then either $\mu = 1$ or $\mu = 2$ or $\mu = n - 3$,

etc.

Generally:

if $\nu = n - k$, where $1 < k \leqq n - 1$ then either $\mu = 1$ or $\mu = 2$ or … or $\mu = k - 1$ or $\mu = n - k$.

Proof: Let $t_0 \in \mathbf{I} = (-\infty, +\infty)$ be an arbitrary firmly chosen point; we chose a basis $[u(t), v(t)]$ of an oscillatory differential equation (1) such that both functions $u(t)$, $v(t)$ and their first derivatives $u'(t)$, $v'(t)$ satisfy the condition (P) at the point t_0. Let $y(t)$ be such a solution of the differential equation (n) that the point t_0, at which this solution together with the function $u(t)$ vanishes, is its ν-tuple zero [we can write ${}^{\nu}t_0$, where $\nu = 1, 2, \dots, n - 1$].

1. Let $\nu = n - 1$; then, by *Corollary of Lemma* 1., every oscillatory solution of (n) vanishing together with its function $u^{n-1}(t)$ at the $(n - 1)$-tuple point ${}^{n-1}t_0$, is exactly of the form

$$y(t) = C_1 u^{n-1}(t),$$

where $C_1 \in \mathbf{R} - \{0\}$ is an arbitrary constant.

If we denote by T_1 the neighbouring zero of the function $u(t)$ lying to the right behind the point t_0 such that $T_1 > t_0$, then

$$y(t_0) = u(t_0) = 0, \qquad y(T_1) = u(T_1) = 0,$$

and it holds for all $t \in (t_0, T_1)$ both $u(t) \neq 0$ [because of the continuity of the function $u(t)$, where for all $t \in (t_0, T_1)$ holds that either $u(t) > 0$ or $u(t) < 0$], and $y(t) \neq 0$; so that no zero of the solution $y(t)$ lies on (t_0, T_1) for all zeros of $y(t)$ coincide with all zeros of the function $u(t)$ and with respect to the form of $y(t)$ being an arbitrary non-zero multiple ($C_1 \neq 0$) of the function $u^{n-1}(t)$, they are $(n - 1)$-tuple zeros. Thus it holds

$${}^{n-1}t_1 = T_1$$

for the first conjugate point from the right to the point ${}^{n-1}t_0$, at which the solution $y(t)$ of the differential equation (n) vanishes.

2. Let $\nu = n - 2$; then, by the *Corollary of Lemma* 1., every oscillatory solution of the differential equation (n) vanishing together with the function $u^{n-2}(t)$ at the $(n - 2)$-tuple point ${}^{n-2}t_0$, is exactly of the form

$$y(t) = u^{n-2}(t)\,[C_1 u(t) + C_2 v(t)],$$

where $C_i \in \mathbf{R}$, $i = 1, 2$, $C_2 \neq 0$, are arbitrary constants. If we denote by T_1 the neighbouring zero of the function $u(t)$ lying to the right behind the point t_0, so that $T_1 > t_0$, then again

$$y(t_0) = u(t_0) = 0, \qquad y(T_1) = u(T_1) = 0,$$

and $u(t) \neq 0$ holds for all $t \in (t_0, T_1)$ [i.e. either $u(t) > 0$ or $u(t) < 0$], while on (t_0, T_1) there always lies exactly one and namely simple zero t' of each function from the double-parametric system

$$y^*(t, C_1, C_2) = C_1 u(t) + C_2 v(t),$$

being always uniquely determined by the choice of constants $C_i \in \mathbf{R}$, $i = 1, 2$, whereby $C_2 \neq 0$, for every such function $y^*(t)$ forms together with the function $u(t)$ a pair of linearly independent solutions (i.e. a basis) of the differential equation (1), whose all (simple) zeros by Sturm separation theorem mutually separate on the interval $\mathbf{I} = (-\infty, +\infty)$. Specially, if $C_1 = 0$, then the solution $y(t) =$

$= C_2u^{n-2}(t)\, v(t)$, $C_2 \neq 0$, possesses exactly one (simple) zero t' on (t_0, T_1), being the zero of the function $v(t)$, i.e.

$$y(t') = v(t') = 0.$$

Hence

$${}^1t_1 = t'$$

for the first conjugate point from the right to the point ${}^{n-2}t_0$ at which the solution $y(t)$ of (n) together with the function $u^{n-2}(t)$ are vanishing.

3. Let $v = n - 3$; then, by *Corollary of Lemma* 1., every oscillatory solution $y(t)$ of (n) vanishing together with the function $u^{n-3}(t)$ at the $(n - 3)$-tuple point ${}^{n-3}t_0$ is exactly of the form

$$y(t) = u^{n-3}(t)\,[C_1u^2(t) + C_2u(t)\,v(t) + C_3v^2(t)],$$

where $C_i \in \mathbf{R}$, $i = 1, 2, 3$, $C_3 \neq 0$, are arbitrary constants. If we denote by T_1 the neighbouring zero of the function $u(t)$ lying on the right of t_0, so that $T_1 > t_0$, then again

$$y(t_0) = u(t_0) = 0, \qquad y(T_1) = u(T_1) = 0,$$

whereby $u(t) \neq 0$ for all $t \in (t_0, T_1)$ in consequence of the continuity of the function $u(t)$. Wheather or not some zeros of the solution $y(t)$ of (n) lie on the interval (t_0, T_1) decides the existence or nonexistence of zeros of the three-parametric function system of the form

$$y^*(t, C_1, C_2, C_3) = C_1u^2(t) + C_2u(t)\,v(t) + C_3v^2(t), \tag{3}$$

where $C_i \in \mathbf{R}$, $i = 1, 2, 3$, $C_3 \neq 0$, are arbitrary constants.

For these constants there may occur exactly three different possibilities:

either $C_2^2 - 4C_1C_3 > 0$ or $C_2^2 - 4C_1C_3 = 0$ or $C_2^2 - 4C_1C_3 < 0$, which – by the trichotomy law of real numbers – mutually exclude.

a) If $C_2^2 - 4C_1C_3 > 0$, then there exist four real constants $D_j \in \mathbf{R}$, $j = 1, \ldots, 4$, such that

$$D_1D_4 - D_2D_3 \neq 0, \qquad D_2D_4 \neq 0$$

(consequently it must simultaneously hold $D_1^2 + D_2^2 > 0$, $D_1^2 + D_3^2 > 0$, $D_2^2 + D_4^2 > 0$, $D_3^2 + D_4^2 > 0$), whereby (3) may be written in the form

$$y^*(t, D_1, \ldots, D_4) = [D_1u(t) + D_2v(t)]\,[D_3u(t) + D_4v(t)],$$

so that – with respect to (3) – there hold the following relations among the previous constants $C_i \in \mathbf{R}$ $(i = 1, 2, 3)$ and those newly introduced $D_j \in \mathbf{R}$ $(j = 1, \ldots, 4)$:

