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PROGRAMMING OF DIFFERENTIAL EQUATIONS WITH SINGULARITIENS OF THE TYPE $\frac{0}{0}$ IN USING THE EXTENSION TO POWER SERIES

KAREL BENEŠ

(Received March 25, 1981)

Dedicated to Prof. Miroslav Laitoch on his 60th birthday

In the technical practice we often meet with differential equations in which indefinite expresions of the type $\frac{0}{0}$ occur. See for instance the differential equation of the 3^{rd} order, whose solution is the function Si (t) as shown below. The indefinite expressions of the type $\frac{0}{0}$ are programmed in the form

$$z(t) \doteq \frac{f(t) + z(0)\,ae^{-\alpha t}}{g(t) + ae^{-\alpha t}}, \tag{1}$$

$$f(0) = 0, \qquad g(0) = 0, \qquad \frac{f(t)}{g(t)} = z(t), \qquad z(0) = \lim_{t \to 0} \frac{f(t)}{g(t)}.$$

The indefinite expressions of the type $\frac{0}{0}$ occur in some differential equations with variable coefficients or in some nonlinear differential equations. The general form of differential equations with variable coefficients has in this case the form

$$\sum_{k=0}^{n} \frac{u_k(t)\, y^{(k)}}{g_k(t)} = F(t), \tag{2}$$

the general form of the nonlinear differential equations of the given type is

$$\sum_{k=0}^{n} \frac{u_k(t)\, f_k(y^{(k)})}{v_k(t)\, g_j(y^{(j)})} = F(t) \qquad j = 0, 1, 2, \ldots, n, \tag{3}$$

where in the equations (2) and (3) some of the expressions $u_k(t)$, $v_k(t)$ and $g_j(y^{(j)})$

may be constants. From now on we shall assume that in (2) and (3) only one broken expression is an indefinite expression of the type $\frac{0}{0}$.

The differential equation

$$ty''' + 2y'' + ty' = 0 \tag{4}$$

is a type of equation (2), which programmed in the form

$$y''' + \frac{2}{t}y'' + y' = 0 \tag{5}$$

with initial conditions $y(o) = 0$, $y'(o) = 1$, $y''(o) = 0$.

The differential equation

$$t^2y'' - t^2y' - y^2 = -2t^2 \tag{6}$$

is a type of equation (3), which programmed in the form

$$y'' - y' - \frac{y^2}{t^2} = -2 \tag{7}$$

with initial conditions $y(o) = 0$, $y'(o) = 1$.

In (5) and (7) there occur singularities of the type $\frac{0}{0}$ for $t = 0$. In machine solving it is necessary to determine the limit of this expression. We shall show next a method using the extension to power series which may be applied in solving this problem.

Assume now the solution of (4) in the form of a power series and determine the derivatives of solutions up to the highest order, i.e. up to the third derivative:

$$y = a_0 + a_1t + a_2t^2 + \ldots + a_nt^n, \tag{8}$$

$$y' = a_1 + 2a_2t + 3a_3t^2 + 4a_4t^3 + \ldots + na_nt^{n-1} \tag{8a}$$

$$y'' = 2a_2 + 6a_3t + 12a_4t^2 + 20a_5t^3 + \ldots + n(n-1)\,a_nt^{n-2} \tag{8b}$$

$$y''' = 6a_3 + 24a_4t + 60a_5t^2 + \ldots + n(n-1)(n-2)\,a_nt^{n-3} \tag{8c}$$

Inserting this into equation (4) we obtain

$$\begin{aligned} &6a_3t + 24a_4t^2 + 60a_5t^3 + \ldots + n(n-1)(n-2)\,a_nt^{n-2} + \\ &+ 4a_2 + 12a_3t + 24a_4t^2 + 40a_5t^3 + \ldots + 2n(n-1)\,a_nt^{n-2} + \\ &+ a_1t + 2a_2t^2 + 3a_3t^3 + 4a_4t^4 + \ldots + na_nt^n = 0. \end{aligned} \tag{9}$$

The coefficients a_1 through a_n will be obtained by comparing the coefficients in particular powers of the independet variable t. In view of the fact that for determining $\lim_{t \to 0} \frac{y''}{t}$ it suffices to determine $y'''(o)$, it is sufficient for us — as follows from (8c) to determine the coefficient a_3. On the ground of the given initial conditions and relation (8), (8a) and (8b) it holds

$$a_0 = 0, \qquad a_1 = 1, \qquad a_2 = 0,$$

comparing the coefficients in the first power t we get

$$6a_3 + 12a_3 + a_1 = 0,$$

i.e.

$$18a_3 = -a_1,$$

$$a_3 = -\frac{a_1}{18} = -\frac{1}{18}.$$

By (8c) it holds $y'''(o) = 6a_3 = -\frac{1}{3}$. From (5) we come to

$$\lim_{t\to 0} \frac{y''}{t} = \frac{-y'''(o) - y'(o)}{2} = \frac{\frac{1}{3} - 1}{2} = -\frac{1}{3}.$$

The value of the expression line $\lim\limits_{t\to 0} \frac{y''}{t}$ can be determined by substituting for y'' according (8b), i.e.

$$\lim_{t\to 0} \frac{y''}{t} = \lim_{t\to 0} \frac{6a_3t + 12a_4t^2 + \ldots + n(n-1)\,a_nt^{n-2}}{t} = 6a_3.$$

Substituting into (5) we get

$$6a_3 + 12a_3 + a_1 = 0,$$

whence, in case of $a_1 = 1$ we determine the walue of $a_3 = -\frac{1}{18}$.

The expression $z = \frac{y''}{t}$ is programmed in the form

$$z \doteq \frac{y'' + z(o)\,ae^{-\alpha t}}{t + ae^{-\alpha t}}, \tag{9a}$$

where $z(0) \lim\limits_{t\to 0} \frac{y''}{t} = -\frac{1}{3}$. The function $y = \text{Si}\,(t)$ is the solution of (4) with the given initial conditions. The programme chart for the solution of equation (5) is shown in Fig. 1.

The above method may be used in programming nonlinear differential equations where the singularity of the mentioned type occurs, as it was the case in soloing equation (6), programmed in the form of (7). Let us assume the solution again in the form

$$y = a_0 + a_1t + a_2t^2 + \ldots + a_nt^n, \tag{10}$$

i.e.

$$y' = a_1 + 2a_2t + 3a_3t^2 + \ldots + na_nt^{n-1} \tag{10a}$$

$$y'' = 2a_2 + 6a_3t + \ldots + na_n(n-1)\,t^{n-2}. \tag{10b}$$

From the given initial condition $y(0) = 0, y'(0) = 1$ now follows $a_0 = 0, a_1 = 1$. By equation (7) it suffices for determining the limit of the expression $\lim_{t \to 0} \frac{y^2}{t^2}$ to determine the value $y''(0)$, i.e. to determine the coefficient a_2. Inserting the relations (10), (10a) and (10b) into equation (6) gives

$$\begin{aligned} &2a_2t^2 + 6a_3t^3 + \dots + n(n-1)\, a_nt^n - a_1t^2 - 2a_2t^3 - \dots - na_nt^{n+1} - \\ &- a_0^2 - a_1^2t^2 - a_2^2t^4 - \dots - a_n^2t^{2n} - 2a_0a_1t - 2a_0a_2t^2 - \\ &- 2a_0a_3t^3 - \dots - 2a_1a_2t^3 - \dots - 2a_{n-1}a_nt^{2n-1} = -2t^2, \end{aligned} \tag{11}$$

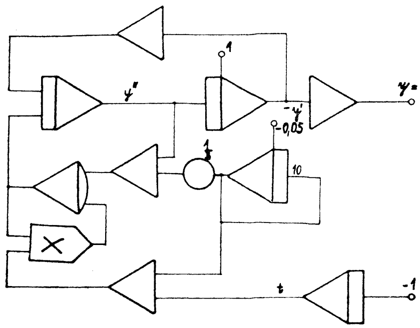

Fig. 1

where $a_0 = 0$, $a_1 = 1$. Comparing the coefficients in the second power t we get

$$2a_2 - a_1 - a_1 = -2,$$

i.e.