$$C_1 = D_1D_3, \quad C_2 = D_1D_4 + D_2D_3, \quad C_3 = D_2D_4.$$

This yields that indeed

$$C_2^2 - 4C_1C_3 = (D_1D_4 + D_2D_3)^2 - 4D_1D_3D_2D_4 =$$
$$= (D_1D_4 - D_2D_3)^2 > 0$$

exactly if $D_1D_4 - D_2D_3 \neq 0$ and besides—in consequence of the assumption that $C_3 \neq 0$—also $D_2D_4 \neq 0$. If we denote

$$y_1^*(t, D_1, D_2) = D_1u(t) + D_2v(t),$$
$$y_2^*(t, D_3, D_4) = D_3u(t) + D_4v(t),$$

then the above established assumptions on constants $D_j \in \mathbf{R}$ (j = 1, ..., 4) mean both the double-parametric function systems $y_1^*(t, D_1, D_2)$, $y_2^*(t, D_3, D_4)$ to be on $\mathbf{I} = (-\infty, +\infty)$ linearly independent not only of each other but each of them with the function $u(t)$, too. Since the functions $y_1^*(t)$, $y_2^*(t)$ and $u(t)$ denote for any (admissible) choice of constants $D_j \in \mathbf{R}$ (j = 1, ..., 4) a triple of always two and two linearly independent solutions of the oscillatory differential equation (1), then every of these solutions possesses simple, mutually separating zeros on $\mathbf{I} = (-\infty, +\infty)$.

Thus, with respect to the solution $y(t)$ of the differential equation (n) [up to an arbitrary non-zero multiplicative constant $C \in \mathbf{R} - \{0\}$] of the form

$$y(t) = u^{n-3}(t)\, y_1^*(t)\, y_2^*(t),$$

all simple zeros of both functions $y_1^*(t)$, $y_2^*(t)$ mutually separate on $\mathbf{I} = (-\infty, +\infty)$ with each other on one hand and together with the $(n - 3)$-tuple zeros of the function $u^{n-3}(t)$ on the other hand. Thereby always exactly one (simple) zero both of the function $y_1^*(t)$ and of the function $y_2^*(t)$ lies on the interval (t_0, T_1), i.e. if we denote these zeros of the functions $y_1^*(t)$, $y_2^*(t)$ by t^* and t^{**}, respectively, then we have either

$$t_0 < t^* < t^{**} < T_1$$

or

$$t_0 < t^{**} < t^* < T_1.$$

In this case the solution $y(t)$ of the differential equation (n) possesses all its $(n - 3)$-tuple zeros and simple zeros; between any two neighbouring $(n - 3)$-tuple zeros there lie exactly two simple zeros.

Hence it holds

$${}^1t_1 = t^* \qquad \text{or} \qquad {}^1t_1 = t^{**}$$

for the first conjugate point from the right to the point ${}^{n-3}t_0$, at which the solution $y(t)$ of the differential equation (n) vanishes together with the function $u^{n-3}(t)$.

b) If $C_2^2 - 4C_1C_3 = 0$, then there exist two real constants $D_j \in \mathbf{R}$, j = 1, 2, such that $D_2 \neq 0$ and in this case (3) may be written as

$$y^*(t, D_1, D_2) = \lambda[D_1u(t) + D_2v(t)]^2, \qquad \text{where } \lambda = \pm 1,$$

so that between the previous constants $C_i \in \mathbf{R}$ (i = 1, 2, 3) in (3) and those newly established constants $D_j \in \mathbf{R}$ (j = 1, 2), the following relations

$$C_1 = \lambda D_1^2,\ C_2 = 2\lambda D_1D_2,\ C_3 = \lambda D_2^2$$

hold.

From this follows that indeed

$$C_2^2 - 4C_1C_3 = (2\lambda D_1D_2)^2 - 4\lambda D_1^2\lambda D_2^2 = 0$$

holds and besides, in consequence of the assumption $C_3 \neq 0$, we have also $D_2 \neq 0$.

Denoting

$$y_1^*(t, D_1, D_2) = D_1u(t) + D_2v(t),$$

such that

$$y^*(t, D_1, D_2) = \lambda y_1^{*2}(t, D_1, D_2),$$

then the established assumption on constants $D_j \in \mathbf{R}$ (j = 1, 2) implies that the double-parametric system $y_1^*(t, D_1, D_2)$ is with the function $u(t)$ linearly independent on $\mathbf{I} = (-\infty, +\infty)$. Since for any (admissible) choice of constants $D_j \in \mathbf{R}$ (j = 1, 2) the functions $y_1^*(t)$ and $u(t)$ denote a couple of linearly independent solutions of the oscillatory differential equation (1), then every function $y_1^*(t)$ belonging to a double-parametric function system $y_1^*(t, D_1, D_2)$ possesses simple zeros mutually separating with all (simple) zeros of the function $u(t)$ on $\mathbf{I} = (-\infty, +\infty)$.

Thus, with respect to the solution $y(t)$ of the differential equation (n) [up to an arbitrary non-zero multiplicative constant $C \in \mathbf{R} - \{0\}$] of the form

$$y(t) = u^{n-3}(t)\, \lambda y_1^{*2}(t)$$

all double-zeros of the function $y_1^{*2}(t)$ are mutually separating on $\mathbf{I} = (-\infty, +\infty)$, with all $(n - 3)$-tuple zeros of the function $u^{n-3}(t)$. [Specially, if $C_1 = C_2 = 0$, $C_3 \neq 0$, is true in (3), then the solution of the differential equation (n) is of the form

$$y(t) = C_3u^{n-3}(t)\, v^2(t),$$

such that $(n - 3)$-tuple zeros of the function $u^{n-3}(t)$ alternate the double zeros of the function $v^2(t)$ on $\mathbf{I} = (-\infty, +\infty)$].

Then exactly one double-zero of the function $y^*(t, D_1, D_2)$ lies on the interval (t_0, T_1), i.e. if we denote it by t^*, then $t^* \in (t_0, T_1)$, hence

$$t_0 < t^* < T_1.$$

In this case the solution $y(t)$ of the differential equation (n) possesses all its zeros, both the $(n - 3)$-tuple zeros and the double zeros. Between each two neighbouring $(n - 3)$-tuple zeros there always lies exactly one double zero.

Hence

$$^2t_1 = t^*$$

holds for the first conjugate point from the right to the point $^{n-3}t_0$, at which the solution $y(t)$ of the differential equation (n) together with the function $u^{n-3}(t)$ vanishes.

c) If $C_2^2 - 4C_1C_3 < 0$, then there exist four complex constants $D_j \in \mathbf{K}$ (where $\mathbf{K}$ denotes a set of all complex numbers), j = 1, ..., 4, such that

$$D_1D_4 - D_2D_3 \neq 0, \qquad D_j \neq 0 \quad \text{for } j = 1, \ldots, 4,$$

whereby (3) may be written as

$$y^*(t, D_1, \ldots, D_4) = [D_1u(t) + D_2v(t)]\,[D_3u(t) + D_4v(t)],$$

so that with respect to (3) the relations

$$C_1 = D_1D_3, \; C_2 = D_1D_4 + D_2D_3, \; C_3 = D_2D_4$$

hold between the real constants $C_i \in \mathbf{R}$ (i = 1, 2, 3) and the complex constants $D_j \in \mathbf{K}$ (j = 1, ..., 4).