$$2a_2 - 2 = -2,$$

$$a_2 = 0, \qquad y''(0) = 0.$$

Inserting to equation (7) gives

$$\lim_{t \to 0} \frac{y^2}{t^2} = 2 + y''(0) - y'(0) = 1. \tag{12}$$

The value of the expression $\lim_{t \to 0} \frac{y^2}{t^2}$ may be determined easier by substituting for y'' by equation (10), i.e.

$$\lim_{t \to 0} \frac{y^2}{t^2} = \frac{a_0^2 + a_1^2t^2 + a_2^2t^4 + \dots + 2a_0a_1t + 2a_0a_2t^2 + \dots + a_n^2t^{2n}}{t^2} = a_1^2 = 1.$$

By the given initial condition we get $a_1 = 1$. The function $y = t$ is the solution of equation (6). The programme chart for the solution of equation (7) is in Fig. 2.

In programming nonlinear differential equations there occur some difficulties in determining the coefficients a_k, for the equation by whose solutions these coefficients are determined, is a nonlinear equation, the equation has a necessary number of roots and solution of the given differential equation need not be one-to-one, as it is the case of the differential equation

$$ty'^2 - y = 0 \tag{13}$$

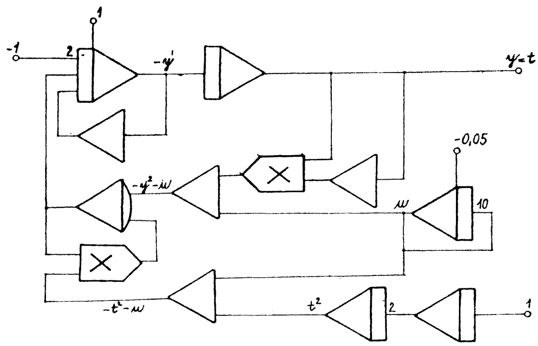

Fig. 2

with the initial condition $y(0) = 0$, programmed in the form

$$y' - \frac{y}{ty'} = 0. \tag{14}$$

If we assume a solution in the form

$$y = a_0 + a_1 t + a_2 t^2 + \ldots + a_n t^n,$$

then

$$y' = a_1 + 2a_2 t + 3a_3 t^2 + \ldots + n a_n t^{n-1},$$

$$y'^2 = a_1^2 + 4a_2^2 t^2 + 9a_3^2 t^4 + \ldots + 4a_1 a_2 t + 6a_1 a_3 t^2 + \ldots + n^2 a_n^n t^{2n-2}.$$

Substituting the above relations into equation (13) we get

$$a_1^2 t + 4a_2^2 t^3 + 9a_3^2 t^4 + \ldots + 4a_1 a_2 t^2 + \ldots + a_0 - a_1 t - a_2 t^2 - \ldots - a_n t^n = 0.$$

If we compare the coefficients we get $a_0 = 0$ (as also follows from the initial condition) and it holds

$$a_1^2 - a_1 = 0, \qquad \text{i.e.} \qquad a_1 = 0; 1.$$

By (14) we have

$$\lim_{t \to 0} \frac{y}{ty'} = y'(0) = a_1 = 0; 1.$$

In case of $a_1 = 0$ is the solution of equation (13) $y = 0$, white $y = t$ in case of $a_1 = 1$.

The programme chart for solving equation (13) in the form of (14) is in the Fig. 3.

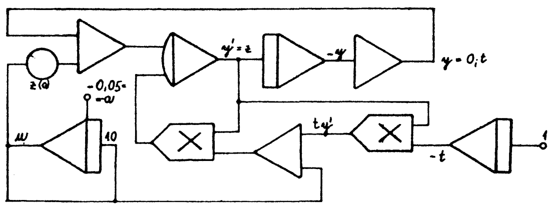

Fig. 3

In programming nonlinear differential equations with singularities of above type it need not be apparent in some cases that a singularity of the type $\frac{0}{0}$ is in question, as it is the case say at the differential equation