From this it follows that, indeed,

$$C_2^2 - 4C_1C_3 = (D_1D_4 + D_2D_3)^2 - 4D_1D_3D_2D_4 = \\ = (D_1D_4 - D_2D_3)^2 < 0$$

exactly if

$$D_1D_4 - D_2D_3 \neq 0$$

and moreover—in consequence of the assumption that $C_3 \neq 0$—there must

$$D_2D_4 \neq 0$$

hold.

Since, according to the inequality $C_2^2 - 4C_1C_3 < 0$ there must hold besides $C_3 \neq 0$ also $C_1 \neq 0$, which implies also

$$D_1D_3 \neq 0,$$

i.e. $D_j \in \mathbf{K} - \{0\}$, j = 1, ..., 4.

We show the existence of complex constants $D_j \in \mathbf{K} - \{0\}$, j = 1, ..., 4, of the above properties to correspond to the existence of the four real constants $E_j \in \mathbf{R}$ (j = 1, ..., 4) such that

$$E_1E_4 - E_2E_3 \neq 0, \qquad E_1^2 + E_3^2 > 0 \qquad \text{and} \qquad E_2^2 + E_4^2 > 0,$$

whereby (3) may also be written as

$$y^*(t, E_1, \ldots, E_4) = \lambda\{[E_1u(t) + E_2v(t)]^2 + [E_3u(t) + E_4v(t)]^2\},$$

where $\lambda = \pm 1$, whence against (3) we see that between the previous constants $C_i \in \mathbf{R}$ (i = 1, 2, 3) and the newly introduced (also real) constants $E_j \in \mathbf{R}$ (j = = 1, ..., 4) there must simultaneously hold

$$C_1 = \lambda(E_1^2 + E_3^2), \; C_2 = 2\lambda(E_1E_2 + E_3E_4), \; C_3 = \lambda(E_2^2 + E_4^2).$$

Hence, the following equalities

$$\begin{aligned} D_1D_3 &= \lambda(E_1^2 + E_3^2), \\ D_1D_4 + D_2D_3 &= 2\lambda(E_1E_2 + E_3E_4), \\ D_2D_4 &= \lambda(E_2^2 + E_4^2), \end{aligned}$$

must simultaneously hold between the complex constants $D_j \in \mathbf{K} - \{0\}, j = 1, \ldots, 4$, and the new real constants $E_j \in \mathbf{R}, j = 1, \ldots, 4$. From this we obtain relations

$$D_1 = \lambda(E_1 + E_3 i), \qquad D_2 = \lambda(E_2 + E_4 i),$$
$$D_3 = E_1 - E_3 i, \qquad D_4 = E_2 - E_4 i,$$

where $i \in \mathbf{K}$ is a complex (pure imaginary) unit. Then indeed

$$C_2^2 - 4C_1C_3 = [2\lambda(E_1E_2 + E_3E_4)]^2 - 4\lambda(E_1^2 + E_3^2)\,\lambda(E_2^2 + E_4^2) =$$
$$= 4\lambda^2(2E_1E_2E_3E_4 - E_2^2E_3^2 - E_1^2E_4^2) =$$
$$= -4(E_1E_4 - E_2E_3)^2 < 0,$$

exactly if $E_1E_4 - E_2E_3 \neq 0$ holds and with respect to $C_1C_3 \neq 0$ there holds moreover

$$E_1^2 + E_3^2 > 0 \qquad \text{and} \qquad E_2^2 + E_4^2 > 0.$$

If we denote

$$y_1^*(t, E_1, E_2) = E_1u(t) + E_2v(t),$$
$$y_2^*(t, E_3, E_4) = E_3u(t) + E_4v(t),$$

then the above assumptions on constants $E_j \in \mathbf{R}$ ($j = 1, \ldots, 4$) imply that both double-parametric systems of functions $y_1^*(t, E_1, E_2)$ and $y_2^*(t, E_3, E_4)$ are linearly independent on $\mathbf{I} = (-\infty, +\infty)$. Since the functions $y_1^*(t)$ and $y_2^*(t)$ form a pair of linearly independent solutions of the oscillatory differential equations (1) for every (admissible) choice of constants $E_j \in \mathbf{R}$ ($j = 1, \ldots, 4$), then every of these solutions has simple mutually separating zeros on $\mathbf{I} = (-\infty, +\infty)$.

From this it especially follows that both functions $y_1^*(t)$ and $y_2^*(t)$ obtained from the systems $y_1(t, E_1, E_2)$ and $y_2(t, E_3, E_4)$ by an arbitrary (admissible) choice of constants $E_j \in \mathbf{R}$, $j = 1, \ldots, 4$, possess no common zero on $\mathbf{I} = (-\infty, +\infty)$.

Since no zero exists on $\mathbf{I} = (-\infty, +\infty)$ at which both double-parametric systems of functions $y_1^*(t, E_1, E_2)$ and $y_2^*(t, E_3, E_4)$ would simultaneously vanish, then the sum of its squares, i.e. the four-parametric function system

$$y^*(t, E_1, \ldots, E_4) = \lambda\{[E_1u(t) + E_2v(t)]^2 + [E_3u(t) + E_4v(t)]^2\}$$

has no zeros on $\mathbf{I} = (-\infty, +\infty)$.

Thus, the system $y^*(t, E_1, \ldots, E_4)$ because of its continuity and because of $\lambda = \pm 1$ is either still positive ($\lambda = 1$) or still negative ($\lambda = -1$) on $\mathbf{I} = (-\infty, +\infty)$.

That is why the solution $y(t)$ of the differential equation (n), being [up to an arbitrary nonzero multiplicative constant $C \in \mathbf{R} - \{0\}$[of the form

$$y(t) = u^{n-3}(t)\,\lambda y^*(t, E_1, \ldots, E_4),$$

has but $(n - 3)$-tuple zeros on $\mathbf{I} = (-\infty, +\infty)$ presenting at the same time the zeros of the function $u^{n-3}(t)$.

[Specially: If in (3) $C_2 = 0$ and $\operatorname{sgn} C_1 = \operatorname{sgn} C_3 \neq 0$ (i.e. simultaneously either

$C_1 > 0$ and $C_3 > 0$ or $C_1 < 0$ and $C_3 < 0$) holds, then the solution $y(t)$ of the differential equation (n) (up to an arbitrary nonzero multiplicative constant $C \in \mathbf{R} - \{0\}$), is of the form

$$y(t) = u^{n-3}(t)\, y^*(t, C_1, C_3),$$

where $y^*(t, C_1, C_3) = C_1 u^2(t) + C_3 v^2(t)$ for all $C_i \in \mathbf{R}$, i = 1,3, fulfilling the above assumptions, denotes a double-parametric system of functions on $\mathbf{I} = (-\infty, +\infty)$, being either positive ($C_1 > 0$ and $C_3 > 0$) or negative ($C_1 < 0$ and $C_3 < 0$), consequently having no zeros here; the only zeros of the solution $y(t)$ of the differential equation (n) are thus the $(n - 3)$-tuple zeros of the function $u^{n-3}(t)$ only].

In this final case the oscillatory solution $y(t)$ of the differential equation (n) has no zero on the interval (t_0, T_1). Since such a solution $y(t)$ vanishes in the whole interval $\mathbf{I} = (-\infty, +\infty)$ exactly at the $(n - 3)$-tuple zeros of the function $u^{n-3}(t)$, it holds: the first conjugate point to the point ${}^{n-3}t_0$ from the right is exactly the neighbouring $(n - 3)$-tuple zero of the function $u^{n-3}(t)$ lying to the right of the point ${}^{n-3}t_0$, i.e.

$${}^{n-3}t_1 = T_1.$$

⋮

k) Let (generally) $v = n - k$, where $1 < k \leqq n - 1$, $n \in \mathbf{N}$, $n > 1$; then, by the *Corollary of Lemma* 1., every oscillatory solution $y(t)$ of the differential equation (n) vanishing together with the function $u^{n-k}(t)$ at the $(n - k)$-tuple zero ${}^{n-k}t_0 \in \mathbf{I} = (-\infty, +\infty)$, is exactly of the form

$$y(t) = u^{n-k}(t)\,[C_1 u^{k-1}(t) + C_2 u^{k-2}(t)\, v(t) + \dots + C_{k-1} u(t)\, v^{k-2}(t) + C_k v^{k-1}(t)],$$

where $C_i \in \mathbf{R}$, i = 1, 2, …, k, $C_k \neq 0$, are arbitrary constants (parameters).

Let us denote by T_1 a neighbouring point of the function $u(t)$ lying on the right behind the point t_0, so that $T_1 > t_0$; then

$$y(t_0) = u(t_0) = 0, \qquad y(T_1) = u(T_1) = 0,$$

whereby for all $t \in (t_0, T_1)$ is $u(t) \neq 0$ true.

Whether between both points t_0, T_1 some zeros of the solution $y(t)$ of the differential equation (n) are lying or not [i.e. whether for all $t \in (t_0, T_1)$ is $y(t) \neq 0$ true] decides the existence or nonexistence of zeros of the k-parametric system of functions having the form

$$y^*(t, C_1, \dots, C_k) = C_1 u^{k-1}(t) + C_2 u^{k-2}(t)\, v(t) + \dots + C_{k-1} u(t)\, v^{k-2}(t) + C_k v^{k-1}(t), \tag{4}$$

always uniquely determined by the choice of all k constants $C_i \in \mathbf{R}$, i = 1, 2, …, k, $C_k \neq 0$.

First we see that no zero of the arbitrary function $y^*(t)$ obtained from the system $y^*(t, C_1, \ldots, C_k)$ by an arbitrary choice of constants $C_i \in \mathbf{R}$, i = 1, 2, ..., k − 1, $C_k \neq 0$ – so far such a point exists – cannot be simultaneously a zero of the function $u(t)$ and reversely [which follows from the assumption that both functions $u(t)$, $v(t)$ form a basis of the differential equation (1), from the condition $C_k \neq 0$ – of *Lemma* 1. – and from the Sturm theorem on mutual separating of all zeros of any two oscillatory linearly independent solutions of the differential equation (1) or their arbitrary natural power up to including the degree $k - 1$, by which the basis (B) of all solutions $y(t)$ of the differential equation (n) is formed].

The k-parametric system of functions $y^*(t, C_1, \ldots, C_k)$ is a homogeneous polynomial of the $(k - 1)^{st}$ degree in the functions $u(t)$ and $v(t)$.

If we restrict the values of the argument t to an open interval (t_0, T_1) only, where $u(t) \neq 0$ [since for all $t \in (t_0, T_1)$ either $u(t) > 0$ or $u(t) < 0$] and because of the assumption $C_k \neq 0$, we can write

$$y^*(t, C_1, \ldots, C_k) =$$
$$= C_k u^{k-1}(t) \left\{ \left[\frac{v(t)}{u(t)} \right]^{k-1} + \frac{C_{k-1}}{C_k} \left[\frac{v(t)}{u(t)} \right]^{k-2} + \ldots + \frac{C_2}{C_k} \frac{v(t)}{u(t)} + \frac{C_1}{C_k} \right\}.$$

Denoting

$$w(t) = \frac{v(t)}{u(t)} \qquad \text{and} \qquad C_i' = \frac{C_{k-i}}{C_k}, \qquad i = 1, 2, \ldots, k - 1,$$

so that

$$y^*(t, C_1', \ldots, C_{k-1}', C_k) = \tag{5}$$
$$= C_k u^{k-1}(t) \, [w^{k-1}(t) + C_1' w^{k-2}(t) + C_2' w^{k-3}(t) + \ldots + C_{k-2}' w(t) + C_{k-1}'];$$

because both functions $u(t)$, $v(t)$ are linearly independent on $\mathbf{I} = (-\infty, +\infty)$ according to the assumption, the function $w(t)$ on the interval (t_0, T_1) doesnot equal to a constant function.

The question regarding the existence of zeros of the solution $y(t)$ of the differential equation (n) on the interval (t_0, T_1), i.e. the existence of such points $t^* \in (t_0, T_1)$ at which $u(t^*) \neq 0$ but $y(t^*) = y^*(t^*, C_1', \ldots, C_{k-1}', C_k) = 0$, reduces to the question of the existence of zeros of the functional polynomial of the $(k - 1)^{st}$ degree in the function $w(t)$ having the form

$$\sum_{j=0}^{k-1} C_j' w^{k-1-j}(t), \qquad \text{where} \qquad C_0' = 1, \tag{6}$$

i.e. practically to the question of solving algebraic equation of the $(k - 1)^{st}$ degree with constant real coefficients

$$w^{k-1}(t) + C_1' w^{k-2}(t) + C_2' w^{k-3}(t) + \ldots + C_{k-2}' w(t) + C_{k-1}' = 0. \tag{7}$$

For our considerations only real solutions of this problem are meaningful, i.e. only the existence (and multiplicity) of real roots $\lambda_j \in \mathbf{R}$, $j \in \{1, 2, \ldots, k-1\}$, of this equation is of interest for us.