$$y'^2 - ty' - 2y = 0 \qquad (15)$$

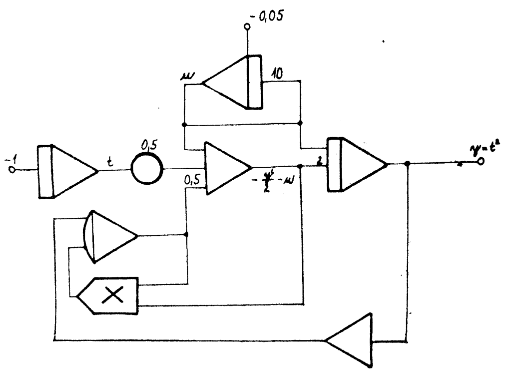

Fig. 4

with the initial condition $y(0) = 0$, programmed in the form

$$y' - \frac{2y}{y'} = t. \qquad (16)$$

Let as now assume a solution of the form

$$y = a_0 + a_1 t + a_2 t^2 + \dots + a_n t^n,$$

then

$$y' = a_1 + 2a_2 t + 3a_3 t^2 + \dots + n a_n t^{n-1},$$
$$y'^2 = a_1^2 + 4a_2^2 t^2 + 9a_3^2 t^4 + \dots + 4a_1 a_2 t + 6a_1 a_3 t^2 + \dots + n^2 a_n^2 t^{2n-2},$$

where $a_0 = 0$.
Inserting the above relations into equation (15) gives

$$a_1^2 + 4a_2^2 t^2 + 9a_3^2 t^4 + \dots - a_1 t - 2a_2 t^2 - 3a_3 t^3 - \dots - 2a_1 t - \\ - 2a_2 t^2 - \dots - 2a_n t^n = 0.$$

If we compare the coefficients, we see that

$$a_1^2 = 0, \qquad \text{i.e.} \qquad a_1 = 0, \qquad y'(0) = 0,$$

so that the expression $\frac{2y}{y'}$ is an indefinite expression of the type $\frac{0}{0}$.

$$\lim_{t \to 0} \frac{2y}{y'} = \lim_{t \to 0} \frac{2a_2 t^2 + 2a_3 t^3 + \dots + 2a_n t_n^n}{2a_2 t + 3a_3 t^2 + \dots + n a_n t^{n-1}} = 0.$$

The programme chart for the solution of (16) is given in Fig. 4.

If in (2) and (3) more indefinite expressions of the type $\frac{0}{0}$ occur, then the relations for the limits of the indefinite expressions are to be determined first, with comparing the coefficients afterwards. For instance, in programming the differential equation

$$t^2 y'' + t y' - y = 3t^2 \tag{17}$$

with the initial conditions $y(0) = 0$, $y'(0) = 0$ we proceed as follows:
The equation is programmed in the form

$$y'' + \frac{y'}{t} - \frac{y}{t^2} = 3. \tag{18}$$

The solution will be assumed to be in the form of a series, i.e.

$$\begin{aligned} y &= a_0 + a_1 t + a_2 t^2 + \dots + a_n t^n, \\ y' &= a_1 + 2a_2 t + 3a_3 t^2 + \dots + n a_n t^{n-1}, \\ y'' &= 2a_2 + 6a_3 t + 12a_4 t^2 + \dots + n(n-1)\, a_n t^{n-2}. \end{aligned} \tag{19}$$

If follows from the initial conditions that $a_0 = a_1 = 0$, so that

$$\lim_{t \to 0} \frac{y}{t^2} = a_2, \qquad \lim_{t \to 0} \frac{y'}{t} = 2a_2. \tag{20}$$

The coefficient a_2 will be determined by inserting the relations (19) into (17) and by comparing the coefficients, i.e.

$$2a_2t^2 + 6a_3t^3 + \dots + n(n-1)\,a_nt^n + \dots + 2a_2t^2 +$$
$$+ 3a_3t^3 + \dots + na_nt^n - a_2t^2 - a_3t^3 - \dots - a_nt^n = 3t^2$$

$$2a_2 + 2a_2 - a_2 = 3$$

$$a_2 = 1.$$

Substitution into relations (20) gives

$$\lim_{t\to 0} \frac{y}{t^2} = 1, \qquad \lim_{t\to 0} \frac{y'}{t} = 2.$$

The programme chart for the solution of (17) in the form of (18) is given in Fig. 5