According to the fundamental theorem of algebra there always exists exactly $k - 1$ (generally complex) constants $\lambda_j \in \mathbf{K}$, $j = 1, 2, \ldots, k-1$, such that by arbitrary firmly chosen coefficients $C'_j \in \mathbf{R}$ ($j = 1,2, \ldots, k-1$) for the decomposition of the polynomial (6) we have

$$w^{k-1}(t) + C'_1 w^{k-2}(t) + \ldots + C'_{k-2} w(t) + C'_{k-1} = \prod_{j=1}^{k-1} [w(t) - \lambda_j], \qquad \text{(w)}$$

whereby

$$\begin{aligned}
C'_1 &= -(\lambda_1 + \ldots + \lambda_{k-1}) = -\sum_{p=1}^{k-1} \lambda_p, \\
C'_2 &= \lambda_1\lambda_2 + \lambda_1\lambda_3 + \ldots + \lambda_1\lambda_{k-1} + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \ldots + \\
&\quad \ldots \\
&\quad + \lambda_{k-2}\lambda_{k-1} = \sum_{\substack{p,q=1 \\ (p<q)}}^{k-1} \lambda_p\lambda_q, \\
C'_3 &= -(\lambda_1\lambda_2\lambda_3 + \ldots + \lambda_2\lambda_3\lambda_4 + \ldots + \\
&\quad + \lambda_{k-3}\lambda_{k-2}\lambda_{k-1}) = -\sum_{\substack{p,q,r=1 \\ (p<q<r)}}^{k-1} \lambda_p\lambda_q\lambda_r, \\
&\vdots \\
C'_{k-1} &= (-1)^{k-1} \lambda_1\lambda_2 \ldots \lambda_{k-1} = (-1)^{k-1} \prod_{j=1}^{k-1} \lambda_j.
\end{aligned}$$

In solving equation (7) there may occur the following cases:

α) equation (7) possesses all roots λ_j, $j = 1, 2, \ldots, k-1$, real, simple, different from one another

β) equation (7) possesses all roots λ_j, $j \in \{1, 2, \ldots, k-1\}$ real, multiple, where denoting their multiplicities by $v_1, \ldots, v_m$ ($v_s \in \mathbf{N}$, $s = 1, \ldots, m$; $m \leqq k-1$), we have

$$v_1 + v_2 + \ldots + v_m = k - 1.$$

γ) Among the roots λ_j, $j = 1, 2, \ldots, k-1$, of equation (7) there occurs one simple complex (imaginary) root at least. Let us denote if by $\lambda_1 = a + ib$, where $a, b \in \mathbf{R}$, $b \neq 0$; i is a pure imaginary unit. Then, respecting the coefficients C'_j, $j = 1, 2, \ldots, k-1$, of the functional polynomial (w) being altogether real, there necessarily exists among the remaining $k - 2$ roots of equation (7) another imaginary root, being complex conjugate to the first root λ_1. If we denote it $\lambda_2 = \bar{\lambda}_1 = a - ib$, then the decomposition of the polynomial (w) has the form

$$[w(t) - (a + ib)] [w(t) - (a - ib)] \prod_{j=3}^{k-1} [w(t) - \lambda_j],$$

i.e.

$$\{[w(t) - a]^2 + b^2\} \prod_{j=3}^{k-1} [w(t) - \lambda_j].$$

Remark that in case when the polynomial $\sum_{j=3}^{k-1} [w(t) - \lambda_j]$ is of odd degree $s = k - 3$ (so that the number $k \in \mathbf{N}$, $k > 3$ is even), then it necessarily has at least one real root, while in case when its degree $s = k - 3$ is even (so that the number $k > 3$ is odd), this polynomial need not have any real root.

δ) Among the roots λ_j, $j = 1, 2, \dots, k - 1$, of equation (7) occurs at least one ν-multiple $\left(\nu \in \mathbf{N}, \nu \leqq \frac{k-1}{2}\right)$ imaginary root; let us denote it again $\lambda_1 = a + ib$, where $a, b \in \mathbf{R}$, $b \neq 0$, i is the pure imaginary unit. Then [respecting again that all coefficients C'_j, $j = 1, 2, \dots, k - 1$, of the functional polynomial (w) are real] there necessarily exists among the remaining $k - 1 - \nu$ roots of equation (7) another imaginary root being complex conjugate to the first ν-multiple root λ_1 and namely with the same multiplicity ν; if we denote it $\lambda_2 = \bar{\lambda}_1 = a - ib$, then the decomposition of the polynomial (w) has the form

$$[w(t) - (a + ib)]^\nu [w(t) - (a - ib)]^\nu \prod_{j=2\nu+1}^{k-1} [w(t) - \lambda_j],$$

i.e.

$$\{[w(t) - a]^2 + b^2\}^\nu \prod_{j=2\nu+1}^{k-1} [w(t) - \lambda_j].$$

A remark analogous to that of γ) would refer to the oddness or evenness of degree $s = k - 2(\nu + 1)$ of the polynomial $\prod_{j=2\nu+1}^{k-1} [w(t) - \lambda_j]$.

Let us remark, that the case α) or γ) may be included into the case β) or δ), when $\nu_1 = \nu_2 = \dots = \nu_{k-1} = 1$ (i.e. $m = k - 1$) or $\nu = 1$. But for completeness of the proof to Theorem 1.1 we must consider the cases α)$-$$\delta$) in detail and by themselves.

1. In case α) the existence of the $k - 1$ real numbers $\lambda_j \in \mathbf{R}$, $\lambda_i \neq \lambda_j$ for $i, j = 1, 2, \dots, k - 1$, $i \neq j$, such that there hold $k - 1$ equations

$$w(t) = \lambda_j, \qquad j = 1, 2, \dots, k - 1$$

denotes the existence of $k - 1$ mutually different points $t_1^*, t_2^*, \dots, t_{k-1}^*$ lying in the interval (t_0, T_1) such that

$$w(t_j^*) = \lambda_j, \qquad j = 1, 2, \dots, k - 1$$

which, with respect to the previous significance of the function $w(t)$ as the quotient $\frac{v(t)}{u(t)}$ of functions $v(t)$ and $u(t)$ on the interval (t_0, T_1) denotes the existence of $k - 1$

pairs of constants $c_{j1}, c_{j2} \in \mathbf{R}$, j = 1, 2, ..., k − 1, for which

$$c_{j1}c_{i2} - c_{j2}c_{i1} \neq 0, \ c_{j2} \neq 0, \qquad i \neq j,$$
$$i, j = 1, 2, \ldots, k - 1$$

holds such that the point t_j^* is a simple zero of any of the functions

$$y_j(t) = c_{j1}u(t) + c_{j2}v(t), \qquad j = 1, 2, \ldots, k - 1,$$

on the interval (t_0, T_1); whereby $\lambda_j = -\dfrac{c_{j1}}{c_{j2}}$, j = 1, 2, ..., k − 1.

Thus the system of functions $y^*(t, C_1, \ldots, C_k)$, cf. (4), occurring by the writing of the solution $y(t)$ of the differential equation (n) sub k), may be written as $2(k-1)$-parametric system having the form

$$y^*(t, c_{11}, c_{12}, \ldots, c_{k-1,1}, c_{k-1,2}) = \prod_{j=1}^{k-1} [c_{j1}u(t) + c_{j2}v(t)]$$

and the very solution $y(t)$ of (n) is [possibly up to an arbitrary nonzero multiplicative constant $C \in \mathbf{R} - \{0\}$] of the form

$$y(t) = u^{n-k}(t) \prod_{j=1}^{k-1} [c_{j1}u(t) + c_{j2}v(t)].$$

Here it holds: any of the functions $y_j(t) = c_{j1}u(t) + c_{j2}(t)$, j = 1, 2, ..., k − 1, by an arbitrary firm choice of constants $c_{j1}, c_{j2} \in \mathbf{R}$, $c_{j2} \neq 0$, always denotes a particular solution of the differential equation (1) and besides, any two of these solutions are with respect to the condition $c_{j1}c_{i2} - c_{j2}c_{i1} \neq 0$, i ≠ j, i, j = 1, 2, ..., k − 1, linearly independent to each other on $\mathbf{I} = (-\infty, +\infty)$. According to Sturm separation theorem, all (simple) zeros of any two linearly independent oscillatory solutions of (1) are separating each other, so that on $\mathbf{I} = (-\infty, +\infty)$

$$t_i^* \neq t_j^*, \qquad i \neq j,$$

i, j = 1, 2, ..., k − 1, holds.