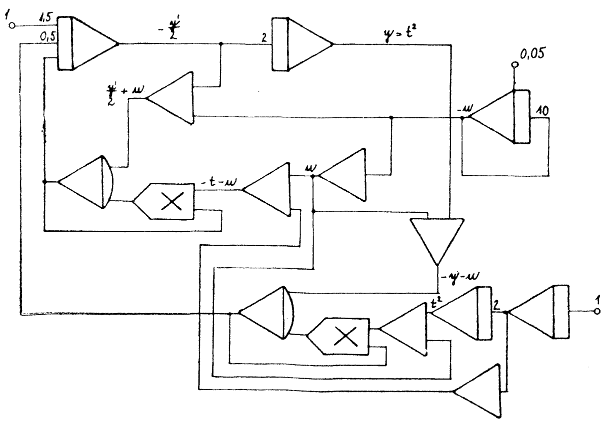

Fig. 5

$u = 0{,}05e^{-10t}$. This procedure can equally well be applied to programming non-linear differential equations, such as

$$ty'^2y'' + y'^3 - 4ty = 3ty'^2 \tag{21}$$

with the initial conditions $y(0) = y'(0) = 0$.
The equation is programmed in the form

$$y'' + \frac{y'}{t} - \frac{4y}{y'^2} = 3. \tag{22}$$

Then, by expending the solution in a series, we get

$$y = a_0 + a_1 t + a_2 t^2 + \ldots + a_n t^n,$$
$$y' = a_1 + 2a_2 t + 3a_3 t^2 + \ldots + n a_n t^{n-1},$$
$$y'' = 2a_2 + 6a_3 t + \ldots + n(n-1)\, a_n t^{n-2},$$

where $a_0 = a_1 = 0$,

$$y'^2 = 4a_2^2 t^2 + 9a_3^2 t^4 + \ldots + 12 a_2 a_3 t^3 + \ldots + n^2 a_n^2 t^{2n-2},$$
$$y'^3 = 8a_2^3 t^3 + 27 a_3^3 t^6 + \ldots + 3 \,.\, 4a_2 t^2 \,.\, 3a_3 t^2 + \ldots + n^3 a_n^3 t^{3n-3},$$

$$\lim_{t \to 0} \frac{y'}{t} = 2a_2, \qquad \lim_{t \to 0} \frac{4y}{y'^2} = \frac{1}{a_2}.$$

Inserting the above relations into (21) and comparing the coefficients at t^3 yields

$$8a_2^3 t^3 + \ldots + 8a_2^3 t^3 + \ldots + (-4a_2^2 t^3) = 12 a_2^2 t^3,$$

i.e.

$$16 a_2^3 - 12 a_2^2 - 4a_2 = 0,$$

i.e.

$$a_2 = 0;\ 1;\ -\frac{1}{4}.$$

The value $a_2 = 0$ is insuitable for machine solving from the given value t, because $\lim_{t \to 0} \frac{4y}{y'^2} \to \infty$, in case of $a_2 = 1$ is $\lim_{t \to 0} \frac{y'}{t} = 2$, $\lim_{t \to 0} \frac{4y}{y'^2} = 1$ and the function $y = t^2$ is the solution. In case of $a_2 = -\frac{1}{4}$ we have $\lim_{t \to 0} \frac{y'}{t} = -\frac{1}{2}$, $\lim_{t \to 0} \frac{4y}{y'^2} = -4$ and the function $-\frac{1}{4} t^2$ is the solution.

The programme chart for the solution of (21) in the form of (22) for the case $a_2 = -\frac{1}{4}$ is shown in Fig. 6.

It is well to point out that in modelling the indefinite expressions of the type $\frac{0}{0}$ on electronic analog computers with a diode multiplier there arises a possibly occurence of a greater error in the modelled quotient in case of small values of the numerator and the denominator. Optimal values m were determined in [1] and [2], at which the dividing circuit works relatively even with small values of the numerator and the denominator as

$$m = 2x_j; \tag{23}$$

the most nufavourable values m are given by

$$m = x_j + x_{j+1}, \tag{24}$$

where x_j stand for the coordinates of the break points of linear functions approximating the quadratic dependences at the diode function transformators of the multiplier.