Since any of the functions $y_j(t)$, j = 1, 2, ..., k − 1, with respect to the assumption $c_{j2} \neq 0$ is linearly independent also of the function $u(t)$ on $\mathbf{I} = (-\infty, +\infty)$, then all zeros of any such function mutually separate moreover with all zeros of the function $u(t)$.

Hence it is that even on any open interval (T_n, T_{n+1}), n = 0, ±1, ±2, ..., where T_n, T_{n+1} are two neighbouring zeros of the function $u(t)$, there always lie exactly k − 1 simple zeros of the system $y^*(t, C_1, \ldots, C_k)$ of the form (4) [for an arbitrary – admissible – choice of constants $C_j \in \mathbf{R}$, j = 1, 2, ..., k] and thus also of the solution $y(t)$ of (n), each of which always belongs to one of the functions $y_j(t)$, j = 1, 2, ..., k − 1, from the system (4) obtained by such a choice of constants $C_j \in \mathbf{R}$.

Thus for the first conjugate point from the right to the point ${}^{n-k}t_0 \in \mathbf{I} = (-\infty, +\infty)$ at which the solution $y(t)$ of the differential equation (n) together with the function $u^{n-k}(t)$ are vanishing, we have

$$ {}^1t_1 = t_j^*, $$

where t_j^* is the simple zero of a particular solution $y_j(t)$ of the differential equation (1), $j = 1, 2, \ldots, k - 1$, being linearly independent of the function $u(t)$ lying in (t_0, T_1), where T_1 denotes the neighbouring zero of the function $u(t)$ lying on the right of the point t_0. Naturally, we assume the point t_j^* under consideration to be the first in the series of all $k - 1$ zeros $t_1^*, t_2^*, \ldots, t_{k-1}^* \in (t_0, T_1)$ belonging always (one at a time) to any particular function $y_j^*(t) = c_{j1}u(t) + c_{j2}v(t)$ from the system $y^*(t, c_{11}, c_{12}, \ldots, c_{k-1,1}, c_{k-1,2})$ obtained by an arbitrary (admissible) choice of constants $c_{j1}, c_{j2} \in \mathbf{R}$ $(j = 1, 2, \ldots, k - 1)$.

2. In case β), to the existence m, $m \in \{1, 2, \ldots, k - 1\}$, real mutually different roots $\lambda_j \in \mathbf{R}$, $j = 1, 2, \ldots, m$, of equation (7) having multiplicities $\nu_1, \nu_2, \ldots, \nu_m$ $(\nu_s \in \mathbf{N}, s = 1, 2, \ldots, m)$ corresponds the existence m of mutually different points $t_1^*, t_2^*, \ldots, t_m^* \in (t_0, T_1)$ such that

$$ w(t_j^*) = \lambda_j, \qquad j = 1, 2, \ldots, m $$

which implies that there exist m pairs of real constants $c_{j1}, c_{j2} \in \mathbf{R}$, $c_{j1}c_{i2} - c_{j2}c_{i1} \neq 0$, $c_{j2} \neq 0$, $i \neq j$, $i, j = 1, 2, \ldots, m$, such that the point t_j^* is a simple zero of any from the functions $y_j(t) = c_{j1}u(t) + c_{j2}v(t)$, $j = 1, 2, \ldots, m$, and consequently ν_s-multiple, $s \in \{1, 2, \ldots, m\}$, zero of the function

$$ Y_j(t) = y_j^{\nu_s}(t) $$

on the interval (t_0, T_1); whereby $\lambda_j = -\dfrac{c_{j1}}{c_{j2}}$, $j = 1, 2, \ldots, m$.

Thus, system (4) of the functions $y^*(t, C_1, \ldots, C_k)$ occuring in the writing $y(t)$ of the differential equation (n) under k) may be written as a $2m$-parametric system having the form

$$ y^*(t, c_{11}, c_{12}, \ldots, c_{m1}, c_{m2}) = \prod_{j=1}^{m} [c_{j1}u(t) + c_{j2}v(t)]^{\nu_j}, $$

hence, the every solution $y(t)$ of the differential equation (n) is [possibly up to an arbitrary nonzero multiplicative constant $C \in \mathbf{R} - \{0\}$] of the form

$$ y(t) = u^{n-k}(t) \prod_{j=1}^{m} [c_{j1}u(t) + c_{j2}v(t)]^{\nu_j}. $$

Since each two from the total m functions $y_j(t) = c_{j1}u(t) + c_{j2}v(t)$, $j = 1, 2, \ldots, m$, are by an arbitrary (firm) choice of constants $c_{j1}, c_{j2} \in \mathbf{R}$ with respect to conditions $c_{j1}c_{i2} - c_{j2}c_{i1} \neq 0$, $c_{j2} \neq 0$, $i \neq j$, $i, j = 1, 2, \ldots, m$, linearly independent (particular) solutions of the differential equation (1) to each other, then all zeros

$$^{\nu_1}t^*, ^{\nu_2}t^*, \ldots, ^{\nu_m}t^* \in (t_0, T_1)$$

of the functions $Y_j(t)$ are mutually separating in (t_0, T_1).

Next, since every of the m functions $y_j(t)$ is with respect to the assumption $c_{j2} \neq 0$, $= 1, 2, \ldots, m$, a solution of the differential equation (1) linearly independent of the function $u(t)$, then besides, all zeros of each function $y_j(t)$ – and so even the function $Y_j(t)$ – are separating with all zeros of the function $u(t)$ and so even of the function $u^{n-k}(t)$.

Thus, even on an arbitrary open interval $(T_n, T_{n+1}) \subset (-\infty, +\infty)$, $n = 0, \pm 1, \pm 2, \ldots$, where T_n, T_{n+1} are two neighbouring zeros of the function $u(t)$, the system $y^*(t, C_1, \ldots, C_k)$ of (4) [for any – admissible – choice of constants $C_j \in \mathbf{R}$, $j = 1, 2, \ldots, k$] and thus also the solution $y(t)$ of (n) has always m mutually different zeros with multiplicities $\nu_1, \nu_2, \ldots, \nu_m$ every of which belongs always to one of the functions $Y_j(t)$ from system (4) obtained by such a choice of constants $C_j \in \mathbf{R}$.

Consequently, the first conjugate point from the right to the point $^{n-k}t_0 \in \mathbf{I} = (-\infty, +\infty)$ at which the solution $y(t)$ of the differential equation (n) together with the function $u^{n-k}(t)$ are vanishing, is exactly that ν_s-multiple point $^{\nu_s}t^*$, $s \in \{1, 2, \ldots, m\}$, from the set of all zeros

$$^{\nu_1}t^*, ^{\nu_2}t^*, \ldots, ^{\nu_m}t^* \in (t_0, T_1)$$

belonging always (one at a time) to any particular function $Y_j(t) = [c_{j1}u(t) + c_{j2}v(t)]^{\nu_j}$ from the system $y^*(t, c_{11}, c_{12}, \ldots, c_{m1}, c_{m2})$ [obtained by an arbitrary – admissible – choice of constants $c_{j1}, c_{j2} \in \mathbf{R}$, $j = 1, 2, \ldots, m$] lying the first from the left in the interval (t_0, T_1), i.e. before all other $m - 1$ zeros of the remaining multiplicities. Then we can write

$$^{\nu_s}t_1 = {}^{\nu_s}t^*.$$

In case $k - 1 = \nu m$, where $\nu \in \mathbf{N}$ and if $\nu_1 = \nu_2 = \ldots = \nu_m = \nu$, then system (4) of functions $y^*(t, C_1, \ldots, C_k)$ and thus also the solution $y(t)$ of the differential equation (n) has in (t_0, T_1) exactly m mutually different zeros $t_1^*, t_2^*, \ldots, t_m^*$ with the same multiplicity ν $\left(\text{for } \nu_1 + \nu_2 + \ldots + \nu_m = k - 1 \text{ is } \nu = \dfrac{k-1}{m}\right)$. Let for their arrangement on (t_0, T_1) holds:

$$t_1^* < t_2^* < \ldots < t_m^*;$$

then the first conjugate point from the right to the point $^{n-k}t_0$ is the point

$$^{\nu}t_1 = t_1^*.$$

Specially for $m = k - 1$, where $\nu = 1$, we get the case α).

Let us remark to the case when the equation (7) has only one $(k - 1)$-multiple root $\lambda \in \mathbf{R}$, i.e. when $m = 1$, corresponds the existence exactly of one and namely $(k - 1)$-tuple zero t^* of the system (4) of functions $y^*(t, C_1, \dots, C_k)$ and thus also of the solution $y(t)$ of the differential equation (n) on interval (t_0, T_1). Hence, the first conjugate point from the right to the point ${}^{n-k}t_0$ then is the point

$$^{k-1}t_1 = t^*.$$

3. In case γ), where among the roots λ_j, $j = 1, 2, \dots, k - 1$, of equation (7) is occurring at least one pair of simple imaginary complex conjugate roots (thus we must assume $k - 1 \geqq 2$, i.e. $k > 2$), then equation (7) may have $k - 1 - 2 = k - 3$ real roots at most (with the sum of their multiplicities $k - 3$ as well). Generally: if among the $k - 1$ roots λ_j of equation (7) there occur exactly p ($p \in \mathbf{N}$) simple imaginary complex conjugate pairs of roots (and therefore $k - 1 \geqq 2p$), let us denote them

$$\lambda_s = a_s + ib_s, \qquad \bar{\lambda}_s = a_s - ib_s,$$

where $a_s, b_s \in \mathbf{R}$, $b_s \neq 0$, $s = 1, 2, \dots, p$, then the sum of multiplicities of the real roots of equation (7) equals to $k - 2p - 1$.

Let us distinguish two possibilities:

γ_1) $k - 1$ is an odd number. Thus we may write $k = 2(q + 1)$, where $q \in \mathbf{N}$, $q \geqq p$. Then there exists at least one real root of equation (7), for the sum of the remaining multiplicities is $2q + 2 - 2p - 1 = 2(q - p) + 1$ [in case of $q = p$ is the remaining single real root of equation (7) simple]. Equation (7) has the form

$$\prod_{s=1}^{p} \{[w(t) - a_s]^2 + b_s^2\} \prod_{j=2p+1}^{2q+1} [w(t) - \lambda'_j] = 0.$$

For the existence of m, $m \in \{1, 2, \dots, 2(q - p) + 1\}$, real, mutually different roots $\lambda'_j \in \mathbf{R}$, $j = 1, 2, \dots, m$, of equation

$$\prod_{j=2p+1}^{2q+1} [w(t) - \lambda'_j] = 0$$

with the multiplicities $\nu_1, \nu_2, \dots, \nu_m$ ($\nu_j \in \mathbf{N}$, $j = 1, 2, \dots, m$) and to them corresponding m mutually different points $t_1^*, t_2^*, \dots, t_m^* \in (t_0, T_1)$ such that

$$w(t_j^*) = \lambda'_j, \qquad j = 1, 2, \dots, m$$

the considerations are analogous to β).

The first conjugate point from the right to the point ${}^{n-k}t_0 \in \mathbf{I} = (-\infty, +\infty)$ at which the solution $y(t)$ of the differential equation (n) together with the function $u^{n-k}(t)$ are vanishing, is exactly that ν_s-multiple point ${}^{\nu_s}t^*$, $s \in \{1, 2, \dots, m\}$, of the set of points

$$^{\nu_1}t^*, {}^{\nu_2}t^*, \dots, {}^{\nu_m}t^*,$$

which with respect to their arrangement in the interval (t_0, T_1) lies on the left from the others as the first of them.

Specially if $q = p$, when equation (7) has exactly one and namely simple real root $\lambda' \in \mathbf{R}$, to which in (t_0, T_1) corresponds one and only one simple zero t^* of system (4) of functions $y^*(t, C_1, \ldots, C_k)$ and thus also the solution $y(t)$ of the differential equation (n), the first conjugate point from the right to the point ${}^{n-k}t_0$ is exactly the point

$$ {}^1t_1 = t^*. $$

γ_2) $k - 1$ is an even number, such that we may write $k = 2q + 1$, where $q \in \mathbf{N}$, $q \geqq p$.

If $q > p$, then there exists at least one real root of equation (7), for the sum of multiplicities of the remaining real roots is $2q + 1 - 1 - 2p = 2(q - p)$; in case of $q = p$, equation (7) has no (real) solution, i.e. there exists no real root to which an existence of at least one point in (t_0, T_1) would correspond, at which the function $w(t)$ would be vanishing. Equation (7) has the form

$$ \prod_{s=1}^{p} \{[w(t) - a_s]^2 + b_s^2\} \prod_{j=2p+1}^{2q} [w(t) - \lambda_j'] = 0, $$

whereby on the existence of its (real) solution just equation

$$ \prod_{j=2p+1}^{2q} [w(t) - \lambda_j'] = 0 $$

decides.

If this equation of the $2(q - p)^{th}$ degree has m [where $m \in \{1, 2, \ldots, 2(q - p)\}$] real mutually different roots $\lambda_j' \in \mathbf{R}$, $j = 1, 2, \ldots, m$, with multiplicities $\nu_1, \nu_2, \ldots, \nu_m$ [$\nu_j \in \mathbf{N}$, $j = 1, 2, \ldots, m$, whereby $\sum_{j=1}^{m} \nu_j = 2(q - p)$], then to them correspond m mutually different points

$$ t_1^*, t_2^*, \ldots, t_m^* \in (t_0, T_1) $$

such that

$$ w(t_j^*) = \lambda_j', \qquad j = 1, 2, \ldots, m $$

[cf. again considerations under β)].