Let the expression $z = \frac{f(t)}{g(t)}$ arise in the initial net for the solution of the given differential equation, where

$$f(t) \doteq at^{\nu}, \qquad g(t) \doteq bt^{\nu} \qquad \text{for } t \to 0,$$

so that

$$\frac{f(t)}{g(t)} \doteq \frac{at^{\nu}}{bt^{\nu}} \doteq \frac{ct^{\nu}}{t^{\nu}}. \tag{25}$$

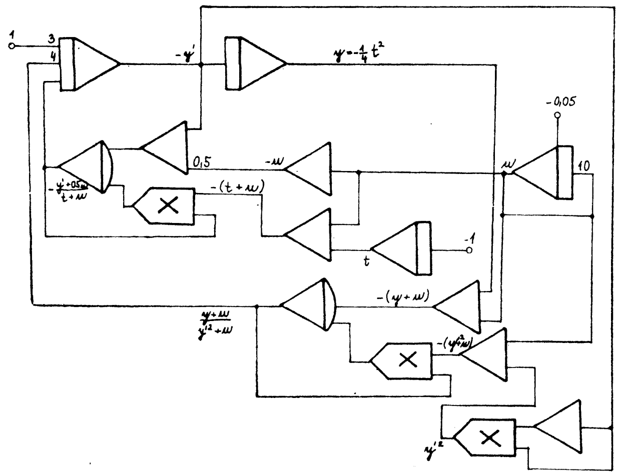

Fig. 6

Modelling instead of the function $z(t) = \frac{f(t)}{g(t)}$ its suintable multiple, i.e.

$$\frac{qf(t)}{g(t)} \doteq \frac{qct^{\nu}}{t^{\nu}}, \tag{26}$$

then by a suitable choice of the coefficient q it possible to model the quotient $\frac{qf(t)}{g(t)}$ with a good accuracy even for very small values t. The coefficient q is to be chosen to fulfil the relation

$$qc = m, \tag{27}$$

i.e. the expression $qc = m$ is to be equal to the small value m by (23). For instance, in programming the differential equation

$$t^2y' - y = 2t^3 - t^2 \tag{28}$$

with the initial condition $y(0) = 0$, whose solution is the function $y = t^2$, we proceed as follows:

Equation (28) is programmed in the form $y' - \frac{y}{t^2} + 2t - 1$, $\lim\limits_{t \to 0} \frac{y}{t^2} = 1$. The expression $z = \frac{y}{t^2}$ is modelled in the form

$$qz = \frac{-q(y + u)}{-(t^2 + u)}, \qquad (c = 1). \tag{29}$$

For the computer MEDA 41 TC the optimal values $m = 0{,}237$, $0{,}447$, $0{,}670$, $0{,}881$ (we choose $m \leqq 1$) by (23) are satisfactory and the values $m = 0{,}342$, $0{,}559$, $0{,}775$, $0{,}987$ by (24) are not satisfactory. Respecting $c = 1$, it holds by (25) $q = m$. Let us choose $q = 0{,}881$ and $q = 0{,}987$. This values were for given computer MEDA TC experimentell precised on the values $q = 0{,}850$ and $q = 0{,}956$, $u =$

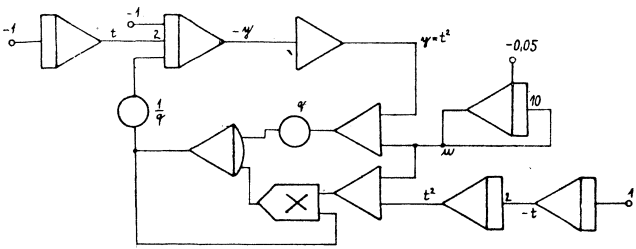

Fig. 7

$= 0{,}050e^{-10t}$. The negative signs in the numerator and the denominator are connected with the requirement for the stability of the counting net in modelling the requirement for the stability of the counting net in modelling the quotient.