The first conjugate point from the right to the point ${}^{n-k}t_0 \in \mathbf{I} = (-\infty, +\infty)$, at which the solution $y(t)$ of the differential equation (n) together with the function $u^{n-k}(t)$ are vanishing, is exactly that ν_s-multiple point ${}^{\nu_s}t^*$, $s \in \{1, 2, \ldots, m\}$, from the set of points

$$ {}^{\nu_1}t^*, {}^{\nu_2}t^*, \ldots, {}^{\nu_m}t^*, $$

which with respect to their arrangement in the interval (t_0, T_1) lies on the left as the first of them.

Specially, if $q = p$, when equation (7) has no real root, the system (4) of functions

$y^*(t, C_1, \ldots, C_k)$ and thus also the solution $y(t)$ of the differential equation (n) has no zero in the open interval (t_0, T_1), so that

$$^{n-k}t_1 = T_1,$$

i.e. the first conjugate point from the right to the point $^{n-k}t_0$ is exactly the first (neighbouring) zero of the function $u(t)$ lying on the right of the point t_0.

In this last (special) case system (4) of functions $y^*(t, C_1, \ldots, C_k)$ has – by any choice of parameters $C_j \in \mathbf{R}$, j = 1, 2, ..., k, $C_k \neq 0$, corresponding to the given conditions – no zeros on the whole interval $\mathbf{I} = (-\infty, +\infty)$ [so that with respect to its continuity on this interval $\mathbf{I}$ there hold either still $y^*(t, C_1, \ldots, C_k) > 0$ or still $y^*(t, C_1, \ldots, C_k) < 0$] and therefore all zeros of solution $y(t)$ of the differential equation (n) coincide with those zeros of the function $u^{n-k}(t)$ and are throughout of multiplicity $\nu = n - k$.

4. In case δ) instead of simple complex conjugate imaginary pairs of roots of equation (7) we consider their possibly multiplicities in a manner completely analogous to that used in γ) including distinguishing two possibilities, where

δ_1) $k - 1$ is an odd number [there always exists at least one real simple or multiple root of the corresponding equation (7)] or where

δ_2) $k - 1$ is an even number [there need not exist any real solution of the corresponding equation (7)].

If we assume – in this most general case – that the equation (7) has exactly p ($p \in \mathbf{N}$) complex (imaginary) mutually different roots

$$\lambda_s = a_s + ib_s,$$

$a_s, b_s \in \mathbf{R}$, $b_s \neq 0$, s = 1, 2, ..., p, with multiplicities $\nu_1, \ldots, \nu_p \in \mathbf{N}$, whereby

$$\sum_{s=1}^{p} \nu_s = M \leqq \frac{k-1}{2}$$

then between remaining roots of (7) there exist again p complex – and namely conjugate – roots

$$\bar{\lambda}_s = a_s - ib_s$$

with corresponding multiplicities equal $\nu_1, \ldots, \nu_s$, for each $\sum_{s=1}^{p} \nu_s = M$ also holds.

Equation (7) is now of the form

$$\prod_{s=1}^{p} \{[w(t) - a_s]^2 + b_s^2\}^{\nu_s} \prod_{j=2M+1}^{k-1} [w(t) - \lambda_j'] = 0.$$

On existence (and multiplicities) of the real roots $\lambda_j' \in \mathbf{R}$, $j \in \{2M + 1, \ldots, k - 1\}$ of (7) only equation

$$\prod_{j=2M+1}^{k-1} [w(t) - \lambda_j'] = 0$$

decides (with total multiplicity $M' = k - 1 - 2M \geqq 0$ remaining for these roots).

The next considerations would virtually follow all those under γ) [together with appealing to the corresponding considerations under β)], where we should get results concerning the existence and multiplicity of the first conjugate point from the right to the point ${}^{n-k}t_0 \in \mathbf{I} = (-\infty, +\infty)$, at which the solution $y(t)$ of the differential equation (n) together with the function $u^{n-k}(t)$ are vanishing.

It turned out that such a point lies in all cases either in an open interval (t_0, T_1) and is of multiplicity $\mu \in \{1, 2, \ldots, k - 3\}$ at most such that

$$1 \leqq \mu \leqq k - 1 - 2M < k - 1$$

always holds; or such a point coincides with the first (neighbouring) zero T_1 of the function $u(t)$ lying to the right of the point t_0 and is of mutliplicity $\mu = n - k$.

By this our *Theorem* 1.1 is completely prooved.

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Souhrn

PRVÝ KONJUGOVANÝ BOD ŘEŠENÍ ITEROVANÉ DIFERENCIÁLNÍ ROVNICE N-tého ŘÁDU

VLADIMÍR VLČEK

V práci je studována iterovaná lineární diferenciální rovnice n-tého řádu (n), jejímž obecným řešením je homogenní polynom $(n - 1)$-ho stupně s n libovolnými (reálnými) koeficienty ve funkcích $u(t)$, $v(t)$, tvořících bázi oscilatorické lineární homogenní diferenciální rovnice 2. řádu v Jacobiho tvaru.

Ke studiu nulových bodů řešení diferenciální rovnice (n) je vybrán systém všech jejích oscilatorických řešení, které se spolu s funkcí $u(t)$ anulují v libovolném pevně

zvoleném bodě $t_0 \in \mathbf{I} = (-\infty, +\infty)$; přitom v *Lemmě* 1 je dokázána nutná a postačující podmínka k tomu, aby t_0 byl právě k-násobným ($k = 1, 2, \ldots, n - 1$) nulovým bodem takového systému řešení dif. rovnice (n).

Po zavedení pojmu tzv. 1. konjugovaného bodu zprava k bodu t_0 pak podstatnou část práce tvoří důkaz věty o existenci a všech možných násobnostech takového bodu do $(n - 1)$-ho řádu včetně.

Резюме

ПЕРВАЯ СОПРЯЖЕННАЯ ТОЧКА РЕШЕНИЯ ИТЕРИРОВАННОГО ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ n-того ПОРЯДКА

ВЛАДИМИР ВЛЧЕК

В работе изучается итерированное линейное дифференциальное уравнение n-того порядка (n), общим решением котогоро является однородный полином $(n - 1)$-ой степени с n произвольными (существенными) коэффициентами в функциях $u(t)$, $v(t)$, осуществляющих базис осцилляционного линейного однородного дифференциального уравнения 2-*го* порядка типа Якобы.

К изучению нулевых точек решений дифференциального уравнения (n) избранна система всех таких его колеблющихся решений, которые вместе с функцией $u(t)$ аннулируются в любой фиксированно выбранной точке $t_0 \in \mathrm{I} = (-\infty, +\infty)$. При этом предположении в *Леме* 1. доказывается необходимое и достаточное условие для того, чтобы t_0 была точно κ-насобная ($\kappa = 1, 2, \ldots, n - 1$) точка такой системы решений дифф. уравнения (n).

После определения так называемой первой точки сопряженной направо от точки t_0, главная часть этой работы состоит из доказательства теоремы об существовании и всех возможных насобностьях такой точки до $(n - 1)$-го порядка включительно.