The programming chart for the solution of the equation (26) is given in Fig. 7. It can be seen from tab. 1 for $q = 0{,}850$ and from tab. 2 for $q = 0{,}956$ that in the case of $q = 0{,}956$ the solution is completely depriciated due to large inaccuracy in modelling the quotient of small values. Since the solution is not konvergent ($y = t^2$), the accuracy of the solution for $q = 0{,}850$ is very good. More accurate modelling of the quotient at the beginning of the solution substantially improves the result in the whole interval in that the solution is looked for.

Tab. 1

t	y_{tab}	y	$\delta(y)$	$y + u$	$t^2 + u$	z_{tab}	z	$\delta(z)$
0,000	0,000	0,000	0,000	0,050	0,050	1,000	1,002	0,002
0,030	0,001	0,001	0,000	0,038	0,038	1,000	0,097	—0,003
0,050	0,003	0,004	0,001	0,033	0,033	1,000	0,997	—0,003
0,100	0,010	0,012	0,002	0,027	0,027	1,000	1,003	0,003
0,200	0,040	0,042	0,002	0,048	0,047	1,021	1,011	—0,010
0,300	0,090	0,091	0,001	0,092	0,090	1,022	1,021	—0,001
0,600	0,360	0,369	0,009	0,369	0,360	1,025	1,029	0,004
1,000	1,000	1,022	0,022	1,021	1,007	1,013	1,014	0,001

$q = 0{,}850$

Tab. 2

t	y_{tab}	y	$\delta(y)$	$y + u$	$t^2 + u$	z_{tab}	z	$\delta(z)$
0,000	0,000	0,000	0,000	0,050	0,050	1,000	0,058	—0,042
0,030	0,001	—0,001	—0,002	0,037	0,038	0,973	0,927	—0,046
0,050	0,003	—0,003	—0,006	0,028	0,033	0,848	0,801	—0,047
0,100	0,010	—0,005	—0,015	0,014	0,024	0,583	0,521	—0,062
0,200	0,040	—0,017	—0,057	—0,010	0,047	—0,212	—0,201	—0,011
0,300	0,090	—0,070	—0,160	—0,069	0,092	—0,750	—0,746	0,004
0,600	0,360	—0,308	—0,668	—0,308	0,361	—0,853	—0,842	0,011
1,000	1,000	—0,450	—1,450	—0,450	1,004	—0,448	—0,441	0,007

$q = 0{,}956$

REFERENCES

[1] K. Beneš: *Zvýšení přesnosti při modelování podílu dvou funkcí.* Elektrotechnický časopis, 1977, č. 7, str. 481—493.

[2] K. Beneš: *Zvýšení přesnosti při modelování podílu malých hodnot u počítačů MEDA 40 TA MEDA 41 TC a ADT 3000.* Elektrotechnický časopis, (To appear).

[3] V. V. Stěpanov: *Kurs diferenciálních rovnic.* Přírodovědecké nakladatelství, Praha, 1950.

Souhrn

PROGRAMOVÁNÍ DIFERENCIÁLNÍCH ROVNIC SE SINGULARITAMI TYPU $\frac{0}{0}$ POUŽITÍM ROZVOJE V MOCNINNOU ŘADU

KAREL BENEŠ

Práce se zabývá určením limit neurčitých výrazů typu $\frac{0}{0}$ při strojovém řešení diferenciálních rovnic. Řešení dané rovnice a jeho derivace se předpokládají ve tvaru mocninné řady. Limity výrazů se určují srovnáním koeficientů u příslušných mocnin *t*.

Резюме

ПРОГРАММИРОВАНИЕ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ С СИНГИЛАРИТАМИ ТИПА $\frac{0}{0}$ РАЗВИТИЕМ В СТЕПЕННЫЙ РЯД

КАРЕЛ БЕНЕШ

В статье описан способ определения пределов неопределенных выражений типа $\frac{0}{0}$ при машинном решении дифференциальных уравнений. Решение данного уравнения и производные уравнения предпологаются в форме степенного ряда. Пределы выражений определяются уравнением коеффициентов у соответствующих степеней т.